Advanced Process Dynamics

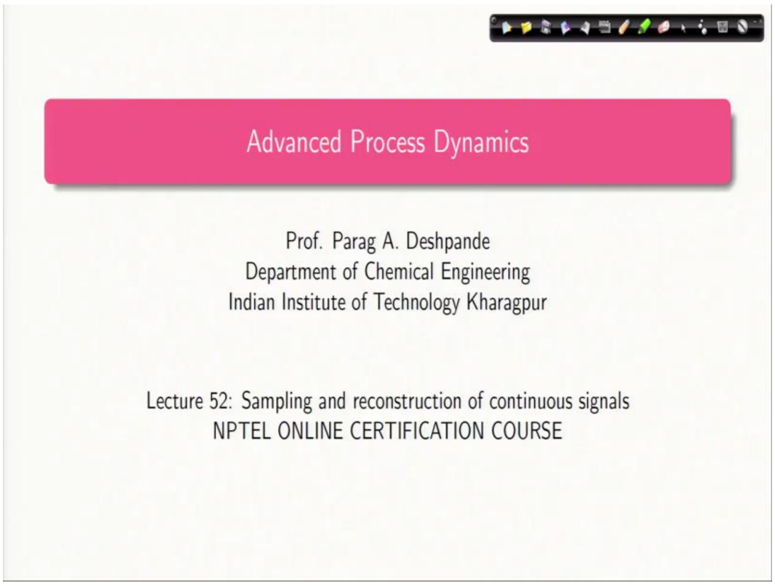
Professor Parag A. Deshpande

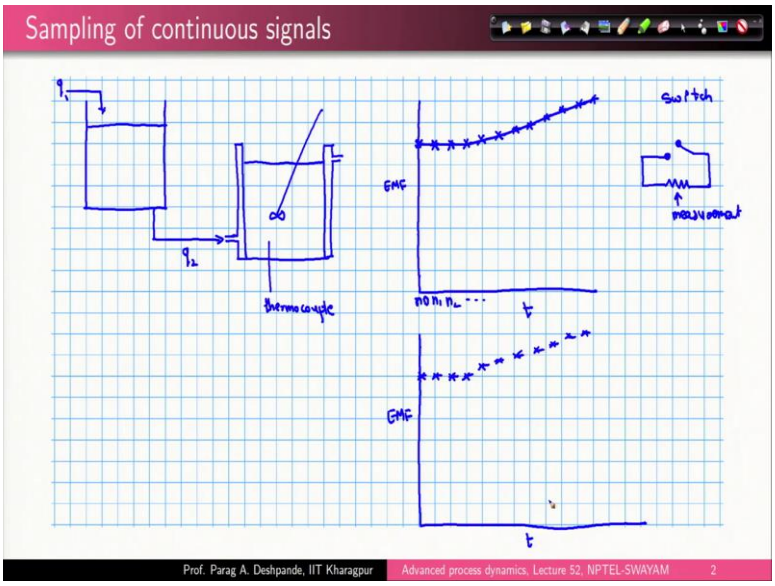
Department of Chemical Engineering

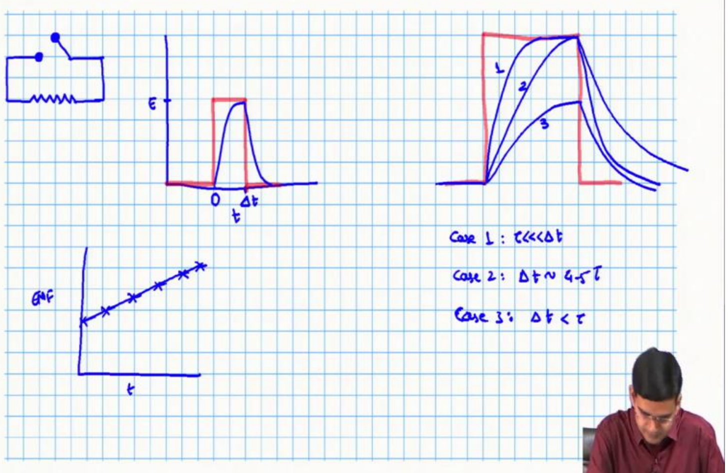
Indian Institute of Technology, Kharagpur

