Advanced Process Dynamics

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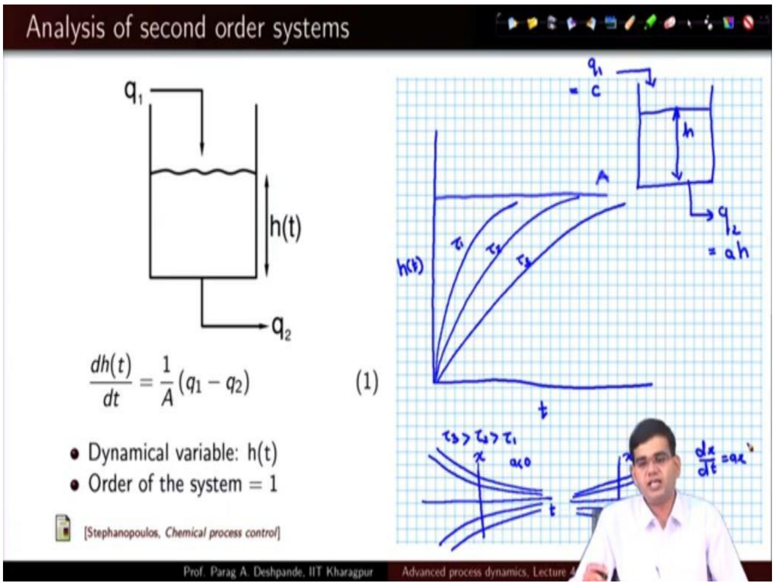
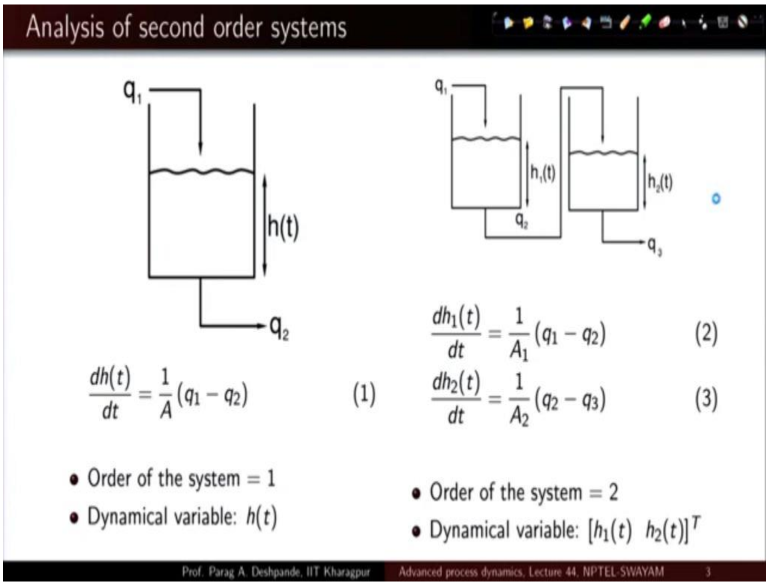
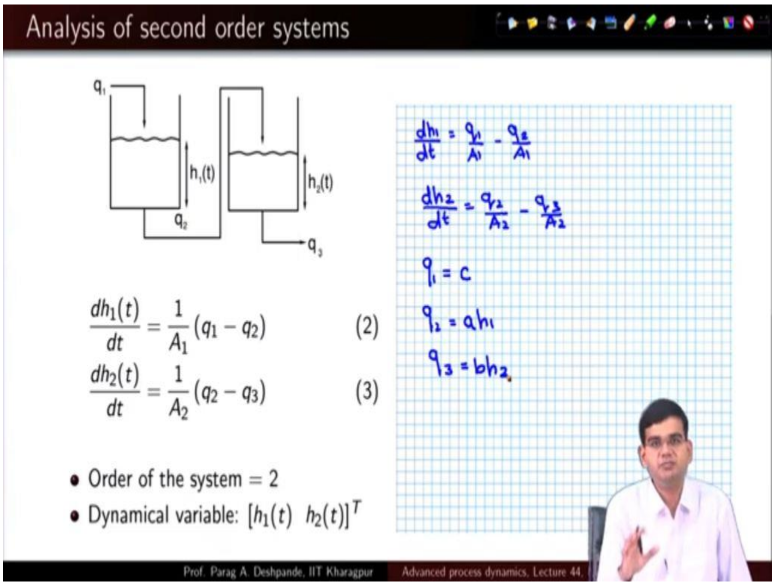
Lecture 44

