Advanced Process Dynamics

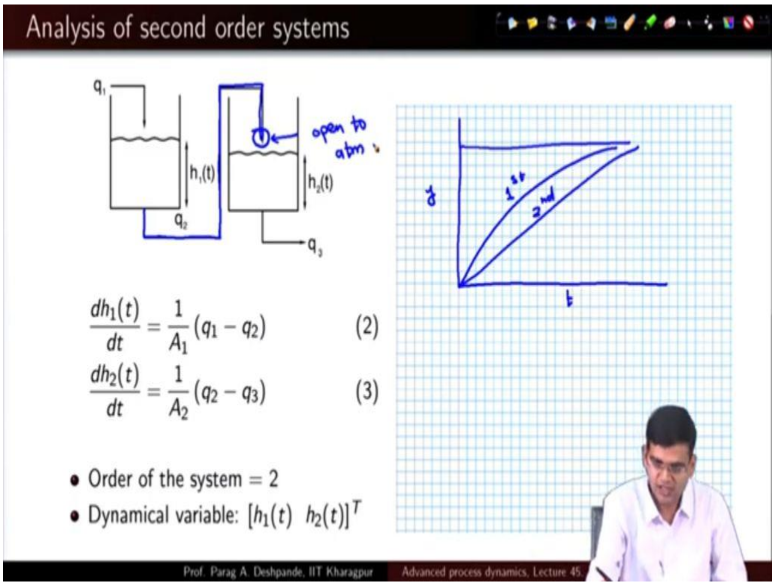
Professor Parag A Deshpande

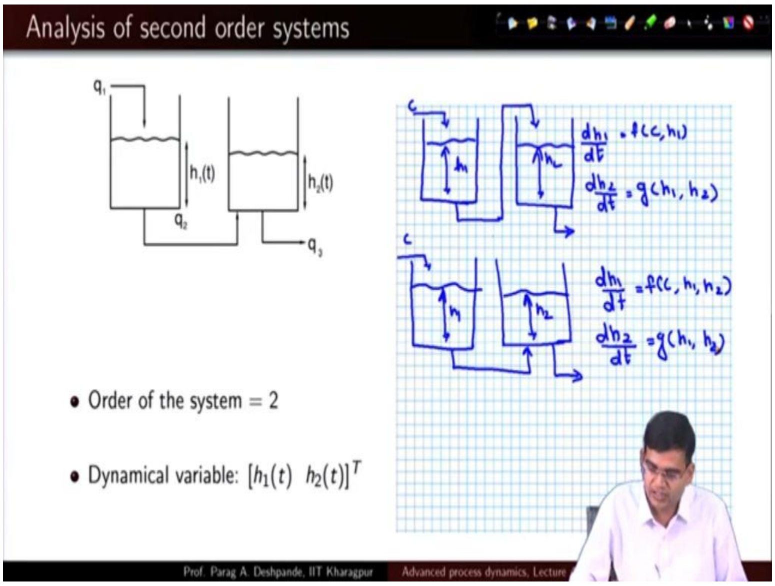
Department of Chemical Engineering

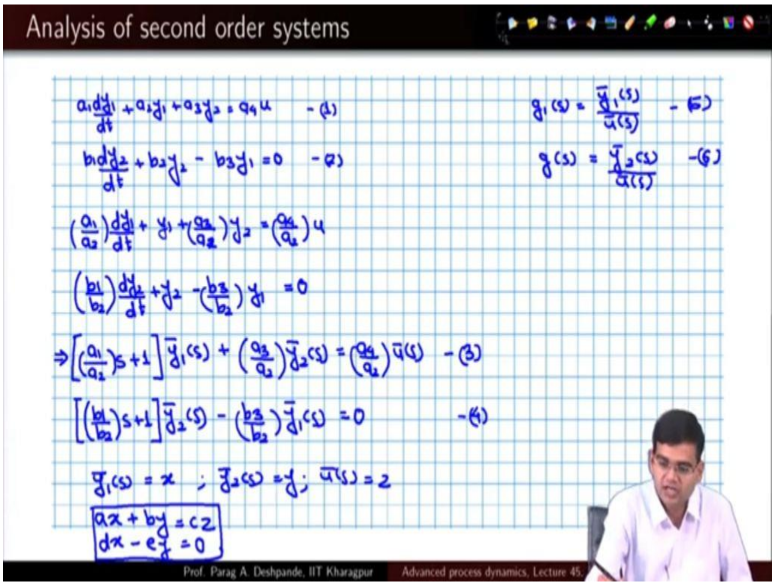
Indian Institute of Technology, Kharagpur

