Advanced Process Dynamics

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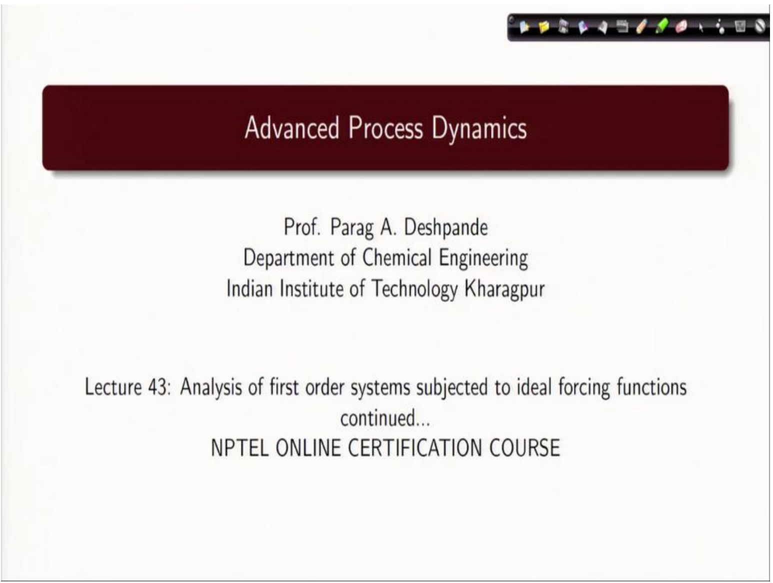
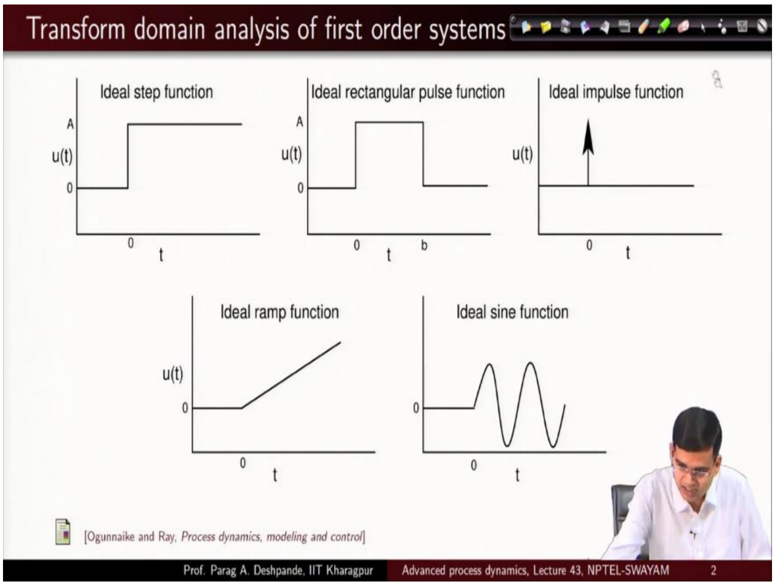
Indian Institute of Technology, Kharagpur

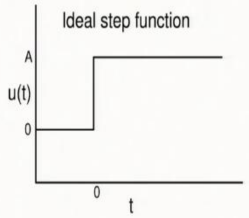
Lecture 43

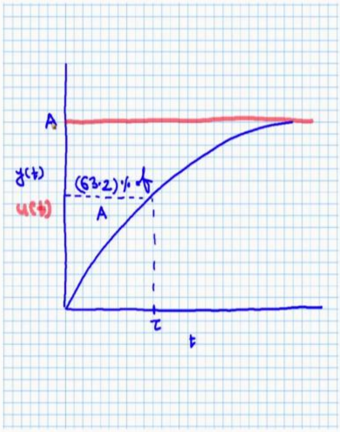
Analysis of first order system subject to ideal forcing functions continued…





Transform domain analysis of first order systems

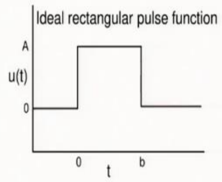




$$u(t) = \begin{cases} 0, & t < 0 \\ A, & t > 0 \end{cases} \quad (1)$$

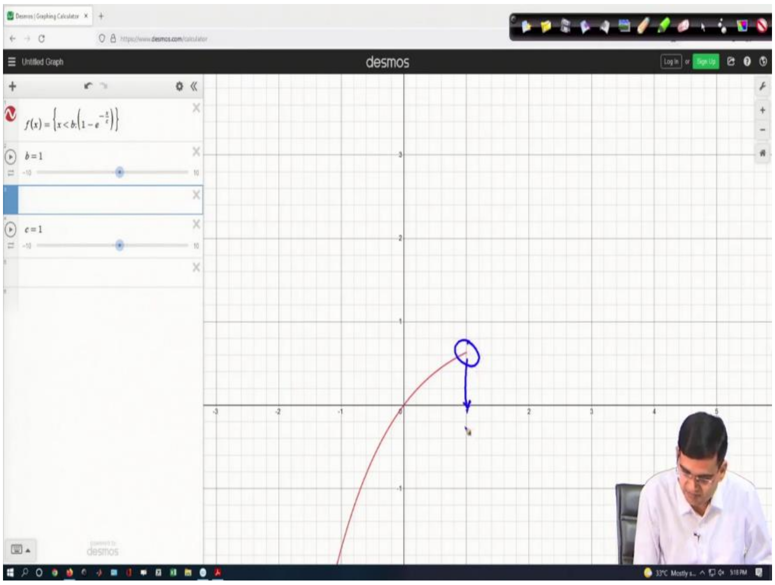
$$y(t) = AK(1 - e^{-t/\tau}) \quad (2)$$