Analysis of response of second order systems






Analysis of second order systems
q1
h1(t)
h2(t)
q2
q3
$$\frac{dh_1(t)}{dt} = \frac{1}{A_1}(q_1 - q_2) \quad (2)$$
$$\frac{dh_2(t)}{dt} = \frac{1}{A_2}(q_2 - q_3) \quad (3)$$
• Order of the system = 2
• Dynamical variable: $[h_1(t) \;\; h_2(t)]^T$
$\frac{dh_1}{dt} = \frac{q_1}{A_1} - \frac{q_2}{A_1}$
$\frac{dh_2}{dt} = \frac{q_2}{A_2} - \frac{q_3}{A_2}$
$q_1 = c$
$q_2 = a h_1$
$q_3 = b h_2$
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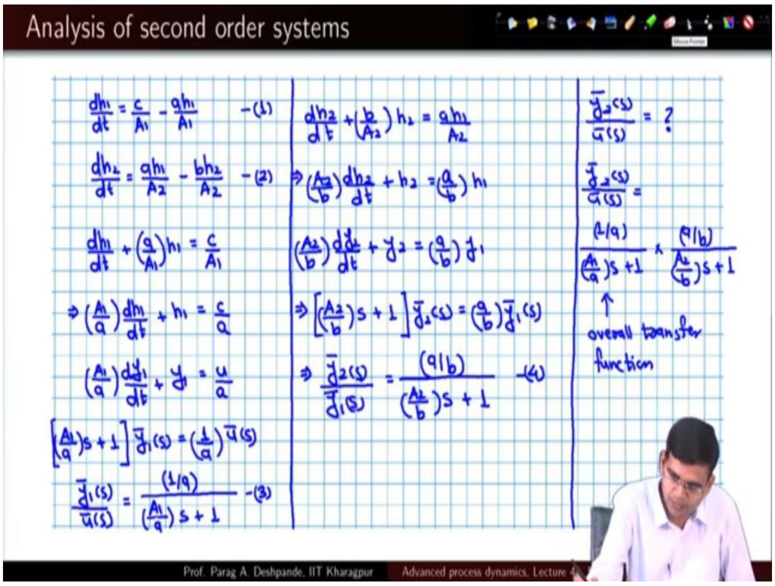
Analysis of second order systems
$\frac{dh_1}{dt} = \frac{c}{A_1} - \frac{a h_1}{A_1}$ -(1)
$\frac{dh_2}{dt} = \frac{a h_1}{A_2} - \frac{b h_2}{A_2}$ -(2)
$\frac{dh_1}{dt} + \left(\frac{a}{A_1}\right) h_1 = \frac{c}{A_1}$
$\Rightarrow \left(\frac{A_1}{a}\right)\frac{dh_1}{dt} + h_1 = \frac{c}{a}$
$\left(\frac{A_1}{a}\right)\frac{dy_1}{dt} + y_1 = \frac{u}{a}$
$\left[\left(\frac{A_1}{a}\right)s + 1\right]\bar{y}_1(s) = \left(\frac{1}{a}\right)\bar{u}(s)$
$\frac{\bar{y}_1(s)}{\bar{u}(s)} = \frac{(1/a)}{\left(\frac{A_1}{a}\right)s + 1}$ -(3)
$\frac{dh_2}{dt} + \left(\frac{b}{A_2}\right) h_2 = \frac{a h_1}{A_2}$
