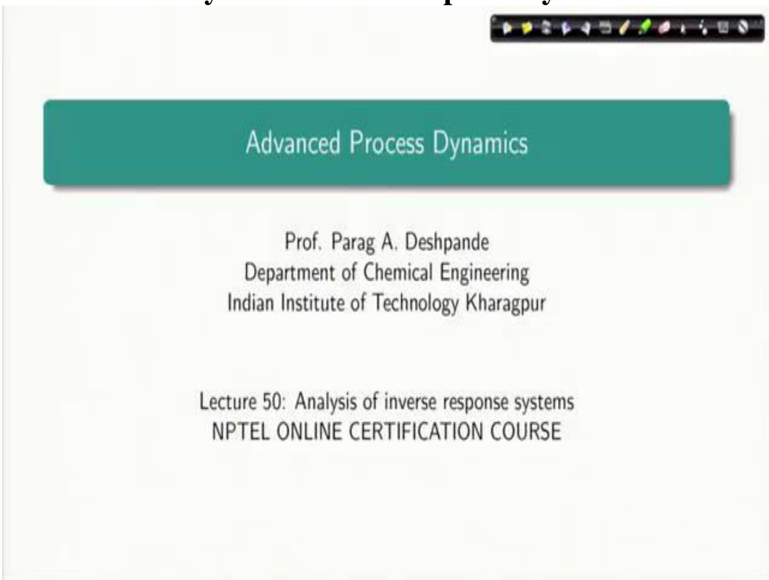
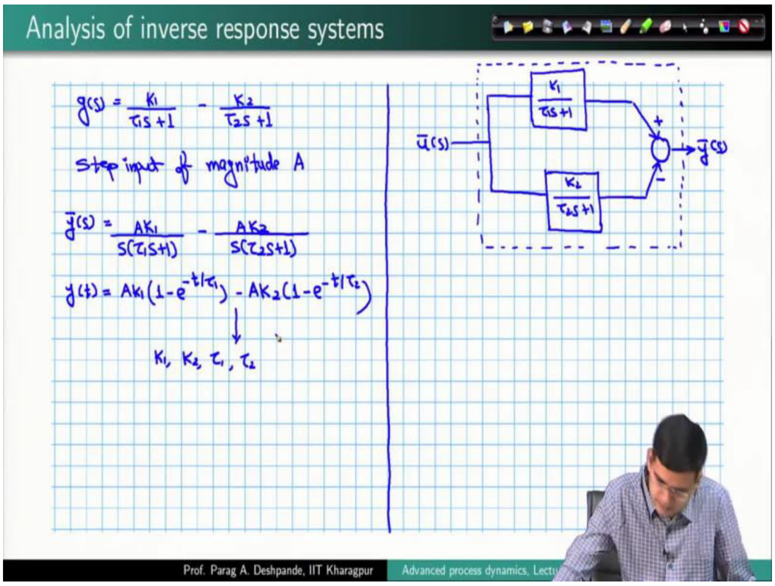
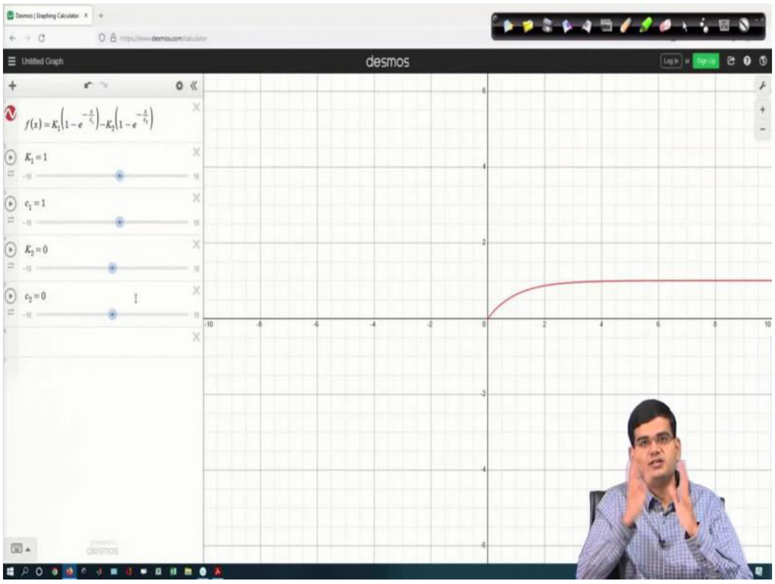
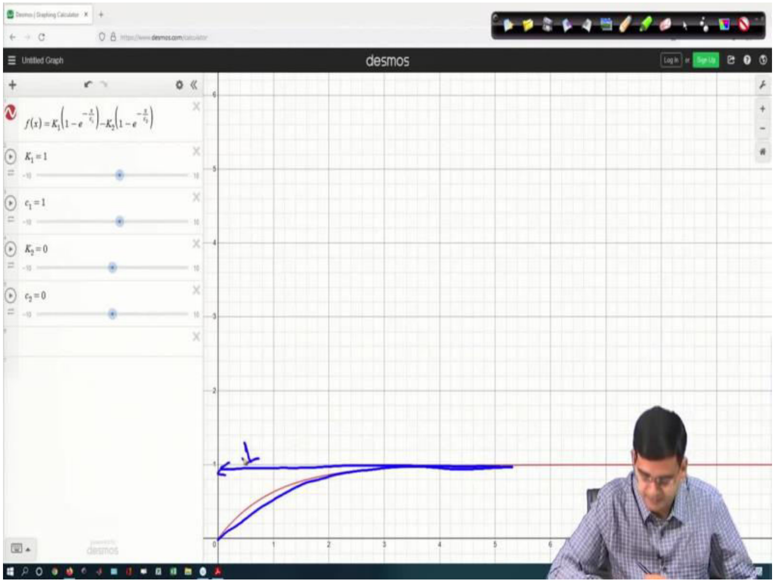
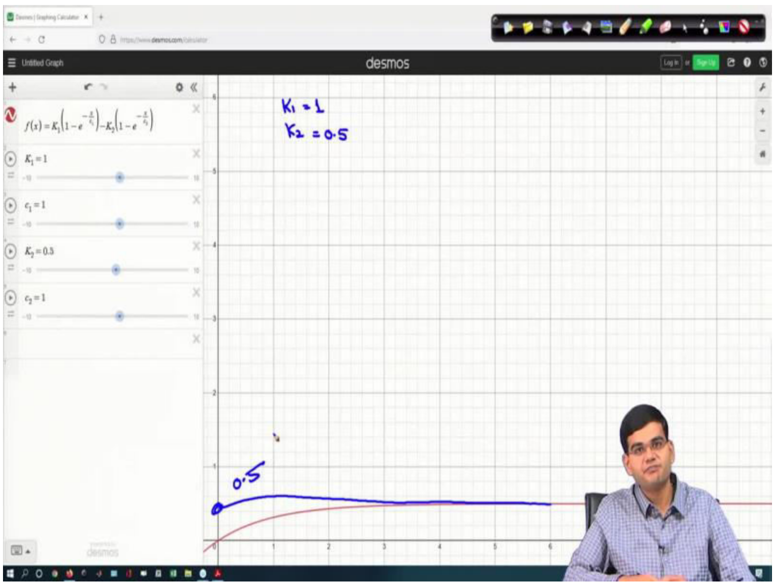
Advanced Process Dynamics
Professor Parag A. Deshpande
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur
Lecture 50
Analysis of inverse response systems