Lecture 52

Sampling and reconstruction of continuous signals







The mathematical framework

$$y^*(nT) = y(nT)\delta(t - nT) \quad (1)$$

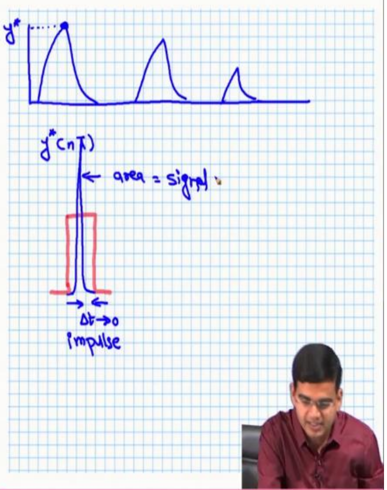
$$y^*t = y^*(0)+y^*(T)+y^*(2T)\cdots \quad (2)$$

$$y^*(t) = y(0)\delta(t) + y(T)\delta(t-T)$$

$$+y(2T)\delta(t-2T) + \cdots \quad (3)$$

$$y^*(t) = \sum_{n=0}^{\infty} y(nT)\delta(t-nT) \quad (4)$$

$$y^*(s) = \sum_{n=0}^{\infty} y(nT)e^{-nTs} \quad (5)$$



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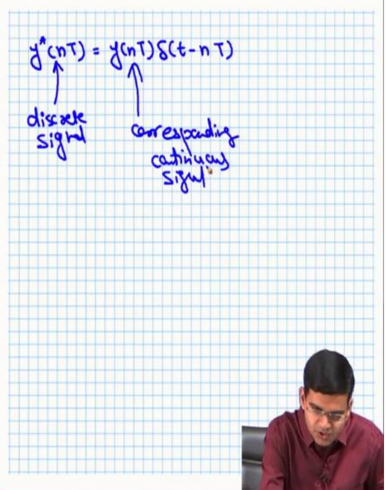

Sampling of continuous signals

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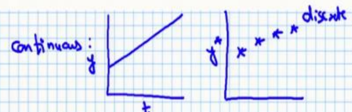


$y(t) = y_0 + mt$

$y^*(t) = y^*(0) + y^*(T) + y^*(2T) + \cdots$

$y^*(nT) = y(nT)\delta(t-nT)$

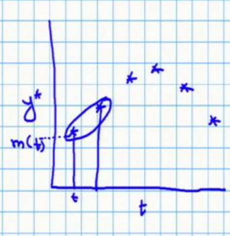
$y^*(t) = \sum_{n=0}^{\infty} y(nT)\delta(t-nT)$

$y^*(s) = \sum_{n=0}^{\infty} y(nT)e^{-sTn}$

Reconstruction of continuous signals

$$m(t) = m(nT) + \left(\frac{dm}{dt}\right)_{nT}(t-nT) + \frac{1}{2!}\left(\frac{d^2m}{dt^2}\right)_{nT}(t-nT)^2 + \cdots \quad (6)$$

Hold element



Reconstruction of continuous signals

$$m(t) = m(nT) + \left(\frac{dm}{dt}\right)_{nT}(t-nT) + \frac{1}{2!}\left(\frac{d^2m}{dt^2}\right)_{nT}(t-nT)^2 + \cdots \quad (6)$$

Zero-order hold:

$m(t) = m(nT) \quad - (1)$

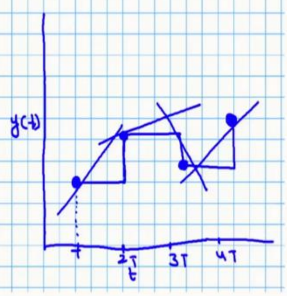
$T \rightarrow 2T$

First order hold:

$m(t) = m(nT) + \left.\frac{dm}{dt}\right|_{nT}(t-nT)$

$m(nT) + \alpha t - \alpha nT$

$m(t) = \alpha t + (m(nT) - \alpha nT) \quad - (2)$



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So, let us take the topic of sampling and reconstruction of continuous signals. To remind ourselves of the reason why we are doing this, we studied in the previous lecture that we need to convert continuous signals to give them as feed to digital computers. So, continuous signals must be converted to discrete time signals. Processing will be taking place inside the computer, so we would need the discrete time models.