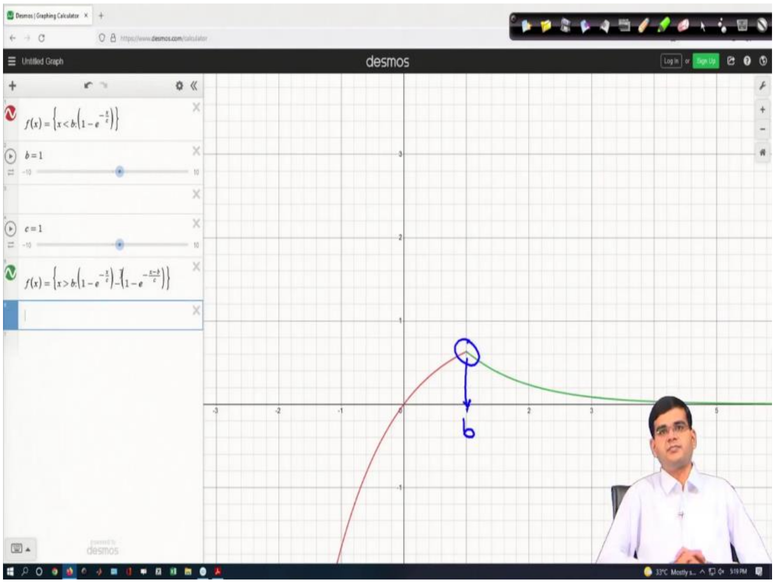
Transform domain analysis of first order systems

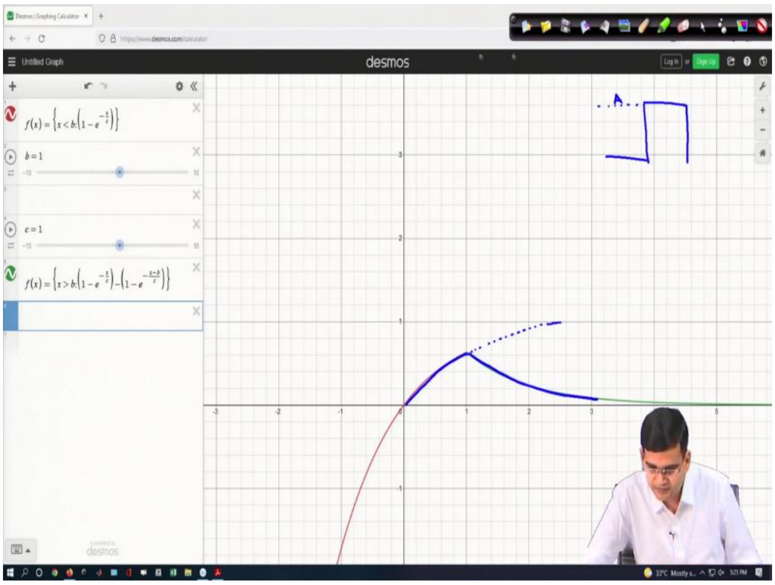


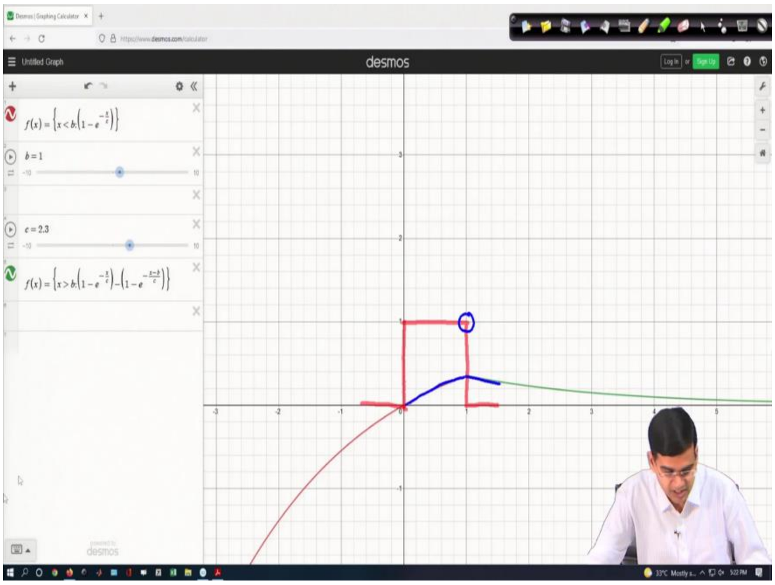
$$u(t) = \begin{cases} 0, & t < 0 \\ A, & 0 < t > b \\ 0, & t > b \end{cases} \quad (3)$$

$$y(t) = \begin{cases} AK(1 - e^{-t/\tau}), & t < b \\ AK[(1 - e^{-t/\tau}) - \\ (1 - e^{-(t-b)/\tau})], & t > b \end{cases} \quad (4)$$




Untitled Graph
desmos
b = 1
c = 1
b


Untitled Graph
desmos
b = 1
c = 1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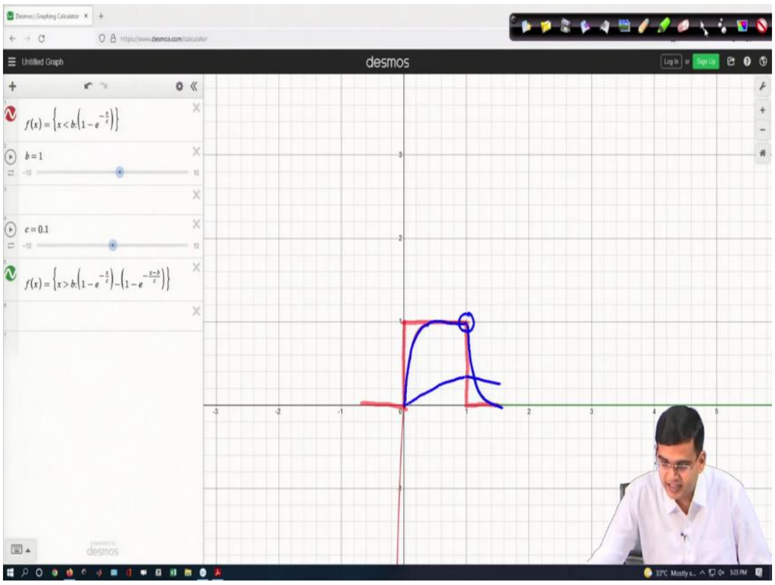
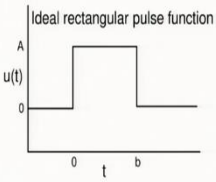
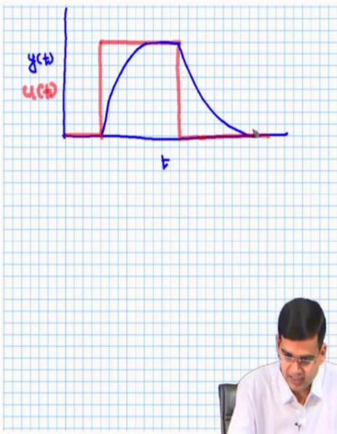
desmos
b = 1
c = 2.3