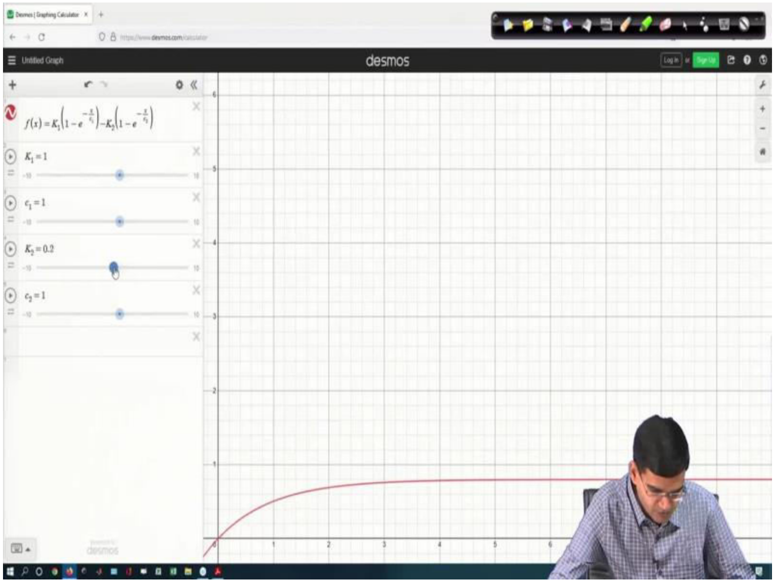


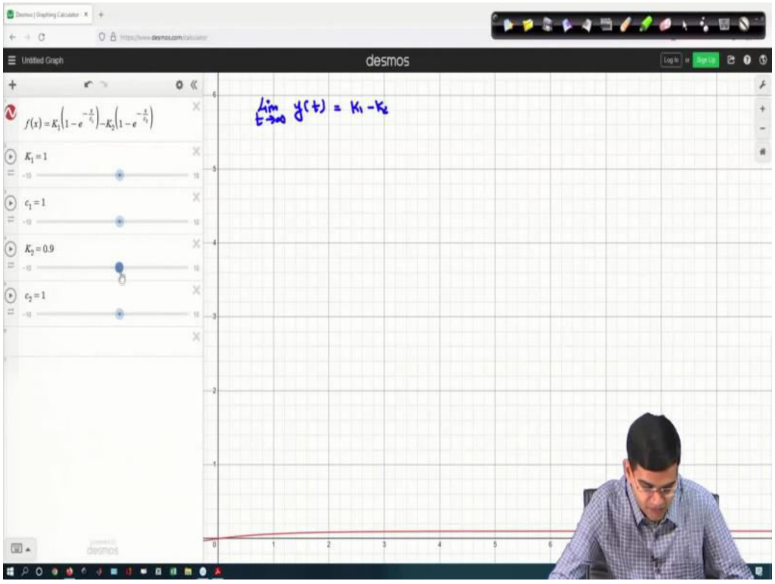


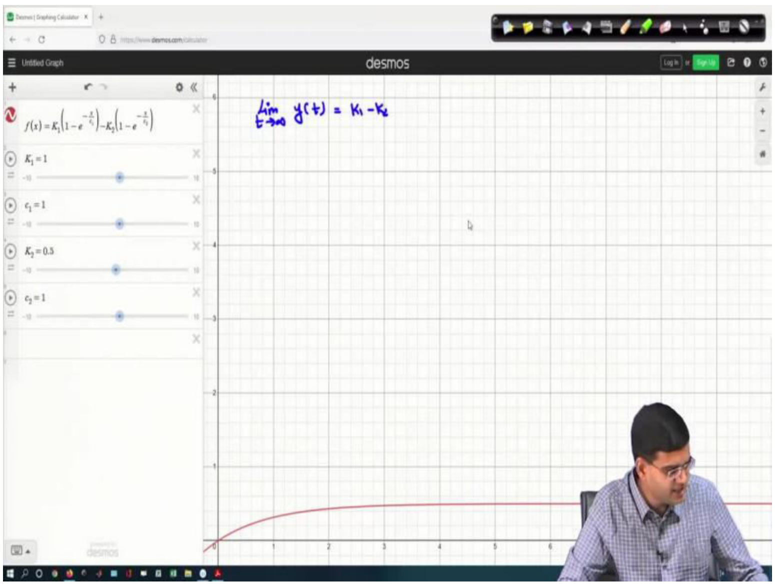

Untitled Graph
desmos
f(x) = K1(1 − e^(−x/c1)) − K2(1 − e^(−x/c2))
K1 = 1
c1 = 1
K2 = 0
c2 = 0