$\Rightarrow \left(\frac{A_2}{b}\right)\frac{dh_2}{dt} + h_2 = \left(\frac{a}{b}\right) h_1$
$\left(\frac{A_2}{b}\right)\frac{dy_2}{dt} + y_2 = \left(\frac{a}{b}\right) y_1$
$\Rightarrow \left[\left(\frac{A_2}{b}\right)s + 1\right]\bar{y}_2(s) = \left(\frac{a}{b}\right)\bar{y}_1(s)$
$\Rightarrow \frac{\bar{y}_2(s)}{\bar{y}_1(s)} = \frac{(a/b)}{\left(\frac{A_2}{b}\right)s + 1}$ -(4)
$\frac{\bar{y}_2(s)}{\bar{u}(s)} = ?$
$\frac{\bar{y}_2(s)}{\bar{u}(s)} = \frac{(1/a)}{\left(\frac{A_1}{a}\right)s + 1} \times \frac{(a/b)}{\left(\frac{A_2}{b}\right)s + 1}$
overall transfer function
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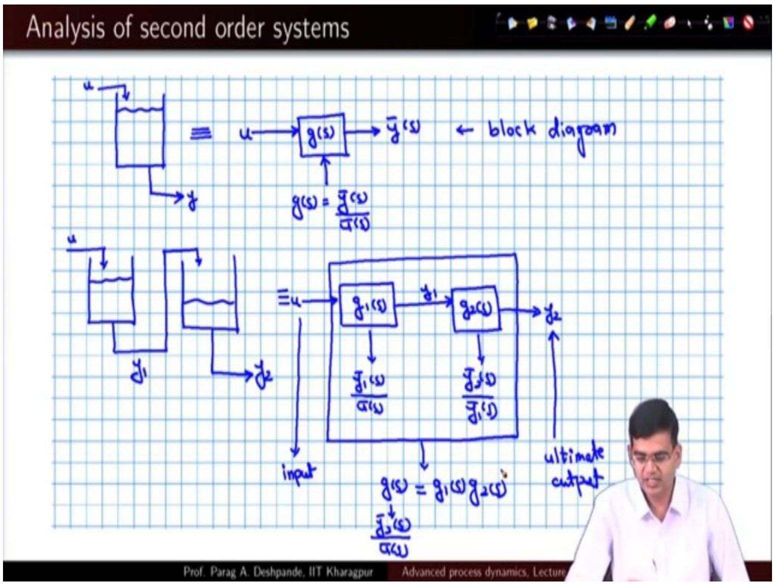
Analysis of second order systems
u
y
≡ u → g(s) → $\bar{y}(s)$ ← block diagram
$g(s) = \frac{\bar{y}(s)}{\bar{u}(s)}$
u
y_1
y_2
≡ u → $g_1(s)$ → y_1 → $g_2(s)$ → y_2
$\frac{\bar{y}_1(s)}{\bar{u}(s)}$
$\frac{\bar{y}_2(s)}{\bar{y}_1(s)}$
input
ultimate output
$g(s) = g_1(s)\, g_2(s)$
$\frac{\bar{y}_2(s)}{\bar{u}(s)}$
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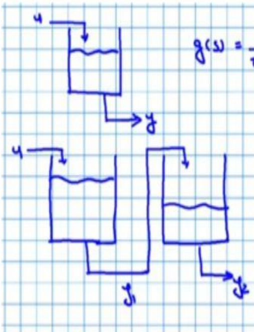
Advanced process dynamics, Lecture