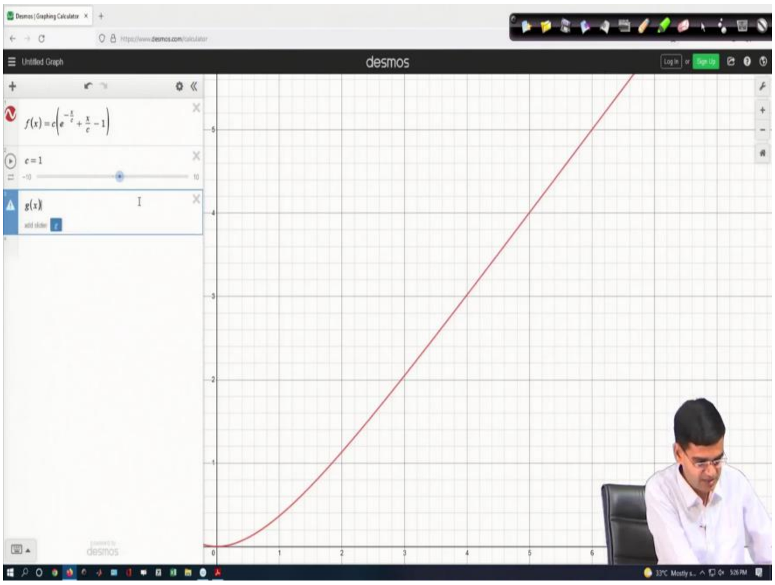
Transform domain analysis of first order systems