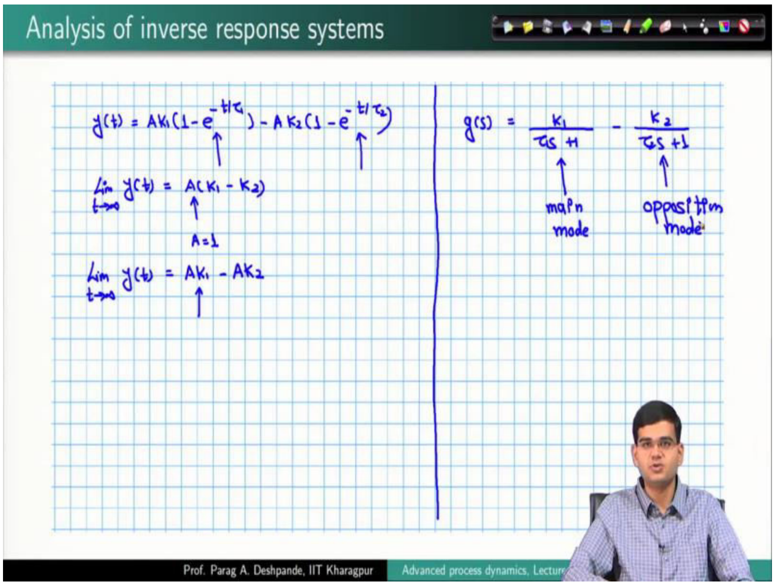

Untitled Graph
desmos
f(x) = K1(1 − e^(−x/c1)) − K2(1 − e^(−x/c2))
K1 = 1
c1 = 1
K2 = 0
c2 = 0
1


Untitled Graph
desmos
f(x) = K1(1 − e^(−x/c1)) − K2(1 − e^(−x/c2))
K1 = 1
c1 = 1
K2 = 0.5
c2 = 1
K1 = 1
K2 = 0.5
0.5


Untitled Graph
desmos
f(x) = K1(1 - e^(-x/c1)) - K2(1 - e^(-x/c2))
K1 = 1
c1 = 1
K2 = 0.2
c2 = 1


Untitled Graph
desmos
f(x) = K1(1 - e^(-x/c1)) - K2(1 - e^(-x/c2))
K1 = 1
c1 = 1
K2 = 0.9
c2 = 1
lim t→∞ y(t) = K1 - K2


Untitled Graph
desmos
f(x) = K1(1 - e^(-x/c1)) - K2(1 - e^(-x/c2))
K1 = 1
c1 = 1
K2 = 0.5
c2 = 1
lim t→∞ y(t) = K1 - K2