We will worry about them in the next lecture. But then once the control action has been decided, the digital signals generated by the digital computer conforming to the control action must be converted back to continuous signal. So, sampling means conversion of continuous time signals to discrete signals and reconstruction of continuous signals means getting continuous signals back from discrete signals. So, let us see what we can or how we can understand this particular aspect.

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In the previous lecture, we took this particular toy example where we have a tank with inlet q_1, inlet is equipped with a centrifugal pump, there is an outlet and outlet flow rate is q_2 and this outlet is fed as and inlet to a jacket and jacket in turn is the cooling jacket of a CSTR. This was our arrangement. We said that we would have a thermocouple to measure the temperature of the reactor and that would be fed as an input signal to your digital computer which would already have an information about the set point the desired temperature of the reactor that controller will determine the difference is called the error and it would implement the control action.

So, now, you have the output from the thermocouple which is EMF of the system. So, how would you convert the signal which is continuous to a digital signal in this particular case. So, suppose you have a signal which is the EMF coming out of the thermocouple and I plotted against time t at steady state you would expect a horizontal line. So, the system was at steady state and then you started to realize that the temperature of my reactor is increasing.

So, let us say that this is the general trend of EMF that you get from the thermocouple. This cannot be fed to a digital computer you want a series of discrete signals. So, what can be one of the ways for me to generate the thermocouple? Well, I can make use of what is called a switch. So, what is the switch? I will have this load here. And then I can imagine that I can operate this switch at regular intervals of time to me make a measurement at the load.

In a continuous signal what would happen the switch would be closed always the current would be continuously flowing, there would be signal that you would get. Now, I have a switch, so I can at my will close the switch get a reading and the moment I open the switch the reading becomes zero which means I do not have any reading. So, therefore, imagine that I have a certain instant of time in which I close the switch, what I should get, well ideally, I should get this reading.

Then imagine that I start from here, first instance of time n_1 I get this let me call this n_0, I get this that n_1, I get this n_2, I get this and then I can imagine that if I keep on doing this then I would get a series of discrete signals, which means that my discrete time signals which would be fed to the computer which would have EMF whether the signal is continuous or discrete EMF would remain EMF would look something like this.

So, looks like there is not much difference between what you got on the top on and what I got on in the bottom except that I do not have the observations at all points of time. But there is a problem. So, what is the problem?

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The problem can be quite easily understood by invoking the concept of response of our system, dynamical response of any system which we have studied in continuous time domain. So, what happens here, I have a switch and I have a load, I am trying to make a measurement across the load. So, now here I have an initial steady state because the circuit is open there is no signal. So, therefore, I have a steady state observation which is zero.

So, this is EMF, E, this is time t, there is no signal, the signal is zero. Now, the moment I switch, I put the switch to a closed position I declare the time as zero. So, this time and this is always possible I have treated this over and again this time is zero. I close the switch, what is going to happen, I instantly get a signal. So, in my continuous time domain, this should have been the magnitude of my signal. And for how long do I keep the switch on?

Imagine that I keep the switch on, because everything is happening physically, it is a physical process of switching on switching off and you want to sample the value of EMF at that particular moment of time. You will keep the switch closed for certain interval of time and let us imagine that that interval of time is Δt. For time Δt, you keep the switch on closed position. So, what would happen between zero and t, between zero and t you should get the same signal which is the value of the signal and then the moment Δt comes you put the switch off again. So, therefore what should happen?

You should return to zero and you should have again the signal going to zero. Now, this is what should happen. And this is the digital output that you should get, but that is not what actually happens. Why does it not happen? Because the system will have some order following which the system will have some dynamics, it is a simple case of a rectangular pulse input the action of switching the switch off maintaining it for some time and putting it off again results in a rectangular pulse input to the system, but the system has to respond.

Because whatever is recording your EMF has to adjust to the new steady state which is the amplitude of your EMF there. So, therefore, now here comes the real tricky situation. We previously studied that if I have a rectangular pulse input function like this, then I have one situation that my response which is given by the blue line would rise like this. And now, since I have a -A it would come like this, this is only one of the responses, the other responses possible response is this. In fact, you also have a third response.