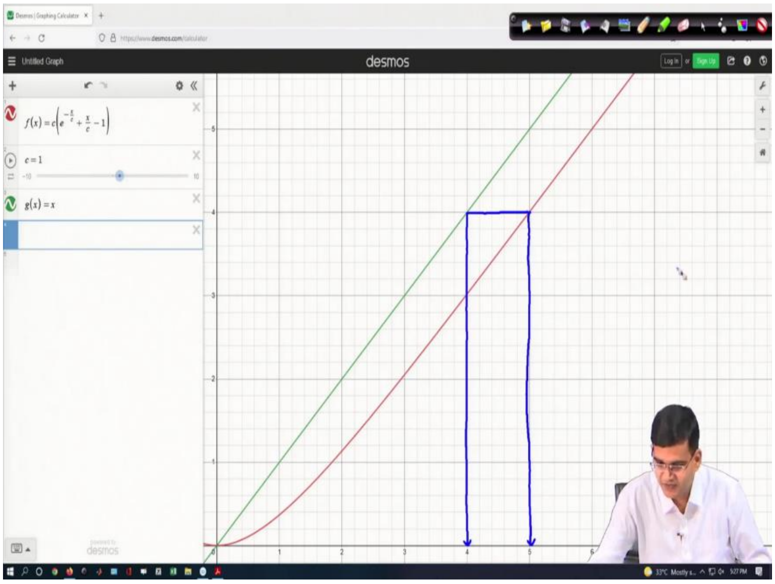


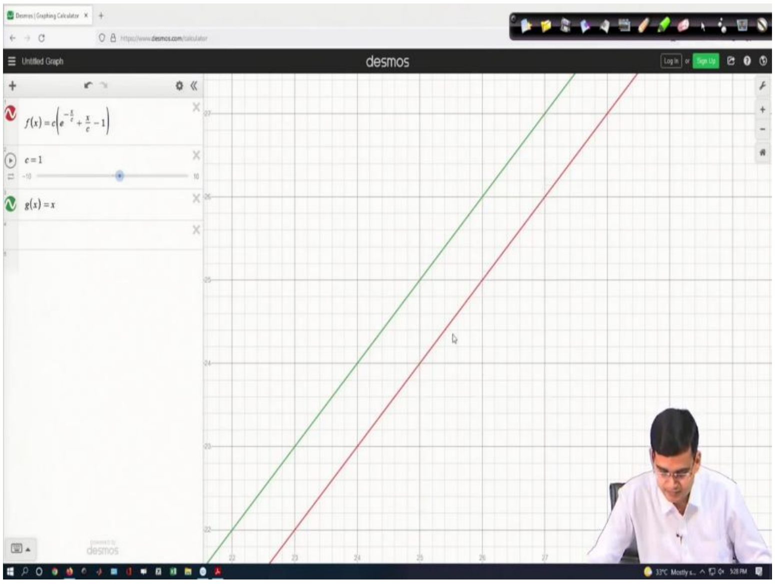


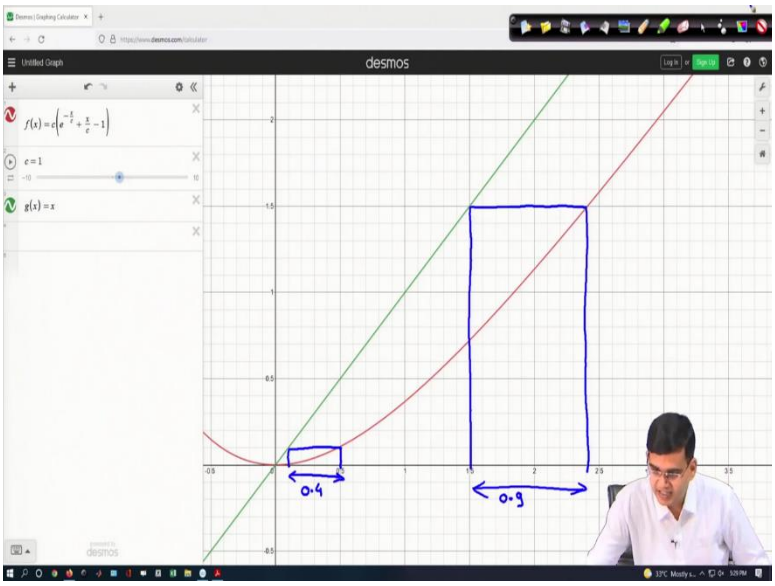
$$u(t) = \begin{cases} 0, & t < 0 \\ A, & 0 < t > b \\ 0, & t > b \end{cases} \tag{3}$$

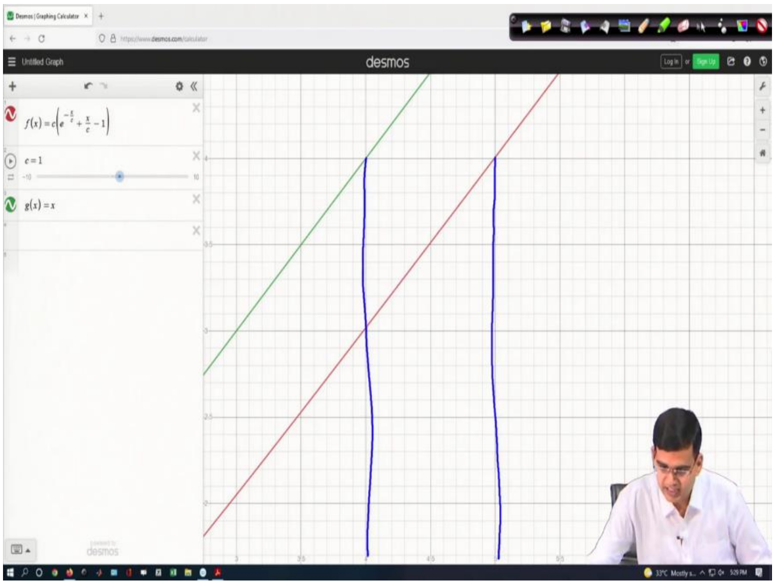
$$y(t) = \begin{cases} AK(1 - e^{-t/\tau}), & t < b \\ AK[(1 - e^{-t/\tau})) - & \\ (1 - e^{-(t-b)/\tau})], & t > b \end{cases} \tag{4}$$

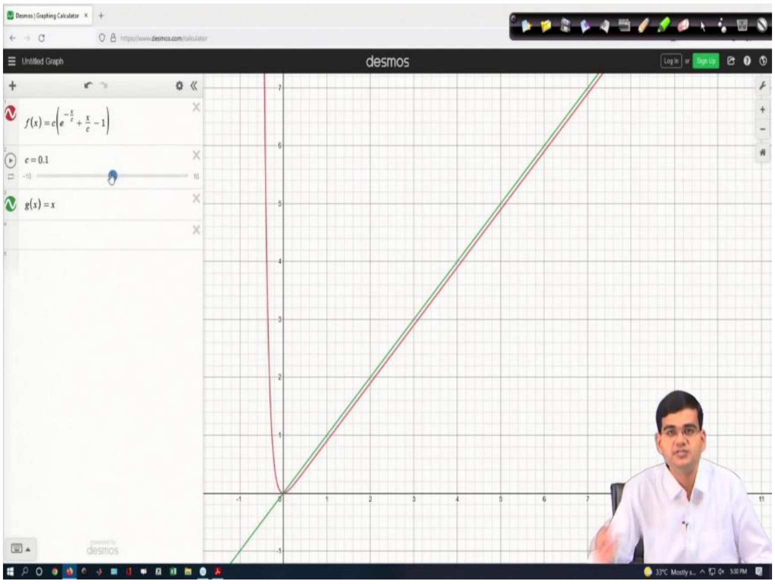



Untitled Graph
desmos
$f(x) = c\left(e^{-\frac{x}{c}} + \frac{x}{c} - 1\right)$
$c = 1$
$g(x) = x$


Untitled Graph
desmos
$f(x) = c\left(e^{-\frac{x}{c}} + \frac{x}{c} - 1\right)$
$c = 1$
$g(x) = x$


Untitled Graph
desmos
$f(x) = c\left(e^{-\frac{x}{c}} + \frac{x}{c} - 1\right)$
$c = 1$
$g(x) = x$
0.4
0.9


Untitled Graph
desmos
$f(x)=c\left(e^{-\frac{x}{c}}+\frac{x}{c}-1\right)$
$c=1$
$g(x)=x$