Analysis of inverse response systems
y(t) = AK1(1-e^(-t/τ1)) - AK2(1-e^(-t/τ2))
lim t→∞ y(t) = A(K1 - K2)
A=1
lim t→∞ y(t) = AK1 - AK2
g(s) = K1/(τ1s+1) - K2/(τ2s+1)
main mode
opposition mode
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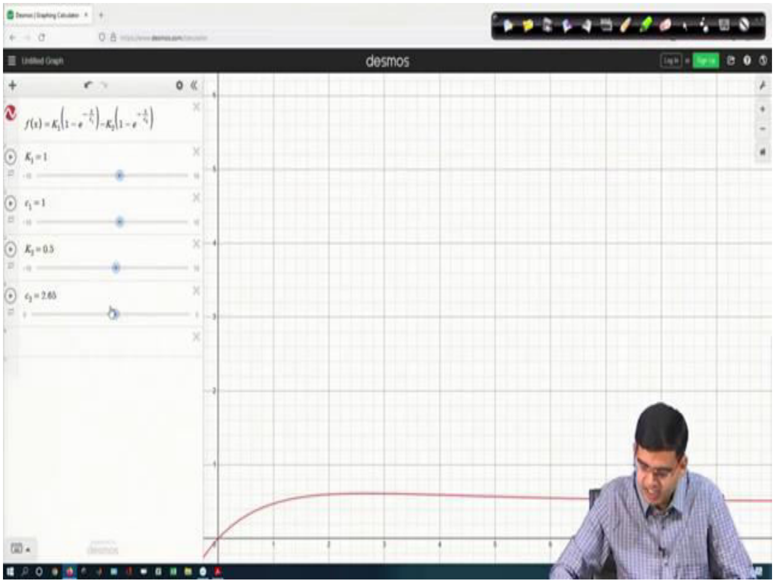
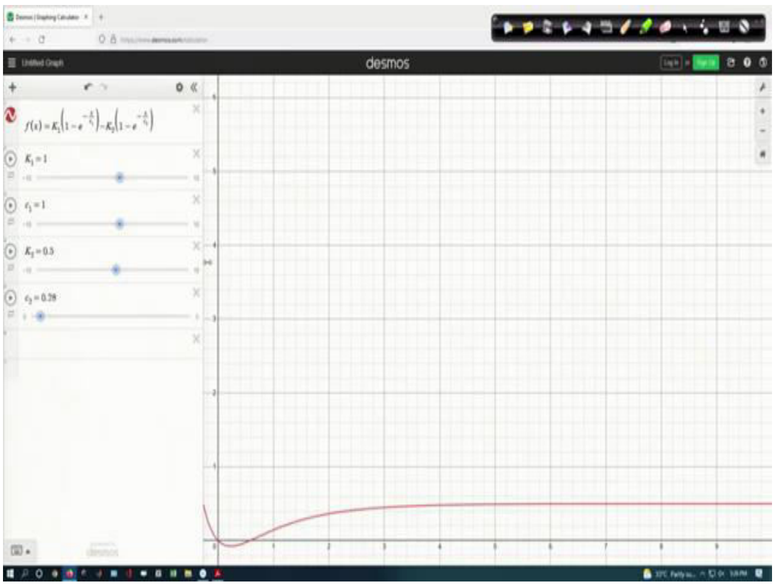
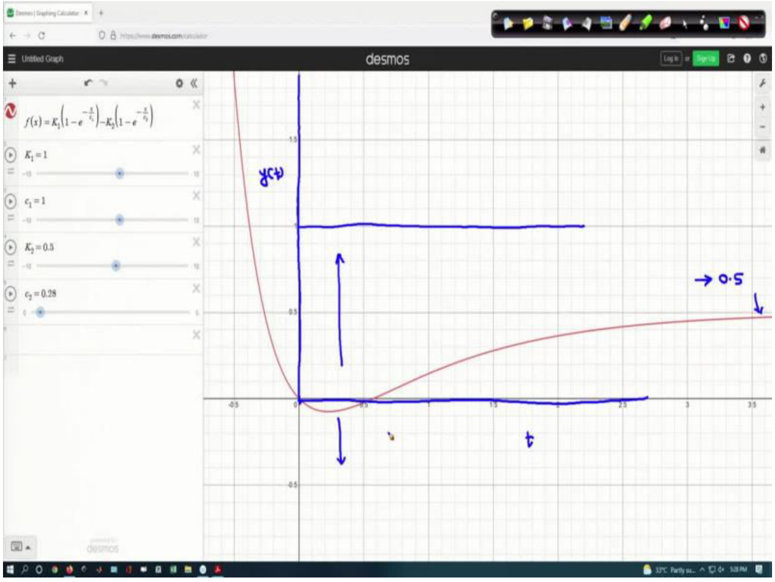
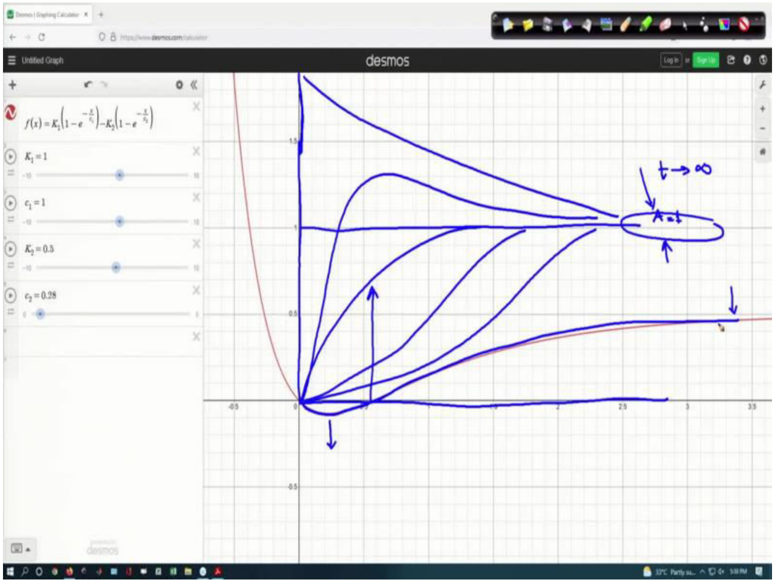
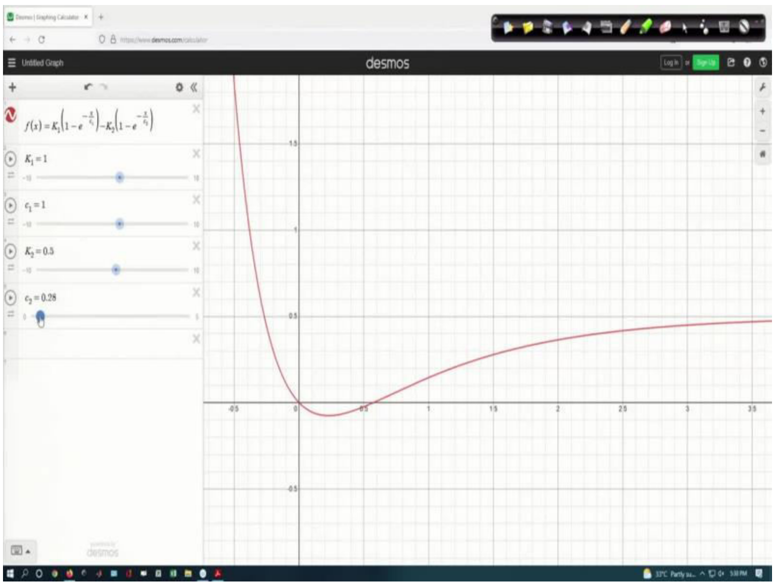
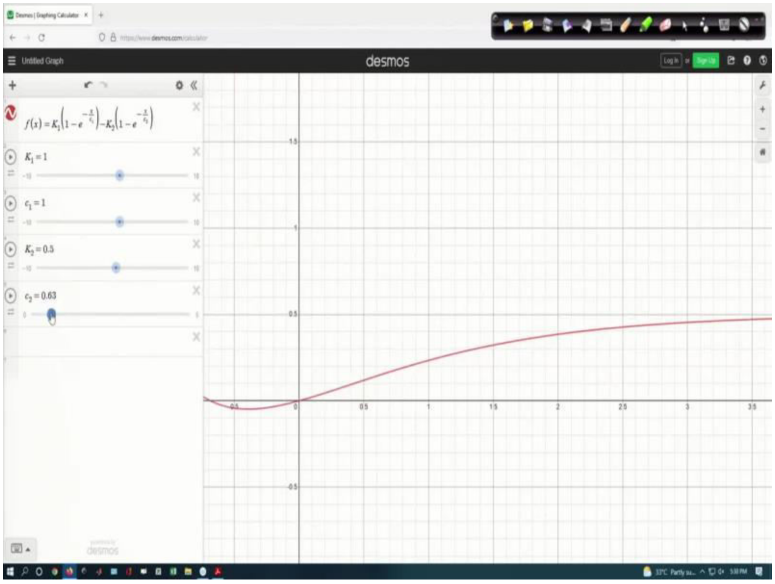

desmos
f(x) = K1(1 - e^(-x/τ1)) - K2(1 - e^(-x/τ2))
K1 = 1
τ1 = 1
K2 = 0.5
τ2 = 2.65