And what do these three responses depend upon? These three responses depend upon the relative magnitudes of Δt and the time constant. So, for 1, 2 and 3,

$$\text{case 1: } \tau <<< \Delta t$$

$$\text{case 2: } \Delta t \sim 4\text{-}5\ \tau$$

$$\text{case 3: } \Delta t < \tau$$

And in case 3, your $\Delta t < \tau$. That is why you have not even reached till the enforced input. So, therefore, now, if I want, if I have a signal EMF versus time which looks like this and I want a digital signal which should look like this I need to worry about the dynamics of the element which is used in your switch, the elements which are recording your EMF and the time for which I am keeping my circuit closed, all these three things must be taken into account.

And therefore, you may see that if you look at situation number 3, case number 3, it is quite apparent that you are going to get absolutely wrong values of your EMF from your system if your $\Delta t < \tau$. So, therefore, sampling of signal itself is not a trivial thing you need to worry about the time constant which is involved in the system which would govern the dynamics of all the components which would record and undergo change in your system.

But in case you do take care of it. Imagine that you have, you have in fact taken care of it. Then how would your signal look like, your signal would look like something like this. So, this is your, the red rectangular pulse is your ideal input, ideally your output should look like this, but you know that that happens only in case of pure gain systems. So, therefore, you would have a signal which would look like. Which is typical output from pulse input function. And if that is the case is there any way for me to model my system? In fact, I have.

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So, what is the underlying mathematical framework? I have my sample signal which is given as a series of rectangular pulse responses and so on. And ideally, they should be rectangles. So, now, when you have a system in continuous domain you can use y(t) as your function. So, you can describe the variation of the signal in time as continuous variable. Now,

$$y^*(nT) = y(nT)\delta(t - nT) \ldots\ldots\ldots (1)$$

And what is the assumption? Now, we have assumed that I have reduced my pulse width Δt to 0. And therefore, what is going to happen? I must have an impulse function. And what would be the area under the impulse function? The area under the impulse function would be equal to the signal, the value of the signal. And that way, you can say that my signal happens precisely at this location and it has an intensity of this much.

And that is how what you convert a continuous time signal to a discrete signal saying the y* is simply a statement of the fact that it is a discrete signal. So, $y^*(nT)$, when n is a natural number, so, at time T, at time 2T, at time 3T and so on. So, here in this case, Δt is nothing but T. So, this would be what

$$y^*(nT) = y(nT)\delta(t - nT)$$

where this is the discrete signal and this is the corresponding continuous signal. So, we have the continuous signal multiply with the derived delta function you will get the magnitude of the discrete signal at that location.

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So, if I have a continuous signal then what would I have, well I will have, suppose a ramp function I have. So, y and t ramp function. I will have y(t) is equal to what?

$$\mathrm{y}(t) = y_0 + mt$$

So, at every location or point of time t, I can determine y(t). How will you? In discrete time how will you do that? So, this becomes y*, these will be discrete points and so on.

So, how would you describe that? You will have to describe that with a series and that series would be

$$\mathrm{y}^*(t) = \mathrm{y}^*(0) + \mathrm{y}^*(T) + \mathrm{y}^*(2T) + \cdots$$

And then in terms of the corresponding magnitudes of the functions in continuous domain, you can convert all of these using the previous formula and the previous formula $y^*(nT) = y(nT)\delta(t - nT)$.

I can substitute each individual part here which will give me nothing but

$$\mathrm{y}^*(t) = \sum_{n=0}^{\infty} y(nT)\delta(t - nT)$$

So, for a continuous signal you have a function, for a discrete time signal you have a series this is the difference. Let me repeat. For a continuous time signal the input signal output signal everything would be functions of time. For discrete time signals you have a series.

And let me repeat y*(t) means the series which has discrete points and this is, how are those discrete points obtain, they are obtained from the intensities of continuous signals and intensities of continuous signals are given by y(nT). Now, the way we used to do Laplace transform for continuous signals you can take Laplace transform of both the sides. So, for here you can write

$$y^*(s) = \sum_{n=0}^{\infty} y(nT)\, e^{-sTn}$$

What property we have used?

We have used the shifting property. So, this will become the Laplace transform. Perhaps this will be useful sometime in future to determine the response of the system subject to certain specific types of discrete input functions. So, how do you convert? To summarize how do you convert continuous time signal to a discrete time signal while you first imagine that you have a switch which would keep your circuit closed for Δt time.