Lecture 45

Analysis of response of second order systems continued…






Analysis of second order systems
a1 dy1/dt + a2y1 + a3y2 = a4u - (1)
b1 dy2/dt + b2y2 - b3y1 = 0 - (2)
g1(s) = ȳ1(s)/ū(s) - (5)
g(s) = ȳ2(s)/ū(s) - (6)
[(a1/a2)s+1] ȳ1(s) + (a3/a2) ȳ2(s) = (a4/a2) ū(s) - (3)
[(b1/b2)s+1] ȳ2(s) - (b3/b2) ȳ1(s) = 0 - (4)
ax + by = cz
dx - ey = 0
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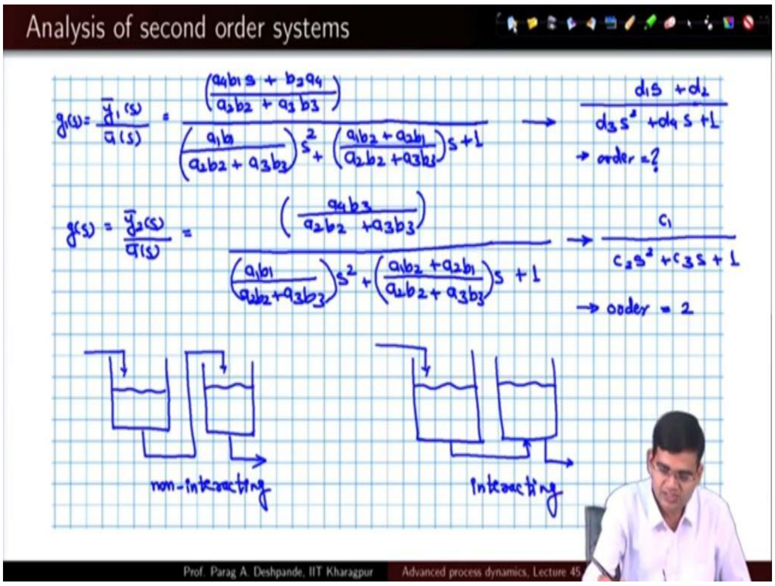
Analysis of second order systems
order = ?
order = 2
non-interacting
interacting
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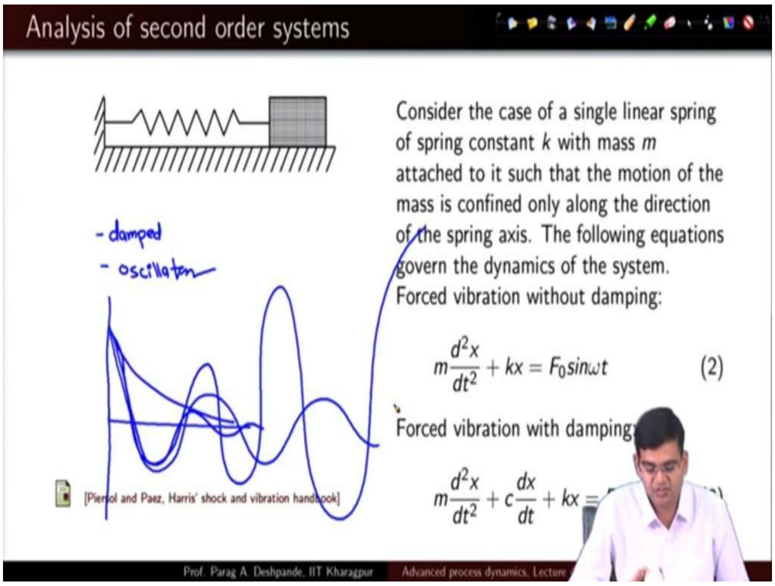
Analysis of second order systems
- damped
- oscillation
Consider the case of a single linear spring of spring constant k with mass m attached to it such that the motion of the mass is confined only along the direction of the spring axis. The following equations govern the dynamics of the system.
Forced vibration without damping:
m d²x/dt² + kx = F0 sinωt (2)
Forced vibration with damping:
m d²x/dt² + c dx/dt + kx =
[Piersol and Paez, Harris' shock and vibration handbook]
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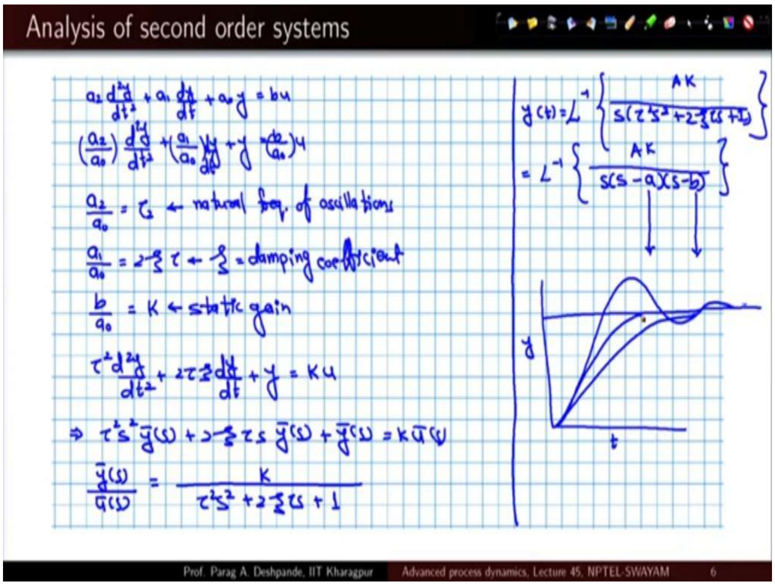