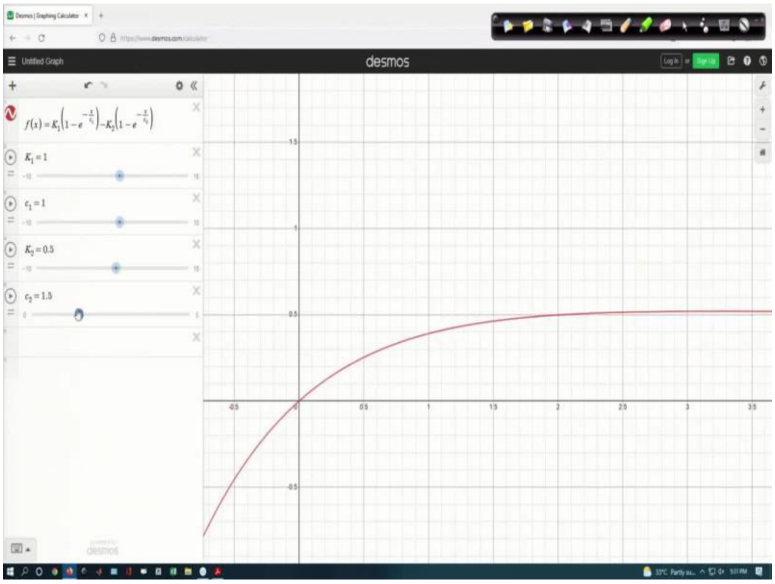

desmos
f(x) = K1(1 - e^(-x/τ1)) - K2(1 - e^(-x/τ2))
K1 = 1
τ1 = 1
K2 = 0.5
τ2 = 0.28

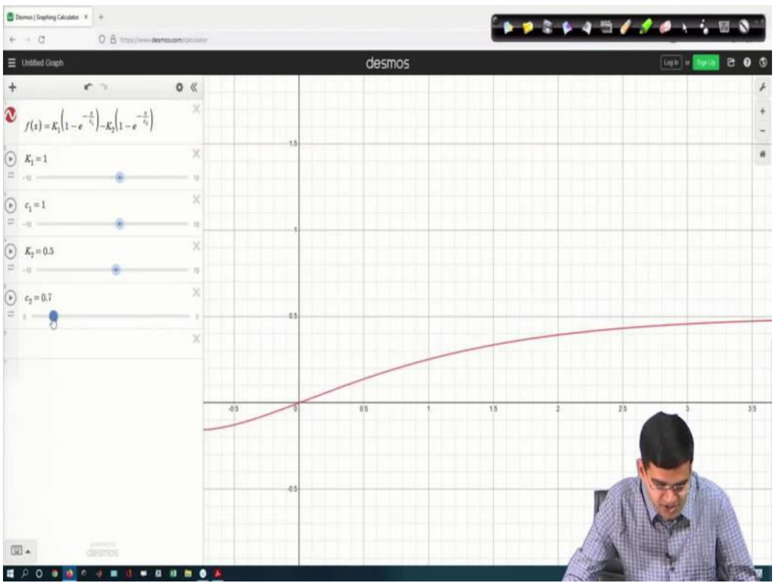

Untitled Graph
desmos
y(t)
t
→ 0.5

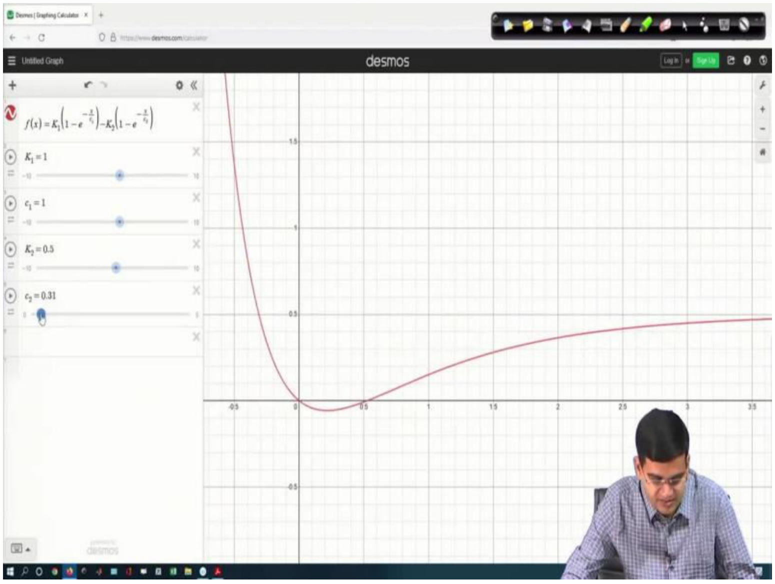

Untitled Graph
desmos
t → ∞

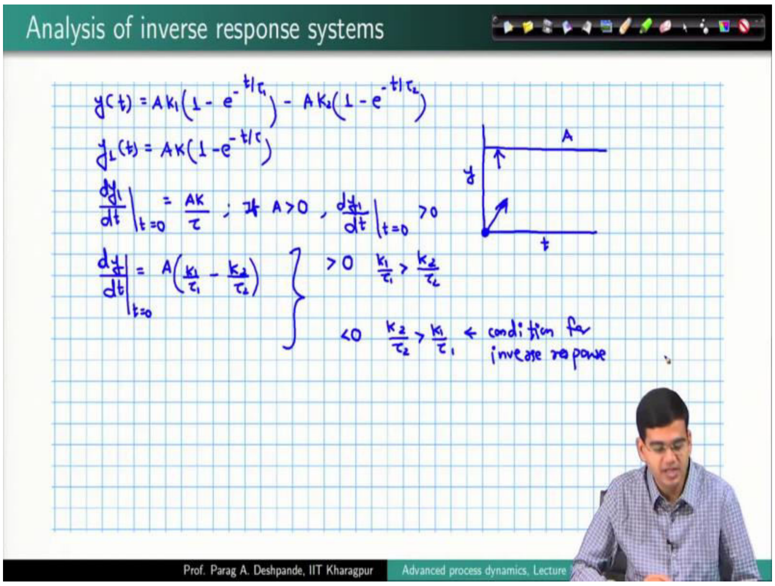

Untitled Graph
desmos


Untitled Graph
desmos
$f(x)=K_1\left(1-e^{-\frac{x}{c_1}}\right)-K_2\left(1-e^{-\frac{x}{c_2}}\right)$
$K_1=1$
$c_1=1$
$K_2=0.5$
$c_2=0.63$


Untitled Graph
desmos
$f(x)=K_1\left(1-e^{-\frac{x}{c_1}}\right)-K_2\left(1-e^{-\frac{x}{c_2}}\right)$
$K_1=1$
$c_1=1$
$K_2=0.5$
$c_2=1.5$