Analysis of second order systems



$$g(s) = \frac{K}{\tau s + 1} \quad -(1) \leftarrow \text{order} = 1$$

$$g_1(s) = \frac{K_1}{\tau_1 s + 1} \quad -(2) \leftarrow \text{order} = 1$$

$$g(s) = g_1(s)\, g_2(s) = \frac{K_1 K_2}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad -(3) \leftarrow \text{order} = 2$$

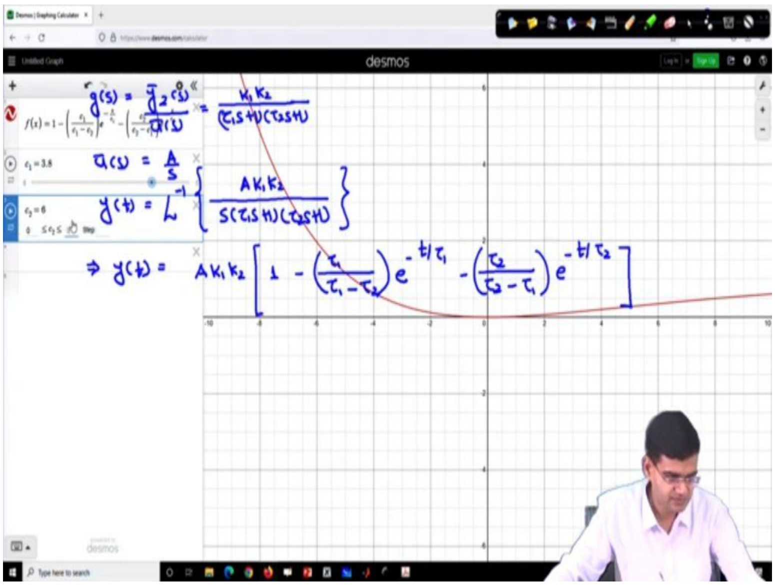
Analysis of second order systems

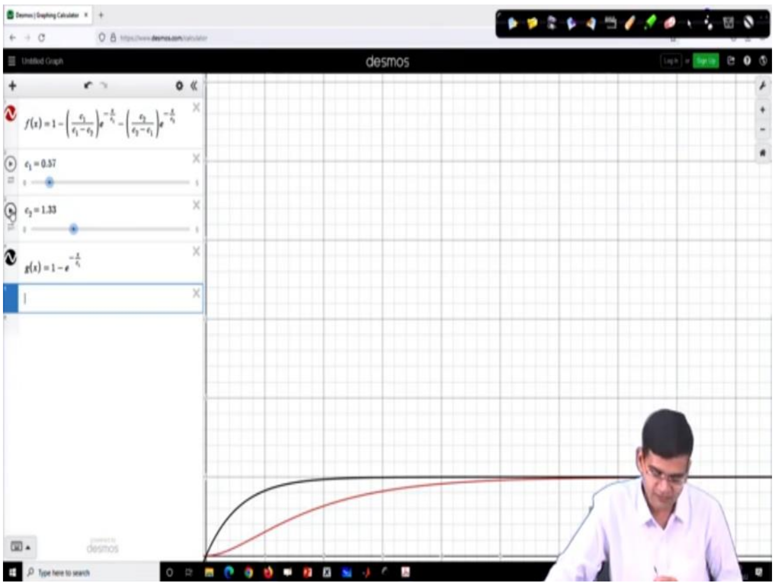
$$g(s) = \frac{\bar{y}_2(s)}{\bar{u}(s)} = \frac{K_1 K_2}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

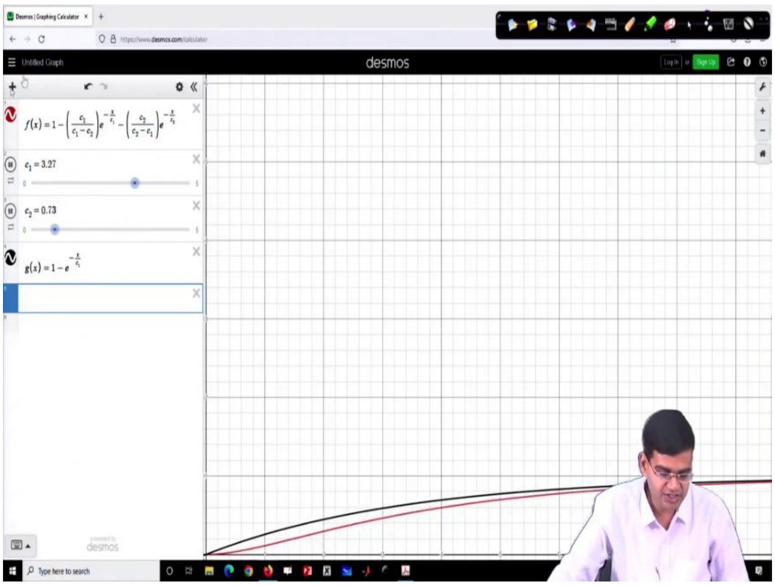
$$\bar{u}(s) = \frac{A}{s}$$

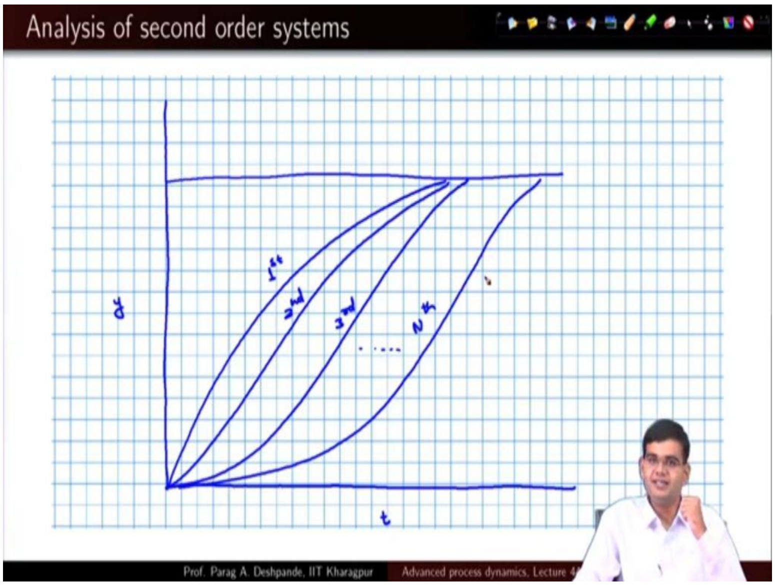
$$y(t) = \mathcal{L}^{-1}\left\{ \frac{A K_1 K_2}{s(\tau_1 s + 1)(\tau_2 s + 1)} \right\}$$