Untitled Graph
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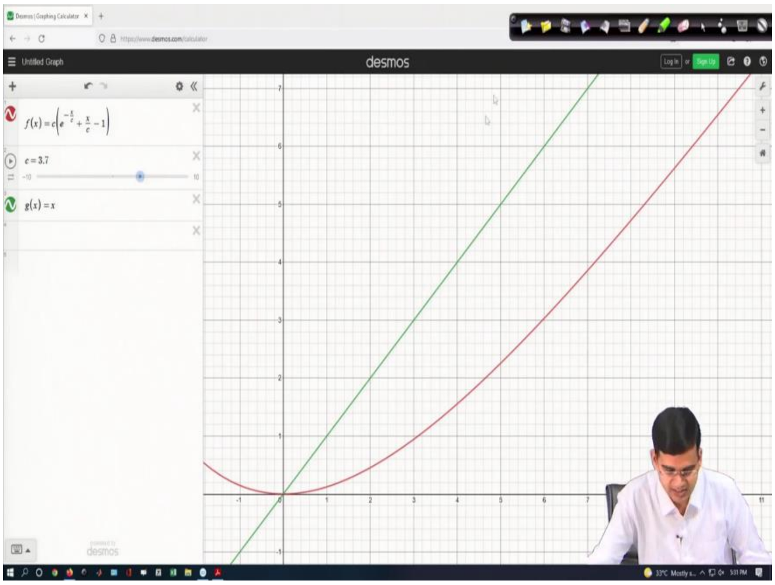
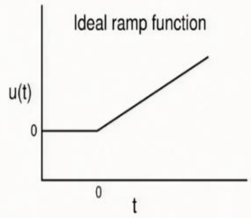
$f(x)=c\left(e^{-\frac{x}{c}}+\frac{x}{c}-1\right)$
$c=0.1$
$g(x)=x$

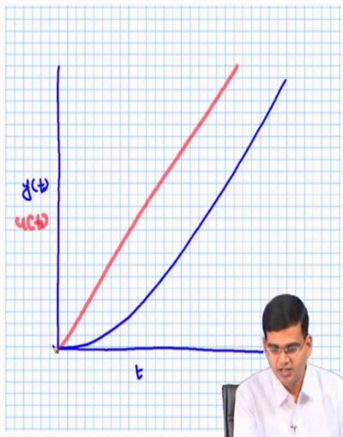

Untitled Graph
desmos
$f(x)=c\left(e^{-\frac{x}{c}}+\frac{x}{c}-1\right)$
$c=3.7$
$g(x)=x$

Transform domain analysis of first order systems

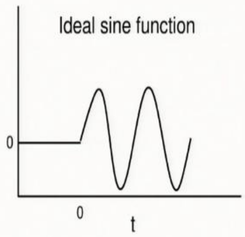


$$u(t) = \begin{cases} 0, & t < 0 \\ At, & t > 0 \end{cases} \quad (5)$$

$$y(t) = AK\tau(e^{-t/\tau} + \frac{t}{\tau} - 1) \quad (6)$$



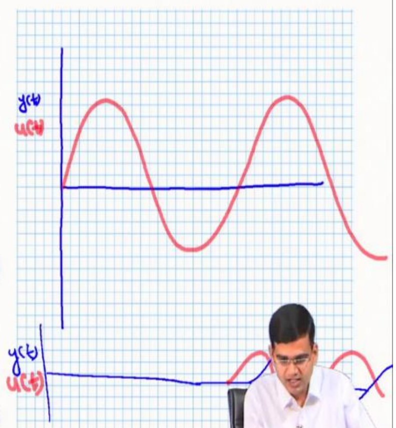
Transform domain analysis of first order systems



$$u(t) = \begin{cases} 0, & t < 0 \\ A\sin\omega t, & t > 0 \end{cases} \quad (7)$$

$$y(t) = AK\left[\cancel{\frac{\omega\tau}{1+\omega^2\tau^2}e^{-t/\tau}} + \frac{1}{\sqrt{1+\omega^2\tau^2}}\sin(\omega t + \phi)\right] \quad (8)$$

$\phi = -\tan^{-1}(\omega\tau)$



Transform domain analysis of first order systems

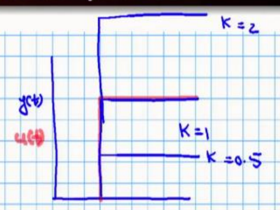
$$a_1\frac{dy}{dt} + a_0 y = bu$$

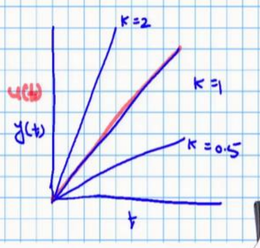
$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{\tau s + 1} \; ; \; \tau \approx 0$$

$$\Rightarrow \frac{\bar{y}(s)}{\bar{u}(s)} = K$$

$$\Rightarrow \bar{y}(s) = K[\bar{u}(s)]$$

$\Rightarrow y(t) = K\,u(t)$ – pure gain





$$a_1 \frac{dy}{dt} + a_0 y = bu$$

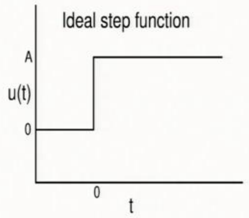
when $a_0 \approx 0$

$$a_1 \frac{dy}{dt} = bu$$

$$\Rightarrow a_1 s \bar{y}(s) = b \bar{u}(s)$$

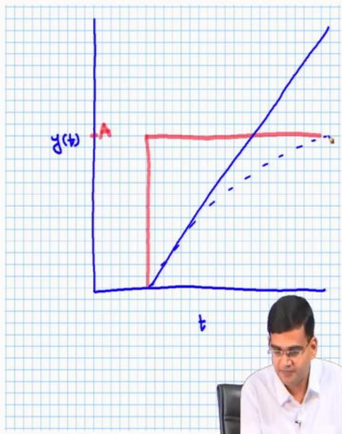
$$\Rightarrow \frac{\bar{y}(s)}{\bar{u}(s)} = \left(\frac{b}{a_1}\right) \frac{1}{s}$$