Untitled Graph
desmos
$f(x)=K_1\left(1-e^{-\frac{x}{c_1}}\right)-K_2\left(1-e^{-\frac{x}{c_2}}\right)$
$K_1=1$
$c_1=1$
$K_2=0.5$
$c_2=0.7$


Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = K_1\left(1 - e^{-\frac{x}{c_1}}\right) - K_2\left(1 - e^{-\frac{x}{c_2}}\right)$
$K_1 = 1$
$c_1 = 1$
$K_2 = 0.5$
$c_2 = 0.31$


Analysis of inverse response systems
$y(t) = AK_1(1 - e^{-t/\tau_1}) - AK_2(1 - e^{-t/\tau_2})$
$y_1(t) = AK(1 - e^{-t/\tau})$
$\left.\frac{dy_1}{dt}\right|_{t=0} = \frac{AK}{\tau}$; If $A > 0$, $\left.\frac{dy_1}{dt}\right|_{t=0} > 0$
$\left.\frac{dy}{dt}\right|_{t=0} = A\left(\frac{K_1}{\tau_1} - \frac{K_2}{\tau_2}\right)$
> 0 $\frac{K_1}{\tau_1} > \frac{K_2}{\tau_2}$
< 0 $\frac{K_2}{\tau_2} > \frac{K_1}{\tau_1}$ ← condition for inverse response
A
y
t
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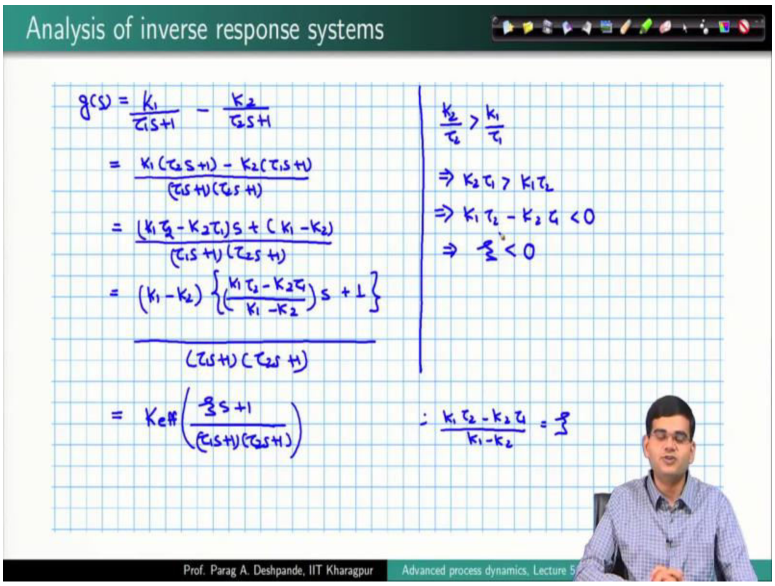
Advanced process dynamics, Lecture


Analysis of inverse response systems
$g(s) = \frac{K_1}{\tau_1 s+1} - \frac{K_2}{\tau_2 s+1}$
$= \frac{K_1(\tau_2 s+1) - K_2(\tau_1 s+1)}{(\tau_1 s+1)(\tau_2 s+1)}$
$= \frac{(K_1\tau_2 - K_2\tau_1)s + (K_1 - K_2)}{(\tau_1 s+1)(\tau_2 s+1)}$
$= \frac{(K_1 - K_2)\left\{\left(\frac{K_1\tau_2 - K_2\tau_1}{K_1 - K_2}\right)s + 1\right\}}{(\tau_1 s+1)(\tau_2 s+1)}$
$= K_{eff}\left(\frac{\xi s+1}{(\tau_1 s+1)(\tau_2 s+1)}\right)$
$\frac{K_2}{\tau_2} > \frac{K_1}{\tau_1}$
$\Rightarrow K_2\tau_1 > K_1\tau_2$
$\Rightarrow K_1\tau_2 - K_2\tau_1 < 0$
$\Rightarrow \xi < 0$
$: \frac{K_1\tau_2 - K_2\tau_1}{K_1 - K_2} = \xi$
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Advanced process dynamics, Lecture 5

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So, today we will look at a very interesting problem of systems exhibiting inverse response.

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So, let us look at one specific form of transfer function. Imagine that you have a process whose transfer function is given as *g(s)* is equal to $\frac{k_1}{(\tau_1 s+1)} - \frac{k_2}{(\tau_2 s+1)}$. Do the individual transfer functions looks familiar? Well, they do because $\frac{k_1}{(\tau_1 s+1)}$ is simply the transfer function of a first order process with static gain K_1 and time constant τ_1.

And now you have minus, it is important, you have a negative sign, and again the transfer function for first order system $\frac{k_2}{(\tau_2 s+1)}$. So, K_2 as the static gain and τ as the time constant. Now, if this is the case, first of all, let us see what is going on. I have two individual transfer functions.

And therefore, if I want to draw an input output type block diagram, then what I can do is that I can have an input to the system which is given us u bar of s as we saw in the previous lecture. I have added 2 with a negative sign, so, I have added a negative of transfer function. So, we will take care of how to do that negative operations, but I will have two streams here.

One will be given as $\frac{k_1}{(\tau_1 s+1)}$, the other one will be given as $\frac{k_2}{(\tau_2 s+1)}$, I must have an adder. I will get an output y a bar of s. And then what I see is that when I get this adder the contribution from the top 1 K_1 τ_1 s + $\frac{k_1}{(\tau_1 s+1)}$ is positive, but the contribution from the negative, from the bottom one is negative. So, I put a negative sign here.

So, this is the input output block diagram for the system. The overall transfer function can be imagined to be within this imaginary block which is shown here. Now, if the transfer function looks something like this, we would like to know the dynamical response of such a system. So, let us consider the case of a step input. So, step input of magnitude A.

So, what will happen? I can write

$$\bar{y}(s) = \frac{Ak_1}{s(\tau_1 s+1)} - \frac{Ak_2}{s(\tau_2 s+1)}$$

So, this will be the Laplace transform of the output variable. And when I invert, again, I know all the terms already, so, it is not very difficult for me to write y of t. So, y of t would be equal to $AK_1(1 - e^{-t/\tau_1}) - AK_2(1 - e^{-t/\tau_2})$.