$$\Rightarrow y(t) = A K_1 K_2 \left[1 - \left(\frac{\tau_1}{\tau_1 - \tau_2}\right) e^{-t/\tau_1} - \left(\frac{\tau_2}{\tau_2 - \tau_1}\right) e^{-t/\tau_2} \right]$$




desmos
Untitled Graph
$f(x) = 1 - \left(\frac{c_1}{c_1 - c_2}\right) e^{-\frac{x}{c_1}} - \left(\frac{c_2}{c_2 - c_1}\right) e^{-\frac{x}{c_2}}$
$c_1 = 0.57$
$c_2 = 1.33$
$g(x) = 1 - e^{-\frac{x}{c_1}}$


desmos
Untitled Graph
$f(x) = 1 - \left(\frac{c_1}{c_1 - c_2}\right) e^{-\frac{x}{c_1}} - \left(\frac{c_2}{c_2 - c_1}\right) e^{-\frac{x}{c_2}}$
$c_1 = 3.27$
$c_2 = 0.73$
$g(x) = 1 - e^{-\frac{x}{c_1}}$


Analysis of second order systems
1st
2nd
3rd
Nth
t
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Advanced process dynamics, Lecture 4

Hello and welcome back. We were studying the dynamical response of first order systems. And we looked at the characteristic features of such systems when subjected to various idealized input of forcing functions. What we will do today is we will take up the example of a second order system, and compare and contrast how the dynamical response or dynamical characteristics of a second order system differs from that of a first order system.

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So, the first order system example that we took yesterday was a single liquid level tank. So, let me quickly draw this tank. We had a tank in which certain level of the liquid h was maintained, the input flow rate was q_1, the output flow rate was q_2; the dynamical equations are in front of you.

$$\frac{dh(t)}{dt} = \frac{1}{A}(q_1 - q_2)$$

And we saw that we could pose these equations very well and develop a transfer function when q_1 was assumed to be a constant. And q_2 was assumed to be a constant times the height or level of liquid in the tank h.

$$q_1 = c\ ;\ q_2 = ah$$

So, we developed the transfer function for the system and the response of such systems when subjected to an ideal step function.

So, I have t on my x-axis, h(t) on the y-axis. And if I have subjected this system to an input of magnitude A, then I know that the dynamical response would look something like this. And for different time constants of my system, I can draw different responses. So, if time constants are τ_1, τ_2, τ_3, then I know that $\tau_3 > \tau_2 > \tau_1$. The smaller the time constant, the faster response. But, what is important to understand here is that, irrespective of the time constant, my system always tries to reach the imposed input which is the forcing function of magnitude A, a step forcing function.

The speed of the response may differ, but you asymptotically reach there. So, two things are there that you your response always has a qualitative behavior which looks something like this. You asymptotically reach the response which must be equal to A and then there are no oscillations in the system. In other words, you have a monotonous behavior. And is this something which is in correspondence with what we observed for the case of our state space analysis. In fact, we did because two typical phase portraits that we saw were like this, that I have t, I have x(t), I have t here, I have x(t) here and the phase lines for the two cases would be like this, as you see here.

This is for a < 0 and this is for a > 0 for the equation of the type $\frac{dx}{dt} = ax.$ So, the behavior was always monotonous; either monotonically increasing or monotonically decreasing. In fact, you in this current case what you did was you were looking for a positive step function of magnitude A, you can have a negative step function of magnitude -A and then you will see that you, you will see a monotonically decreasing behavior in your system. So, looks like the two behaviors are consistent.

In fact, what you drew here h(t) versus t is nothing but a phase portrait, where you have drawn the response rather than absolute values and various phase lines would actually correspond to very similar behavior, which you saw otherwise during the state space domain analysis. So, if this be the case, if we take an example of a second order system. Then, what we want to know is that in case where we are doing the analysis in transform domain, will I still observe analogous responses or the family of responses which we saw in the previous lectures during the state space domain analysis. So, let us see if that is the case.