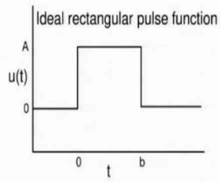
– Pure cap



$$u(t) = \begin{cases} 0, & t < 0 \\ A, & t > 0 \end{cases} \tag{9}$$

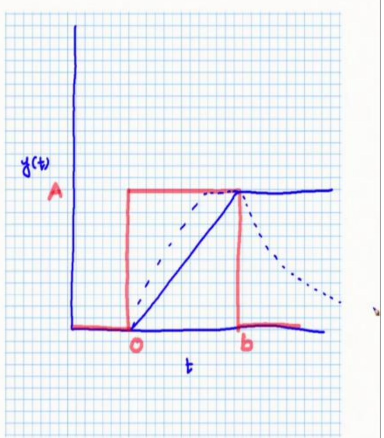
$$y(t) = AKt \tag{10}$$

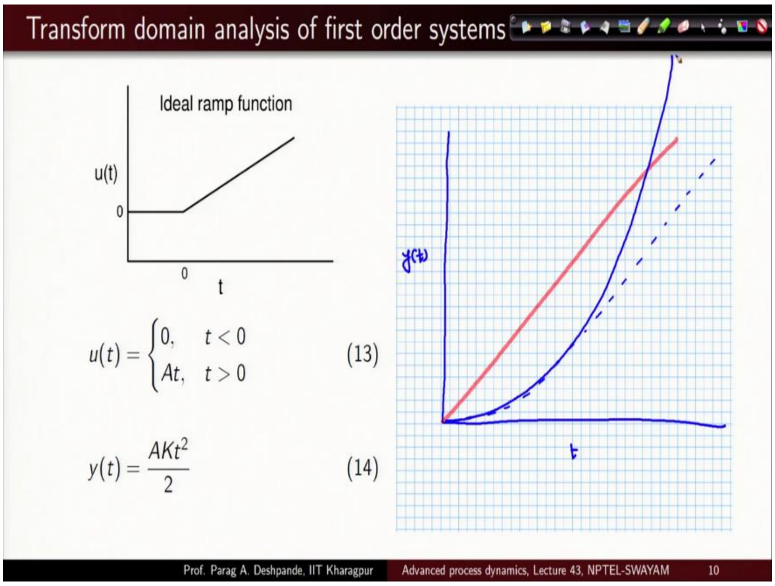


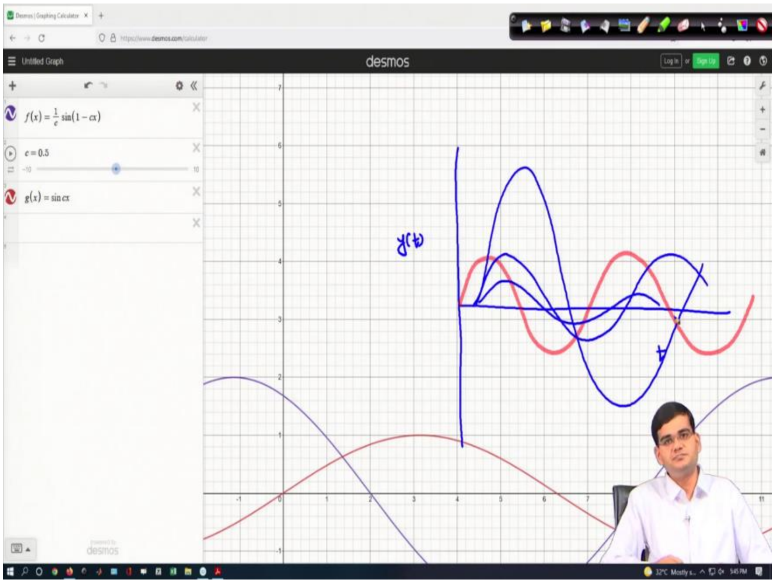


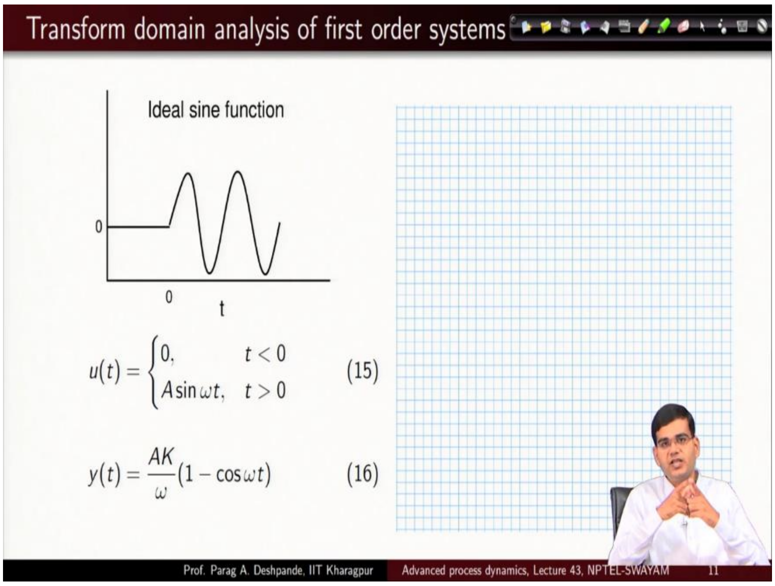
$$u(t) = \begin{cases} 0, & t < 0 \\ A, & 0 < t > b \\ 0, & t > b \end{cases} \tag{11}$$

$$y(t) = \begin{cases} AKt, & 0 < t < b \\ AKb, & t > b \end{cases} \tag{12}$$









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So, we are studying the Analysis of Dynamics of first Order Systems subject to various ideal forcing functions.

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In the previous lecture, we looked at the ideal step function and the response of the system subject to ideas to functions. We have other ideal forcing functions as well which you can see in front of you, rectangular pulse function, impulse function, ramp functions, and functions and so on.

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Let us take some more examples from these functions. In the previous lecture the response of an ideal step function was obtained like this. So, in the deviation variable form if this is y(t) and u(t) so, let me make this like this u(t) then I knew that my u(t) is like this and with time my y(t) would catch it up like this. This is the general behavior of the system and if you declare this as τ the time constant, so, then you will see that this is 63.2 if 63.2% of A, if this is A. So, this is what we saw about the ideal step function in the previous lecture.