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So, let us continue our discussion on second order dynamics. In the previous lecture, we saw the system which is in front of you, and we derived the overall transfer function for the system. And what we saw was that if I draw the dynamical response of the system as y versus t, and if this is the step magnitude of the step input; then we made two observations. The first observation was that you always have a response which would be more sluggish compared to the response of the first order system. So, in this particular diagram, the top one is the first order response, the bottom would one be the second order response.

And we saw that in no case did the response show any oscillatory behavior. In this particular setup, what was done was that you have you have a tank, in which you give the input. And the output from the first tank was fed as the input for the second tank; but let, let us go into more details about the construction. The outlet of the first tank is open to atmosphere. So, in this diagram when you see that a stream comes out as q_2, this pipe here is open to atmosphere, open to atmosphere; and then you drain the outlet in the second tank. So therefore, if you simply remove the second tank from the system, you will start spilling the liquid. You will start draining the liquid; and nothing will happen to the dynamics of the first tank.

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Now, we as may as well have another arrangement, look here. Now, the outlet from the first tank is going to the second tank; but it is fed in such a manner that the outlet is no more open to the atmosphere. Rather, it goes to the, to such a portion of the tank where you already have the liquid. So, in principle depending upon h_1 and h2, you can have the flow of the liquid either from tank one to tank two, of in case where $h_2 > h_1$. We will know that the flow would be from tank two to tank one.

If this if such is the construction of the system, what would be the transfer function is something which would we would be interested in. So, let us look into two constructions. We have first construction in which the liquid is fed to the first tank; the outlet becomes open to atmosphere, and then you put that liquid in the second tank. So, in our notations, u, y_1, y_2 or to be more intuitive; let us use the usual notations, so let us do one thing. Let us do this, let us use c the constant flow rate, h_1 and h_2. And therefore, in this case h_1 liquid level in the first tank would be a function of what? It would be a function of c.

Definitely not in, nothing to argue about that. It would also be dependent upon h itself ....h_1; let me write

$$\frac{dh_1}{dt} = f(c, h_1)$$

So, time rate of change of h_1 will depend upon h_1; and it would also depend upon the input. And

$$\frac{dh_2}{dt} = g(h_1, h_2)$$

$\frac{dh_2}{dt}$ would depend upon what? It would be some different function. It would now depend upon h_1, because the flow rate from tank one would depend upon h_1. So therefore, since q_2 is a function of h_1, $\frac{dh_2}{dt}$ will become a function of h_1, and also a function of h_2.