So, you will approximate it as a rectangular pulse, you know the response and then as $\Delta t \rightarrow 0$, you get an impulse function and then you convert your continuous function to series with the help of a derived delta function. So, this is how we have converted the continuous signal to a discrete signal. Now, imagine that you have converted this signal of temperature to discrete signal and you have fed it to the computer, the computer will now take action.

So, what is going to happen? The action recommended by the computer because of the digital control will also be digital. And in our particular case what happened, in our case it was the robotic arm which used to move. Now, if you want to increase the flow rate, the robotic arm has to move continuously ideally, you would not like it to move in and give this in a jerk manner, you would like to move it continuously. So, therefore, this discrete signal which is provided by the computer has to be changed to a continuous signal. And that is why the conversion of discrete to continuous signal is important. So, let us look how we convert the digital signal to a continuous signal.

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We do it with the help of what is called the hold element. So, we have what is called the hold element. So, what is the basic concept? The basic concept is very simple that if you have y* here......t, and you have some discrete signal here, then between these two instances of time, at this point of time you do not have any signal, and what you want to do, you would like to fill this gap, the whole problem now reduces to the way of determining how to connect these two points.

And that is what gives rise to the concept of hold element. So, if I have m here, if this is m(t), this is some instant of time t, then what I do is I do a Taylor series expansion. So, the Taylor series expansion is in front of you.

$$m(t) = m(nT) + \left(\frac{dm}{dt}\right)_{nT} (t - nT) + \frac{1}{2!}\left(\frac{d^2m}{dt^2}\right)_{nT} (t - nT)^2 + \cdots$$

When I do this Taylor series expansion, what I get is an infinite series, and now I can truncate the series at different locations. So, imagine that I have truncated the series right after the first term on the left-hand side.

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So, I have what I get is a zero-order hold. Now, it is a hold, which means the system is holding the signal there. So, imagine that now I have y, I do not have y*, I have y and t. At any given moment of time if this is my signal, or let us say at T, 2T, 3T, 4T, my different signals are these. Let us say my four different signals are these, and I want to change them to continuous signals.

So, if I have zero-order hold, then

$$m(t) = m(nT) \dots \dots \dots (1)$$

which means it is not a function of time at all. Which means that if I go from T to 2T, the intensity of my signal remains the same, it does not depend upon time t at all. It is independent of time, which means that while going from t to capital T, I maintain a constant signal and the moment I reach 2T the intensity becomes this.

Similarly, when I go from 2T to 3T, I maintain this signal and then the intensity suddenly becomes this and so on. Well, you have in one way converted your signal to a continuous signal because now the value of the signal y(t) exists at all points of time. But it is not really very elegant because you really do not have a continuous nature of the function here. So, what about the first order hold?

So, how would first order hold look like? I would retain the first derivative. So,

$$m(t) = m(nT) + \left(\frac{dm}{dt}\right)_{nT} (t - nT)$$

So, let us imagine that

$$\frac{dm}{dt} = \alpha$$

So,

$$m(t) = m(nT) + (\alpha t - \alpha nT)$$

which means

$$m(t) = \alpha t + (m(nT) - \alpha nT) \dots \dots \dots (2)$$

It is an equation of a straight line. So, intercept is given, the slope is given to you.

So, therefore, for a first order hold while going from one signal to the other signal between them the signal would be a straight line, the signal would be a straight line of the slope which is given to you and it must extend the intercept which is given here. So, therefore, for every, so between any two signals you will have to satisfy all of these conditions.

I am not calculating exact slope and the intercept I am just trying to give you an idea that now you will have to construct straight lines which would conform to this particular condition which is given by equation (2), and here this was equation (1). So now, you can see equation (2) versus equation (1). There is little difference in the behavior.

And then if you keep on increasing the number of terms in the Taylor series expansion you may expect more and more continuity in your signals as the output. So, what we learned today is that it is possible to convert a continuous signal to a discrete time signal using what the technical of sampling here with the help of a switch and it is also possible to convert a discrete time signal to a continuous signal with the help of what is called a hold element.

Out of the three processes which are required for handling discrete time dynamical response only one step is left which is to convert a continuous model to a discrete time model. Let us consider that case in the next lecture. Till then good bye.