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Let us move ahead to a new forcing function, ideal rectangular pulse function. So, my rectangular pulse is like this

$$u(t) = \begin{cases} 0, & t < 0 \\ A, & 0 < t < b \\ 0, & t > b \end{cases}$$

that you for time t < 0, here, there was no disturbance and from for b for 0 to b you provide it a positive A step input and then again, a negative -A step and put at b so that you come back to the original 0. So, this is the rectangular pulse function the expression for the pulse function is in front of you. t < 0, 0 between 0 and b so, this is a correction this has to be less than between 0 and b it is A and for t > b, it is again 0. So, therefore, the response following the usual method I leave the method the details for you to work out the responses given like this.

$$y(t) = \begin{cases} AK\left(1 - e^{-\frac{t}{\tau}}\right), & t < b \\ AK\left[\left(1 - e^{-\frac{t}{\tau}}\right) - \left(1 - e^{-\frac{(t-b)}{\tau}}\right)\right], & t > b \end{cases}$$

So, let us try to plot this.

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So, we have f(x) since we know the significance of A and K, we will keep A and K as unity and see the effect of only the time constant. So,

$$f(x) = 1 - e^{-\frac{x}{c}}$$

This is what we saw in the previous case as well but this was true only for x < b. So, for x < b, this is the case. So, when my b is 1 so, you are not just keeping on evolving you can find only till this point and this is b.

Then next again, I will have f(x). f(x) is equal to now, x > b, what is going to happen?

$$f(x) = \left(1 - e^{-\frac{x}{c}}\right) - \left(1 - e^{-\frac{(x-b)}{c}}\right)$$

So, let us now try to develop the understand how this response looks like. From 0 times 0 to b you had a +A step input you can imagine this as +A step input and if you see the behavior here, you see the same behavior. If you did not give this -A and what would have happened was that it would have continued like this could have continued. But now, since you have given a -A, this arm comes and therefore, now, the question is what is the significance of time constants C, let me do this analysis.

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So, what I will do is this that I will increase the value of the time constant, what has happened. See, you gave it you gave the system a positive step of A, you maintained it for some time there and you would have liked the system to have gone there, stayed there and when you wanted the system come back you gave a -A so, you would like would have liked it to come back. Here, you have given a unit step so, therefore, your forcing function would look like this this is this red highlighter gives you the rectangular pulse.

But what has happened is instead of reaching till this ultimate value the system when the system starts coming down because of the effect of -A and that has become the case because you did not give the system sufficient time or the in other words, the time constant was very large.

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If I make the time constant very small then see what happens the system indeed reaches to that point. So, now, you see what has happened is that the system has reached this point. And if b the duration of time is larger than 5 times your time constant then you would know that your system would already have reached 99.3% of the ultimate value. And in fact, that is the case here. So, you have time constant, $\tau = 0.1$ and you have $b = 1$ which means you have 10 time constant, that you have provide the system with 10 time constants and therefore, the system has perfectly reached this point and then again start it start coming down and it starts coming down again pretty fast because the time constant is very small. So, this is the ideal rectangle rectangular input.

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So, let me draw this. Let me draw couple of such plots. So, y(t) versus t and this is u (t) would be like this ideal rectangular forcing function and y(t) following the up +A would rise like this and then it would come like this. So, let me extend this also, now how sharply would it rise and how sharp it would come down whether it would be able to catch that up or not it will depend upon the time constant. So, now, you can do an analysis of ideal ramp function.

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So, what is an ideal ramp function?

$$u(t) = \begin{cases} 0, & t < 0 \\ At, & t > 0 \end{cases}$$

and the response is given as what is there in front of you.

$$y(t) = AK\tau\left(e^{-\frac{t}{\tau}} + \frac{t}{\tau} - 1\right)$$

Let us try to plot the response. So,

$$f(x) = c\left(e^{-\frac{x}{c}} + \frac{x}{c} - 1\right)$$

Again, I will not worry about the other three quadrants. Let me worry about the first quadrant and this is how the dynamics looks what how does your input function itself look, how does your forcing function look.

Let me draw the forcing function as well.

$$g(x) = x$$

A has been considered as unity. So, it is simply x.

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So, you have the forcing function moving as a straight-line constant gradient, and then following it with some lag you can see that there is a definite lag here. There is a lag here with that lag your response follows the forcing function. Now, when I drew these two vertical lines, you saw that they landed perfectly at 4 and 5. Is it a coincidence or it has some reason behind it? Let us try to understand the reason why they ended up at perfect 4 and 5.

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So, the first characteristic feature of such a system is that your response would follow your forcing function that would be a time lag and when you see at very large time intervals see here, these 2 are 2 perfectly parallel lines green is your forcing function, red is your response, perfectly parallel and then you drop any 2 verticals they would be exactly equally spaced, but would they be always equally spaced.

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That will not be the case because you see here, if I see the behavior of the system here during the initial stages. I dropped pair of 2 verticals. You see here. The differences 0.4 time units. References 0.4 time units. Here, 1.5 and 2.4 the differences 0.9 time units. So, at initial stages the difference is small and then the difference that is growing but the different saturates at larger time intervals to exactly unity.

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Here, if you see 4 and 5 so, what is going on, this is another way of indicating the time constant because if you see you will realize that the difference between these 2 parallel lines depends upon the time constant. Let me make time constant small these 2 lines are coming very close. If I make time constant very large this difference will be large. So, therefore, what is the meaning of unit time constant, the difference between the response and the difference between the forcing functional response if it is unity then in one time in the first time constant their difference would be 63.2 % of the ultimate difference.

And after 4 or 5 time constants…. after 5 time constants, you will see that the difference between the lag bit of the forcing function and the response would become unity that is another meaning of time constant. So, this is how you will see how the system would behave.

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So, let me draw this here. So, this is y (t)…. t this is also u(t) and u(t) is a ramp function and what you see is that you will have the response which will look like this.

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Finally, you can also have a look into the response of the sine function, I encourage you to punch this incredibly long expression in Desmos by the way, you will need this information that

$$\phi = -\tan^{-1} \omega\tau$$

Then, you can punch in this and you will find the response versus time, what we can see here is something interesting. So, this is y(t) and u(t) and what you see here is that u(t) is a sine function so, it will look something like this. what is there in the response in the response you will find that there is this sin ωt with some lag (sin (ωt + ϕ)) which means that if the input is a sine curve the output also would be a sine curve.