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So, I have a system in which I have a first order system with a single tank. And I saw previously that I can make a second order system from this by adding another tank downstream to the first tank and again taking out an outlet stream which is which has a volumetric flow rate q_3 in this case. So, q_1 comes to tank one, q_2 goes out of tank one, acts as a feed to two; q_3 comes out of tank three. And I have two components of my dynamical vector in this case; h_1, the level of liquid in tank one and h_2, the level of liquid in tank two. So, let us see if we can do a similar analysis, and what kind of features we get in this particular case.

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So, I have the equations like this,

$$\frac{dh_1(t)}{dt} = \frac{1}{A_1}(q_1 - q_2)$$

and

$$\frac{dh_2(t)}{dt} = \frac{1}{A_2}(q_2 - q_3)$$

These are my dynamical equations. And what are the different relationships between a given flow rate and the liquid level? I can write here q_1, in fact, q_1 is the inlet flow rate; so, it would be simply for the time being assume that it is a constant; so,

$$q_1 = c$$

q_2 will depend upon the outlet; because its outlet from tank one, depend upon the level of liquid in tank one. So, I can write this as

$$q_2 = ah_1$$

and q_3 can be written as some other wall coefficient. So, b and this time it would depend upon the level of liquid in tank two; So,

$$q_3 = bh_2$$

So, let us now do analysis which is similar to the analysis which we did for a single tank.

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So, I have the equation as

$$\frac{dh_1}{dt} = \frac{c}{A_1} - \frac{ah_1}{A_1} \quad \ldots\ldots\ldots (1)$$

and

$$\frac{dh_2}{dt} = \frac{ah_1}{A_2} - \frac{bh_2}{A_2} \quad \ldots\ldots\ldots (2)$$

So, let me do some rearrangements to put the equations in suitable form. So, let me write this as equation (1); let me refer this equation as (2). So, I can write this as

$$\frac{dh_1}{dt} + \frac{ah_1}{A_1} = \frac{c}{A_1}$$

If we remember the procedure from yesterday's discussion, I will make the coefficient of h_1 as unity.

So, this becomes

$$\frac{A_1}{a}\frac{dh_1}{dt} + h_1 = \frac{c}{a}$$

And now I will convert this equation to a form which has the departure variables or which would give me (h_1 - h_s). And if c acts as the input function, So, (c_1 - c_s). So, that corresponding equation in deviation variable form would become

$$\frac{A_1}{a}\frac{dy_1}{dt} + y_1 = \frac{u}{a}$$

Now, it is not very difficult from here to get an expression for the Laplace transformation. So, I will have

$$\left[\left(\frac{A_1}{a}\right)s + 1\right]\overline{y_1}(s) = \frac{1}{a}\bar{u}(s)$$

From where I get an expression

$$\frac{\overline{y_1}(s)}{\bar{u}(s)} = \frac{\left(\frac{1}{a}\right)}{\left(\frac{A_1}{a}\right)s + 1} \ldots\ldots\ldots (3)$$

So, what is the physical significance of equation (3)? Equations (3) is the transfer function for the dynamics of the tank one alone. So, if that is the case, I have $\overline{y}(s)$, which means the dynamical variable in deviation form for tank one, divided by the input for tank one. So, this will give me the dynamics only of tank one. And then I can say that from this equation, $\frac{1}{a}$ would be my static gain of the system and the time constant of a system would be $\frac{A_1}{a}$.

So, this is what I get then, I also need to do one thing. I now need to do a similar analysis for equation (2). So, I will do a rearrangement for equation (2) from where I will get

$$\frac{dh_2}{dt} + \frac{bh_2}{A_2} = \frac{ah_1}{A_2}$$

I will do some rearrangement. So, I can write this as

$$\frac{A_2}{b}\frac{dh_2}{dt} + h_2 = \frac{a}{b}h_1$$

I will put this in deviation variable form and in fact, I have forgotten the area. So, let me, no sorry area gets cancelled out. So, area will not appear there. So, it is simply So, from here I will now put the equation to deviation variable form. So, I can write this as

$$\frac{A_2}{b}\frac{dy_2}{dt} + y_2 = \frac{a}{b}y_1$$

And now I will take the Laplace transformation. So, this will be

$$\left[\left(\frac{A_2}{b}\right)s + 1\right]\overline{y_2}(s) = \frac{a}{b}\overline{y_1}(s)$$

In other words,

$$\frac{\overline{y_2}(s)}{\overline{y_1}(s)} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{A_2}{b}\right)s + 1} \dots\dots\dots (4)$$

Let me refer this as equation number (4). So, let us try to understand what does equation (4) mean, equation (4) has $\left(\frac{\overline{y_2}(s)}{\overline{y_1}(s)}\right)$. So, if you remember our arrangement, you had tank two in which the input to tank two was flow rate two; the and outlet was flow rate q_3. So, that means y_2 is acting y_1 is acting as the input, and y_2 is acting as the output; so, therefore, this is the Laplace transform.