The time rate of change will depend upon h_2; this is pretty straightforward. Now, I have another arrangement in which I have c, I have h_1; and I have an arrangement like this h_2. So therefore, now

$\frac{dh_1}{dt}$ in this case would be a function of c pretty intuitive; it would be a function of h_1 intuitive. But, as I said whether $h_2 > h_1$; or $h_1 > h_2$ will decide the flow between the two tanks. So therefore, now it will also be dependent upon h_2, which was not the case in the first case.

$$\frac{dh_1}{dt} = f(c, h_1, h_2)$$

And

$$\frac{dh_2}{dt} = g(h_1, h_2)$$

So, now I have a set of equations which would be different in this particular case. Now, because $\frac{dh_1}{dt}$ is not only dependent upon c and h_1; it also depends upon h_2. So, therefore, I need to come up with, I need to devise a method to write a set of equations which would satisfy these conditions. So, let us write our equations in the deviation variable form.

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So, I can write the first equation as

$$a_1 \frac{dy_1}{dt} + a_2 y_1 + a_3 y_2 = a_4 u \text{ ....(1)}$$

The only forcing function in the system is u. What about the dynamical variable y_2? Now, for y_2 of the level of liquid in the second tank, h_1 is acting as a forcing function; because if you change h_1, h_2 will change. There is no direct input which comes as the input to the second tank; the input only input is the output from the first tank.

So, therefore, now I can write this as

$$b_1 \frac{dy_2}{dt} + b_2 y_2 - b_3 y_1 = 0 \text{ ....(2)}$$

So, now I have two equations for obtaining my transfer function, equation (1), equation (2). Let me do one thing; let me try to determine the transfer function. I will put the equations in the standard form.

I will write this as

$$\left(\frac{a_1}{a_2}\right)\frac{dy_1}{dt} + y_1 + \left(\frac{a_3}{a_2}\right)y_2 = \left(\frac{a_4}{a_2}\right)u$$

Similarly, I will put equation (2) in the standard form

$$\left(\frac{b_1}{b_2}\right)\frac{dy_2}{dt} + y_2 - \left(\frac{b_3}{b_2}\right)y_1 = 0$$

Time to take the Laplace transform. So, I can write

$$\left[\left(\frac{a_1}{a_2}\right)s + 1\right]\overline{y_1}(s) + \left(\frac{a_3}{a_2}\right)\overline{y_2}(s) = \left(\frac{a_4}{a_2}\right)\bar{u}(s) \dots\dots\dots (3)$$

The Laplace transform of the first equation. And the Laplace transform of the second equation will become

$$\left[\left(\frac{b_1}{b_2}\right)s + 1\right]\overline{y_2}(s) - \left(\frac{b_3}{b_2}\right)\overline{y_1}(s) = 0 \dots\dots\dots (4)$$

Let me write this as equation (3) and equation (4).

Now, what do I need to solve for my transfer function will involve two quantities. A transfer function would be the one

$$g_1(s) = \frac{\overline{y_1}(s)}{\bar{u}(s)} \dots\dots.. (5)$$

And my overall transfer function

$$g(s) = \frac{\overline{y_2}(s)}{\bar{u}(s)} \dots\dots\dots (6)$$

Now, I have two equations and therefore, I can solve for $\overline{y_1}(s)$ and $\overline{y_2}(s)$ in terms of $\bar{u}(s)$ and get my transfer function. Transfer functions (5) and (6); let me repeat what I just said. You have this equation of the form if I write

$$\overline{y_1}(s) = x; \;\; \overline{y_2}(s) = y; \;\; \bar{u}(s) = z$$

Then, I have a equation of the form

$$ax + by = cz$$

and

$$dx - ey = 0$$

This is the functional form of the two equations given by equation (3) and (4). Where,

$$a = \left(\frac{a_1}{a_2}\right)s + 1$$

$$b = \frac{a_3}{a_2}$$

$$c = \frac{a_4}{a_2}$$

$$d = \left(\frac{b_1}{b_2}\right)s + 1$$

$$e = \frac{b_3}{b_2}$$

So, you will solve for x, you will solve for y; which means you will solve for $\overline{y_1}(s)$, and you will solve for $\overline{y_2}(s)$ and you will get this transfer function. So, I am writing the final expression here. The final expression that you should get by doing this mathematical jugglery is something which is front of you.