But when there would be a phase difference, but this is also accompanied with another term which means that there would be some disturbance apart from your sinusoidal nature, but that disturbance or that behavior has $e^{-\frac{t}{\tau}}$ which means that there would be initial transients and as at for at very large intervals of time.

When the effect of $-\frac{t}{\tau}$ will die down, your...this part will not be there at all. All you will be left with will be sin ωτ and therefore, what us going to happen is you will have something like this that you will have somewhere here not initially, you will have something like this. It would be a phase difference. So, y(t) and u(t), I encourage you to put it in Desmos and verify yourself that this is actually the case.

And after how much time will can you be assured that this portion will become insignificant again the importance of time constant after time 5 time constants this particular term will become insignificant and you will have effect only from the sin (ωt + ϕ) term.

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Now, one more thing about the importance of gain of the system. So, what we saw is that we had the equation of this form

$$a_1\frac{dy}{dt} + a_0 y = bu$$

And then, for this particular system you had

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{(\tau s + 1)}$$

Now imagine that you have a system where you have some gain for your system and you have negligible time constant which means $\tau \approx 0$ ……very small-time constants.

So, for that case what will happen

$$\frac{\bar{y}(s)}{\bar{u}(s)} = K$$

which means what which means what

$$\bar{y}(s) = K\,\bar{u}(s)$$

which means

$$y(t) = K\,u(t)$$

What is this? This is a case of pure gain system. And what is the physical significance? The physical significance is very simple here that if you have y(t) and u(t), then if this is your input function, then your output function response will simply catch up here, for k = 1. This would be this…. for k = 2, this would be this…… for k = 0.5 and so on. There will not be any delay between the input and output. So, I can write I can draw this quickly for a ramp function for k = 1, what you can see is u(t) uses as ramp function, this is the ramp function.

Then what you can see is y(t) versus t would be simply this k = 1, when k = 2 , the slope will become double…. k = 0.5, the slope will become half. Let me draw it k = 2, k = 0.5. So, there is no lag between the forcing function and the response. And this happens when the time constant is very, very, very small … $\tau \approx 0$.

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Let us see another specific case where what you have is called a pure capacity system. So, in this case, I have

$$a_1 \frac{dy}{dt} + a_0 y = bu$$

and

$$a_0 \approx 0$$

Then, what happens

$$a_1 \frac{dy}{dt} = bu$$

and

$$a_1 s\, \bar{y}(s) = b\, \bar{u}(s)$$

that gives

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \left(\frac{b}{a_1}\right)\frac{1}{s}$$

So, this will be the functional form of your transfer function and such systems are called pure capacity systems.

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The solution method will remain the same you will write the Laplace Transform for the forcing function and take the inverse Laplace transformation, analyze the dynamics. But these particular systems of a very interesting dynamics and when you have come across the systems you need to be really careful because your intuitions regarding their usual responses will not work. So, I have the solutions in front of you. So, for a pure capacity system, when you subject a pure capacity system to an ideal step function what happens.

So, let us see you have a pure capacity system.

$$u(t) = \begin{cases} 0, & t < 0 \\ A, & t > 0 \end{cases} \quad \ldots\ldots\ldots\ldots.. (9)$$

and

$$y(t) = AKt \quad \ldots\ldots\ldots\ldots.. (10)$$

y(t) is what you want…. versus t and you have an ideal step function. So, imagine that you have done this. This is A… this is what you would like. So, looking at this particular response given by equation (10) your responses given as AKt it is a straight line. This is going to be your response completely different, because otherwise you would have expected the response to have to have settled down like this. This is does not happen, because the system is a pure capacity system, you need to be really careful.

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What about an ideal rectangular pulse function. Let us draw this y(t)…. t… this is b….. this is 0….. this is A. So, this is the input what would you expect, exponential rise and then exponential decay. But now, the system is pure gain capacity system.

$$u(t) = \begin{cases} 0, & t < 0 \\ A, & 0 < t < b \\ 0, & t > b \end{cases}$$

and

$$y(t) = \begin{cases} AKt, & 0 < t < b \\ AKb, & t > b \end{cases}$$

So, therefore, you see for time between 0 and b you have a straight-line solution Akt, so, it would be this and for t > b, b is a constant. So, therefore, it is this a completely different behavior otherwise what would you have expected you would have expected some behavior like this. But instead of decaying, you do not see the system to come down, it can it becomes constant.

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Then, a ramp function, we just saw the response for ramp function y(t)...t this is the forcing function, the response, the general response you would expect to be like this. But what is the actual response

$$y(t) = \frac{AKt^2}{2}$$

Your actual response would be like this. Again, very different.

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And finally, sine function... y(t) versus t so, the ideal sine input would be like this and then you have

$$u(t) = \begin{cases} 0, & t < 0 \\ Asin\omega t, & t > 0 \end{cases}$$

and

$$y(t) = \frac{AK}{\omega}(1 - cos\omega t)$$

And then, what is really interesting is to see this. You see here, my function is

$$f(x) = \frac{1}{c}\sin(1 - cx)$$

What my ideal forcing function is

$$g(x) = \sin\ cx$$

Now, we see the behavior, as changed here c and you can see that you have... depending upon ω, your response can, it will have some phase lag, but the period of oscillation will be the same. So, it can be same like this, it can be more or it can be less with same period of oscillation.

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So, this behavior is very different from what you would intuitively expect as the general dynamical behavior of the system. So, therefore, it is important to realize the importance of functional form of your transfer function. And for pure gain systems for pure capacity systems and otherwise, you can expect different responses of the system subject to different forcing functions.

So, we will stop here today and in today’s lecture, we took the examples of the systems which are which were first order. In the next lecture, we will continue our discussion and take a second order dynamics. Till then, goodbye.