But, what I want to do is now this, I can I can have a number of tanks in my system in a process and just for example. And all I would be interested in is the output from the last tank and output of the last tank would depend upon the level of liquid in the last tank. So, therefore, I would be interested in the level of liquid in the last tank. I may not worry about what is going on in the intermediate tanks. All of them would result into some dynamics which would affect the liquid level in the final tank.

But, I would ideally like to control only the input feed which is the input to the first tank and I would like to know the effect at the last tank. So, that means what do I want? I want $\frac{\overline{y_2}(s)}{\overline{u}(s)}$.

What should this be equal to? Well, equation (3) has $\frac{\overline{y_1}(s)}{\bar{u}(s)}$ and equation (4) has $\frac{\overline{y_2}(s)}{\overline{y_1}(s)}$ I will simply multiply these two equations; I will get the desired quantity. So, I will get

$$\frac{\overline{y_2}(s)}{\bar{u}(s)} = \frac{\left(\frac{1}{a}\right)}{\left(\frac{A_1}{a}\right)s + 1} + \frac{\left(\frac{a}{b}\right)}{\left(\frac{A_2}{b}\right)s + 1}$$

So, therefore, I see that my overall transfer function, my overall transfer function is simply a multiplication of individual transfer functions.

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So, let us try to put this system in form which can be easy to visualize. So, when we have a single tank, so I have a single tank input output. My input is u in the deviation variable form. Output is y. So, what I infer from here is that I give an input to my system, and I feed it to a box which is which does a mathematical operation. And that mathematical operation happens to be g(s)... the transfer function, the first order transfer function

$$g(s) = \frac{\bar{y}(s)}{\bar{u}(s)}$$

and what I get as the output is $\bar{y}(s)$. So, this is the this is the block diagram for the system.

Now, I have two tanks. So, I have the input in terms of deviation variable and other deviation variables, I have the input to the first tank. From the first tank, I have the output which is y_1; y_1 is the output from first tank acts as the input for the second tank, and here you have y_2. Now, y_1 is an has been represented as an output that happens to be the liquid level in this case. So, this is a...... do not think that y_1 is a flow or y_1 is whether it is a flow or a level of liquid and so on. We are trying to convert the variables and the representation which you see here in terms of block diagram.

So, all we are trying to do but this particular representation is that if there is some disturbance u; which means some change in the inlet flow rate, some change in the outlet flow rate will happen. It will happen, because of the change in the liquid level and so on. So, what is going on here? Your ultimate input to the system is u and this input acts on the first tank with the transfer function $g_1(s)$ and it gives the output as y_1, output from the first tank. So, this would be what?

$$g_1(s) = \frac{\overline{y_1}(s)}{\bar{u}(s)}$$

And then y1 is fed to the second tank as the input and the output that you get is y_2.

So, here you will have

$$g_2(s) = \frac{\overline{y_2}(s)}{\overline{y_1}(s)}$$

But, if you want if you want this to be the input and this to be the ultimate output, then what would you like to do? You would like to draw a larger block here. Now, in this larger block, you will have some mathematical operation like the way you have a mathematical operation g(s) for the first order system. You have again a single input u, single output y_2 and within this, there is a larger mathematical operation which takes place. And then the corresponding single transfer function would become

$$g(s) = \frac{\overline{y_2}(s)}{\bar{u}(s)}$$

And this would be nothing but

$$g(s) = g_1(s)\, g_2(s)$$

So, this is the block diagram representation of this entire scheme. Now what is interesting to be noted here is that I have this.

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I have my first system u ...... y; my transfer function is

$$g(s) = \frac{K}{(\tau s + 1)} \;\ldots\ldots\ldots (1)$$

I have now the second system u, y1, y2 and I will have $g_1(s)$ which gives the dynamics of liquid level change in the first tank. It would be

$$g_1(s) = \frac{K_1}{(\tau_1 s + 1)} \ldots \ldots \ldots (2)$$

And g(s) which is the overall transfer function indicating the dynamics of my system would be

$$g(s) = g_1(s)\ g_2(s)$$

And this would be equal to

$$\mathrm{g}(s) = \frac{K_1 K_2}{(\tau_1 s + 1)(\tau_2 s + 1)} \ldots \ldots \ldots (3)$$

K_1, K2 simple multiplication of individuals static gains. And if this be the case, I want to know what is the dynamics or what is the order of my system?