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$$g_1(s) = \frac{\overline{y_1}(s)}{\bar{u}(s)} = \frac{\left(\frac{a_4 b_1 s + b_3 a_4}{a_2 b_2 + a_3 b_3}\right)}{\left(\frac{a_1 b_1}{a_2 b_2 + a_3 b_3}\right)s^2 + \left(\frac{a_1 b_2 + a_2 b_1}{a_2 b_2 + a_3 b_3}\right)s + 1}$$

So, this is the transfer function giving the dynamics of the first tank will analyze this; this is an interesting case; but, before that let me write down the overall dynamics of the system.

$$g(s) = \frac{\overline{y_2}(s)}{\bar{u}(s)} = \frac{\left(\frac{b_3 a_4}{a_2 b_2 + a_3 b_3}\right)}{\left(\frac{a_1 b_1}{a_2 b_2 + a_3 b_3}\right)s^2 + \left(\frac{a_1 b_2 + a_2 b_1}{a_2 b_2 + a_3 b_3}\right)s + 1}$$

The expressions look a little complicated, but actually there is nothing complicated in it; because a_1,a_2, a_3,a_4, b_1, b_2, b_3, they are all constants. So, you all of these expressions which look a little complex will have some numerical values and those values are to be plugged in, nothing greater than that. But, what we need to understand is something which is very interesting; let us look at the overall transfer function. The overall transfer function is of the form which is,

$$g(s) = \frac{c_1}{c_2\, s^2 + c_3 s + 1}$$

So, this is the form of the transfer function. What is the order of the dynamics of the system overall? For such a case, I look into the denominator, I look the degree of the polynomial; and I decided upon that. So therefore, the order of the overall dynamics of the system is 2. And the moment I see 2 here, I know that the dynamics of my system, the overall dynamics of my system is going to be slower compared to the dynamics of single tank, if the second tank were not there. Whether I use this formulation or the particular format, which I use previously the particular arrangement; anyway, got this.

But, what I see is something interesting here in the first transfer function; if I plug in all of this, I get

$$g_1(s) = \frac{d_1 s + d_2}{d_3 s^2 + d_4 s + 1}$$

So, first of all, the first question which arises is that what is the order? Because, if I just see the denominator, then I see the degree of the polynomial in the denominator. And then from there itself, you can see that the tank one which used to be an independent unit in the previous case; and that is why it used the previous cases referred to as the non-interacting case, used to have the order 1.

Here if you just look at the denominator, the order is not 1; it has to be at least 2. But, there is something in the numerator as well. So, how would you define the order of a system when you have some polynomial in the numerator and some polynomial in the denominator as well? We will take this particular topic in the next lecture. But before that, what we need to look into is into this particular aspect is that, now your first tank is no more an independent unit; and that is why it is called an interacting system. So, you cannot expect simply first order dynamics even in the first tank; forget about the entire dynamics.

You do not have simple first order dynamics, even in the first tank; and therefore, the arrangement which is given like this is called an as called a non-interacting system. Because, if you remove any of the tanks downstream, the dynamics of the tanks upstream does not change in material; they are not interacting in fact. So, the downstream dynamics does not affect the upstream dynamics; the upstream dynamics will always affect the downstream dynamics. So, that is not the meaning of non-interacting; the meaning of non-interacting is that the downstream dynamics is not affecting the upstream dynamics.

On the other hand, you have a system like this, where the dynamics of the second tank is affecting the dynamics of first tank. And therefore, this is called the interacting system; and the transfer function form is very different. We will study the dynamics of such systems a little later. But, what you saw in general was that the dynamics of a second and higher order is sluggish compared to the dynamics of first order system. And it is guaranteed to be sluggish as well as monotonous, when you have the case when you simply have the multiplication of individual transfer functions as the overall transfer function.

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Now, we had a look into this particular aspect of second order dynamics, where you have the spring mass system. And in this spring mass system, we saw that in fact you can have damped behavior; and you can have oscillatory behavior. In fact, what you saw was that the oscillations could die down; the oscillations could the, the response can be such that not oscillations would die down. The response would be such that there is there is no oscillation; the oscillations could die down. The oscillations may increase in magnitude amplitude; and finally, they must be sustained in magnitude, this all four.

In the previous case, we saw only monotonous behavior; but, then how would you get this all of these responses from a second order system in transform domain. Let us let us look into that.