And now I would like to know the dynamical behavior of the system. Let me see that in this I have four parameters. I have K_1, I have K_2, I have τ_1, I have τ_2. I can set A as arbitrarily some value and I know the effect of K in general because since I have now K_1 and K_2, I would worry about K_1 and K_2 both, but A, I will set as 1 for my analysis. So, I would worry about K_1, K_2, τ_1 and τ_2.

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Let us see the dynamical response of the system by plotting it on this plotter. I have f(x) is equal to K_1. $1 - e^{-x/c}$. Let us call τ_1 as C_1 - K_2 $(1 - e^{-x / c2})$. So, let us see the dynamical response. Let us clear this first. To understand the dynamical response, we must realize that I have set A as unity. So, first of all, when I make K 1, 0 and K_2 also 0, I have simply the first term $K_1(1 - e^{-t/2\tau_1})$, first order response.

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So, is this the first order response to a step input function of magnitude unity? Well, it does, perfectly. You see here, we start from here, asymptotically reach 1. This is 1. Now, let me do one thing, let me change K_2 from 1 to 0.5 and have the same time constant unity. So, time constant τ_1 and time constant τ_2 are both unity, but K_1 is so, my K_1 is 1 and K_2 is 0.5.

When I have only the first order response, I know that my ultimate value of y(t) would be unity. What do I see here? My ultimate value of y(t) here is 0.5. Now, this is strange, because in every single case which we saw till now, as t tends to infinity, you always got a response which would conform to the magnitude of your step input. Here, what is going to happen is you are not reaching the applied input at all, you are stopping somewhere in between. What that in between is, let me see.

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Let me make K_2 as 0.25, 0.3, for example. I reach here, 0.7. Let me make this 0.9 I reach here at 0.9. So, looks like what is going on is that I have y of t at limit t tends to infinity is $K_1 - K_2$.

This is going on. I can adjust various values and you will see that this is always conformed. This is always conformed.

In fact, I can make $K_2 > K_1$ and you can see that this $K_1 - K_2$ is always satisfied. So, my ultimate response, so, the first point which I see specifically in this particular case is that as t tends to infinity, I do not reach my applied input, it goes down. And by how much? $K_1 - K_2$.

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Can I analytically get this solution? Turns out that it is possible, because I have the solution as

$$y(t) = AK_1(1 - e^{-t/\tau_1}) - AK_2(1 - e^{-t/\tau_2})$$

So, this is an exponential term decaying, this is an exponential term decaying. So, therefore, limit t tends to infinity y of t would be what? The exponential terms will become 0. So, I have $A(K_1 - K_2)$ And in our case, A was equal to 1.

So, therefore, therefore, what is going to happen? My limit t tends to infinity $y(t) = A(K_1 - AK_2)$. As simple as that. If you do not have the other part of the transfer function, you would have only this much, AK_1. So, this means what? This means that if I have a transfer function which looks like this g of s is equal to $K_1 / (\tau_1 s + 1) - K_2 / (\tau_2 s + 1)$, then if K_2 τ_2 s + 1 is not there, you would have reached the full response, but it is opposing your response to reach the ultimate fate.

So, therefore, therefore, this is called the main mode of your system, and this is called the opposition mode of your system. So, the first transfer function is called the main mode of your system, and the second transfer function is called the opposition mode of your system. Let us see if we can see something more.

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Let us go back to our analysis. Let me set K_1 as 1, let me set K_2 as 0.5 so that I know that ultimate response is going to be 0.5, it is fine. Now, let me tweak around the value of c_2. I will also keep τ_1 constant, and let me change the value of τ_2. I know that τ_1 and τ_2 are positive. So let me make it between 0 and, 0 and, say, 5. Now, I have the value of K_2 as 0.5. And now, let me change the value of c_1.

See what is going on. I am keeping on increasing the value of c_2. Sorry, not c_1, c_2 the time constant, τ_2. My ultimate value remains 0.5. The only thing which is happening is that there occurs a change in the speed of response. There occurs change in the speed of response, you see here.

But then, a funny thing happens here. Let me make it smaller. It is important that you see what is going on here. At this stage, what happens is, so let me plot it you have the response y(t), you have the time t. What is your ultimate response? See this is tending to, tending to 0.5, which is expected because K_1 is 1, and K_2 is 0.5, quite expected.

But what is weird here, what is interesting rather, what is interesting here, what you see here is that your enforced input was positive, unit step input. The enforced input was positive. And although you put a unit step input, your response was negative, at least initially, at least initially. Now, what happens to your fate of the system as time t tends to infinity? Well, does not matter what was the initial dynamics of your system. Ultimately, you reached 0.5, which was anyway your analytical solution.

That is not a problem. The problem is that initial dynamics showed an inversion in your response. Let me emphasize why am I seeing an inversion in the response because, let me plot on this particular plot all the type of responses which you have seen till today. So, this is the enforced input, A is equal to 1.

What was your first order response? Let us say this is your first order response. If this is your first order response, what would happen to your second order response. This will become your second order response. If this is your second order response, what will happen to your third order response, this will become your third order response.

Then, if you add zeroes to your system, what is going to happen? Probably, this is going to be a response. But if you have a lead-lag system, what will be a possible response, this can as well be a possible response. So I have in the single plot, made everything. And then what I can observe is that as t tends to infinity, all of them tend to come here, all of them tend to reach here.

First thing which has happened in this particular case is that you do not reach here, you reach here, and that is simply a $K_1 - K_2$. But what is even more interesting to see here is that in all previous cases, you always had, right from the beginning, the response of the system, which went up. But now, for the first time, you have the response, which is going down, the response which is going down.

Now, the system realizes that it is going in a wrong direction, and therefore it inverts and then comes back again, to catch up to this value. That is well agreed, that is not an issue. But you still need to see one thing that you have an inverse response. And why does that happen, or rather, when does that happen? So, let me look at the condition when it happens.

I have maintained constant k_1, I have means maintained constant c_1, I have maintained constant K_2, and now I am tweaking around the value of c_2, which means I am changing the time constant c_2. And as long as time constant $\tau_2 > K_2$, you always have positive response, see here. The inverse response happens only when your τ_2 becomes smaller than K_2. But is that the generic reason behind that or is that the generic condition? Let us try to derive the condition.