So, I have equation (1), equation (2), equation (3). If I look at equation (1), I have one single tank; the order of my system is 1. How do I do that? How do I know that? I look at the degree of the polynomial in the in the denominator of my transfer function; $\tau s + 1$, first degree polynomial ...order one. The overall dynamics for a two-tank system, here order not very difficult to see. The degree of the polynomial in the denominator is 2. So, the order is 2. And we in fact know that it is a second order system; because we derived this transfer function considering two first order ODEs, perfect correspondence.

But, what is the order of the system? If you consider only the first tank of a two-tank system; the order the order is again 1. So, the dynamics of only the first tank is first order; whereas, the overall dynamics of the system is second order. Now, the question is how would the second order dynamics look like? So, the transfer function is here in front of in front of you.

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So, the transfer function is

$$\mathrm{g}(s) = \frac{\overline{y_2}(s)}{\bar{u}(s)} = \frac{K_1 K_2}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

And my input function is a step input; so,

$$\bar{u}(s) = \frac{A}{s}$$

So, what would be y(t)?

$$y(t) = L^{-1}\left\{\frac{AK_1K_2}{(\tau_1 s + 1)(\tau_2 s + 1)s}\right\}$$

You will do a partial and do the inverse Laplace transformation. So, I am writing the final step for you; please work this out at home.

This should look like

$$y(t) = AK_1K_2\left[1 - \left(\frac{\tau_1}{\tau_1 - \tau_2}\right)e^{-\frac{t}{\tau_1}} - \left(\frac{\tau_2}{\tau_2 - \tau_1}\right)e^{-\frac{t}{\tau_2}}\right]$$

Very simple derivation nothing very difficult in it, please work this out. This is what you will get as the answer. What I would emphasize now or spend my time now is to see the nature of this solution. So, let us plot this solution in the Desmos calculator. What we have in front of us is f(x). I know the importance of A and K_1 and K_2. So, let us get rid of that. Let us make everything unity; let us denote τ_1 as c_1.

So,

$$f(x) = 1 - \left(\frac{c_1}{c_1 - c_2}\right)e^{-\frac{x}{c_1}} - \left(\frac{c_2}{c_2 - c_1}\right)e^{-\frac{x}{c_2}}$$

So, let us see how does this look like? These values must always be positive; let me make from 0 to 5, let me make this from 0 to 5. And I am worried only about the first quadrant, so let us look at this; so, this is the dynamical behavior, dynamical evolution. Let me make the time constant small that, so that you can see a quick evolution of the system; this is how the system is evolving.

Looks similar to what we studied in the previous lecture; but then let us compare it against of the dynamical response of a first order system. So, the dynamical response of a first order system is simply given as

$$g(x) = 1 - e^{-\frac{x}{c_1}}$$

c_1 is just the time constant of for the first order system and then let me do one thing. I have already seen that what you can see as the black curve, which is the response of a first order system is faster than the response of a second order system. Let me do an animation, and see if is there any situation where this red curve, which is the response of a second order system becomes faster than the response of a first order system.

So, let me do an animation and let us watch the animation. You will see that in every case, the black curve is always on top, the red curve is always at the bottom. The meaning of this is that in each and every situation, the dynamical response of a second order system is slower than the dynamical response of a first order system. This is one important feature that we saw. So, let us do one thing. Let us quickly draw the response.

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What we see here is this, that I have t versus y. And if this is the imposed forcing function, then if this is the first order response, then second order response under every condition would always be something like this. And if this is second order, now you can extend this analysis; the third order would be like this, even more slower. The fourth order would be even more slower and so on; so, this is the N^{th} order dynamic. So, as the order of your system increases, the dynamics become slower and slower and slower; this is one important observation. Final one more observation is that you never saw any oscillation in the system.

You always saw that the response of the second order system was slower than the response of the first order system. But, never was the response oscillatory, which means that in second order system as well, you saw only monotonous response. But, this is quite contrary to what we saw in the state space domain analysis that you could have response which was oscillatory. So, why does this happen? Whenever you have systems where the transfer function is simply multiplication of individual transfer functions; you will never observe any oscillations in the system.

What are the cases when you will observe oscillations; we will take that up in the next lecture. Till then, goodbye.