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So, imagine that I can write a general second order system as:

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{\mathrm{dy}}{\mathrm{dt}} + a_0 \mathrm{y} = \mathrm{b}u$$

The general second order system. First spring mass system also spring mass system with damping with external forced, I mean forced oscillations; this is a perfect example. So, I will try to determine the transfer function; I will do this in this exact same manner as I did previously. I will make the coefficient of y as unity. So, I can write this as

$$\left(\frac{a_2}{a_0}\right)\frac{d^2y}{dt^2} + \left(\frac{a_1}{a_0}\right)\frac{\mathrm{dy}}{\mathrm{dt}} + \mathrm{y} = \left(\frac{\mathrm{b}}{a_0}\right)u$$

I am assuming the equation to be in a deviation variable form. And for this particular case, as I had assigned $\frac{a_1}{a_0} = \tau$, time constant and $\frac{b}{a} = K$.

In this system, I assigned

$$\frac{a_2}{a_0} = \tau^2$$

Basically, means natural frequency of oscillations in the system.

$$\frac{a_1}{a_0} = 2\xi\,\tau$$

where,

ξ is the damping coefficient; and

$$\frac{\mathrm{b}}{a_0} = K$$

K is the static gain of the system. So, I can write this as

$$\tau^2\frac{d^2y}{dt^2} + 2\xi\,\tau\frac{\mathrm{dy}}{\mathrm{dt}} + \mathrm{y} = Ku$$

Now, the next step would be to determine the, to do the Laplace transformation. So, this would be equal to

$$\tau^2 s^2\,\bar{y}(s) + 2\xi\,\tau\,\bar{\mathrm{y}}(s) + \bar{\mathrm{y}}(s) = K\,\bar{u}(s)$$

From where I get the second order transfer function

$$\frac{\bar{\mathrm{y}}(s)}{\bar{u}(s)} = \frac{K}{\tau^2 s^2 + 2\xi\,\tau s + 1}$$

So, is it a second order transfer function or is the order second order corresponding to this transfer function? You can look at the denominator of the transfer function, look at the degree of the polynomial; and this in fact is a second order polynomial. So, therefore, the system corresponding to this transfer function is in fact the second order transfer function. But now, how would I determine the response? It is not very difficult; I will determine the response in the exact same manner as I determine the response for a first order system.

So, what I will do is for example, for a response for subjected to a step input, what I will do? I will have y(t), which will be given us Laplace inverse of what?

$$y(t) = L^{-1}\left\{\frac{AK}{s(\tau^2 s^2 + 2\xi\ \tau s + 1)}\right\}$$

And how would I determine the response? I will need to do partial fractions. So, I assumed that there is a way that I will be able to split the quadratic polynomial; and I would be able to factorize it. So, what will happen?

$$y(t) = L^{-1}\left\{\frac{AK}{s(s-a)(s-b)}\right\}$$

Assuming that I will be in a position to do this factorization; and this factorization is always possible.

So, I will put this expression in this form; and then I will do further partial fraction inverted determine the response. And when I do that, then depending upon a and b, the roots of the polynomial I will get different behaviors. So, therefore, when a and b are complex, you will get. So, this is y, this is t, and this is enforced input; when a and b are complex and they will be complex conjugates, you will get an oscillator behavior. Do you see an oscillatory behavior for complex eigen values in the previous state space domain analysis? Yes, we do see. When you have a system where they are purely imaginary, you will see when they are purely real; then what will happen?

In that case, real distinct roots, you will see a monotonous behavior. And then you will see one particular case where when the two roots become equal, and then that is the critically damped response; and that is the fastest response that you can obtain from the system, which is second order. So, in fact whether you use transform domain analysis or state space domain analysis, you would always see that there is a correspondence between state space domain analysis and transform domain analysis.

So, we will stop here today. We looked into the into one particular case where the transfer function was of the form $g_1(s) = \frac{d_1 s + d_2}{d_3 s^2 + d_4 s + 1}$; which means that there was, there was a polynomial in the numerator as well as a polynomial in the denominator. And the very definition of the order of the system became questionable; because you not only have the polynomial in the denominator following which with the help of the degree of the polynomial, you could determine the order; you have a polynomial in the numerator as well. So, is there a new method using which we may need to define the order of the system. We will take this up in the next week. Till then, goodbye.