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So, I have in my system, this

$$y(t) = AK_1(1 - e^{-t/\tau_1}) - AK_2(1 - e^{-t/\tau_2})$$

Let me do one thing, let me compare it against. So, compare it against y 1 of t, which has only the first order dynamics. So, it would be $AK(1 - e^{-t/\tau})$. What I would like to know is the value of d y 1 by d t at t is equal to 0. What is this going to be equal to?

This is going to be equal to AK / τ, that is it, AK / τ, at t = 0. And if A > 0, which means you are given a positive step, then dy_1/dt at t = 0 for a positive gain will always be greater than 0, which means that if I plot y versus t, then here, you will always have a positive slope, always positive. And this is your A. So, this is your direction of A, positive, and this is the direction of your response, positive, in every case.

Now, what happens here.

$$Y(t) = y(t) = AK_1(1 - e^{-t/\tau_1}) - AK_2(1 - e^{-t/\tau_2})$$

$$\frac{dy}{dt} = \frac{Ak_1}{\tau_1} - \frac{Ak_2}{\tau_2}$$

And from here you can get the condition for inverse response. So, now, this $\frac{dy}{dt}$ and this must be at t = 0, the response right at the time when you introduce a disturbance in $\frac{dy}{dt}$ is equal, $\frac{dy}{dt}$ at time t is equal to $\frac{Ak_1}{\tau_1} - \frac{Ak_2}{\tau_2}$.

So, for this to be greater than 0, the usual response K_1 / τ_1 would be greater than K_2 / τ_2. And if this is less than 0, what would happen? Well, the derivative will become negative, the gradient would become negative. That is the condition for inverse response, K_2 / τ_2 would become greater than K_1 / τ_1. And this is the condition for inverse response.

Now, the question is that what does the inverse response behavior depend upon? It does not simply depend upon K_1, it does not depend upon K_2, it simply does not depend on τ_1 and τ_2. It depends upon this inequality where K_2 / τ_2 as whenever K_2 / τ_2 becomes greater than K_1 / τ_1, you get an inverse response, which means, that before trying to catching up to the applied difference in A K_1 - A K_2, the system will go in the, respond in the opposite direction.

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Now, I can do one more analysis. Under what condition do you get this inverse response, and what kind of systems do you get this? So, let us look at the transfer function,

$$g(s) = \frac{k_1}{\tau_1 s+1} - \frac{k_2}{\tau_2 s+1}$$

So, I can simplify this as

$$\frac{k_1(\tau_2 s + 1) - k_2(\tau_1 s + 1)}{(\tau_2 s + 1)\,(\tau_1 s + 1)}$$

which can be further simplified

$$\frac{(k_1 - k_2)\{(\frac{k_1\tau_2 - k_2\tau_1}{k_1 - k_2} + 1)\}}{(\tau_2 s + 1)\,(\tau_1 s + 1)}$$

So, we have already seen that the effective gain of the system is K_1 - K_2. So, therefore, I can write this as what? $K_{effective}$, which is given as K_1 - K_2, times what? If I say that $\frac{k_1\tau_2 - k_2\tau_1}{k_1 - k_2}$ as a quantity called ζ, then I can write this as (ζs + 1) / (τ_1 s + 1)(τ_2 s + 1).

So, this is going to be a transfer function of your system. Now, for an inverse response, I know that K_2 / τ_2 should be greater than K_1 / τ_1, which means $K_2 \tau_1$ must be greater than $K_1 \tau_2$, which means, which means $K_1 \tau_2 - K_2 \tau_1$ would be less than 0, in other words ζ would be less than 0. So, now, I have a condition for inverse response, a system which is of the order 2,1. See the transfer function, the transfer function is $(\zeta s + 1)/(\tau_1 s + 1)(\tau_2 s + 1)$.

So, a system of the order 2,1 such that ζ is less than 0, will exhibit an inverse response behavior. And if this is the case, then the way we have been doing this analysis of transfer function, the response, step response and various responses can be done in this case also. You have $(\zeta s + 1)/(\tau_1 s + 1)(\tau_2 s + 1)$ for unit step, you will multiply it by A by s do a partial fraction, do an inversion and you can determine the behavior, you can determine the analytical solution.

From the analytical solution, by plotting the analytical solution, you can in fact find out that as long as $K_2 / \tau_2 > K_1 / \tau_1$, your initial response will always be negative for the positive enforced input. And therefore, this particular response is very interesting, as well as very important to be understood. So, what we saw in all the cases is that when you have an enforced input which is positive, your response is always towards the direction of the enforced input.

But what is the meaning of inverse response? That you have enforced your system to have a positive input, but the system is going in the other direction. And this response is important to understand, because imagine that you have a system where you are running a reactor and you have a control system, when associated with the control system, you will have a transfer function, you need to make sure that you do not have an inverse response behavior in your transfer function.

Why? Because, if the temperature of your reactor starts increasing and you want to have a corrective action, the direction also should be appropriate. If the direction of your control action is in the inverse, then what is going to happen? It is going to worsen the effect of the input. Now, as time t tends to infinity, one can understand that the system dynamics will be such that everything is fine, because the direction of your dynamics and the direction of your input is the same.

But during the initial stages, your control action is such that you should reduce the temperature, but the temperature would start increasing because of the inverse response. Let us take the example of flight dynamics, the flight, the aircraft suddenly starts coming down. It is a dangerous situation, you would like the control system to take it up. If you have an inverse response, well, it is going to be very dangerous, because your system will respond such that there would occur further decrease in the altitude, a potentially very dangerous situation.

So, therefore, you must identify the response or the presence of inverse response in your system. And in all the cases you saw that the system not only ultimately reaches the enforced input, the direction also is appropriate. So, what we learned here is during the analysis of transform domain systems, what we come across is a concept called transfer function, and one can look at the transfer function and the nature of the transfer function. and by looking at the transfer function itself one can have at least a qualitative idea about how the system will evolve in time.

Till today, we took all the examples which were continuous domain. From the next week onwards, we will take an example of a system and try to understand the underlying system in which you would like to understand the dynamics of the process in discrete time. So, we will meet again next week. Till then, goodbye.