Advanced Process Dynamics

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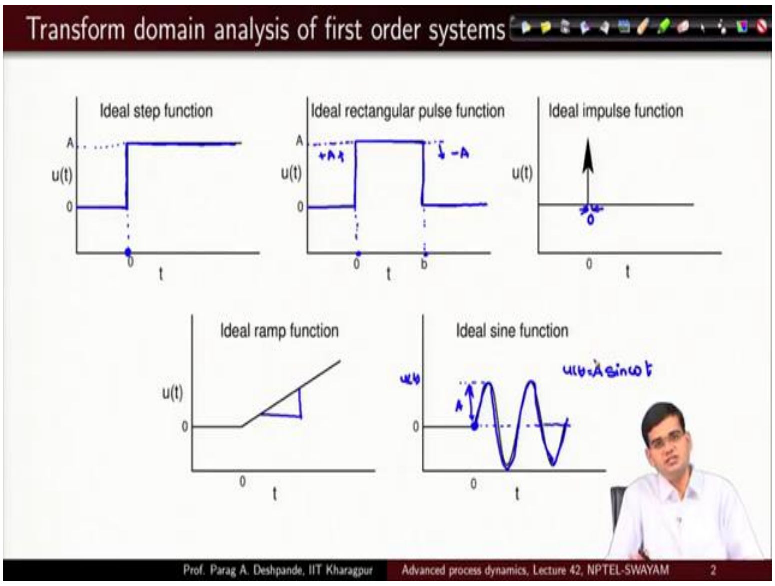
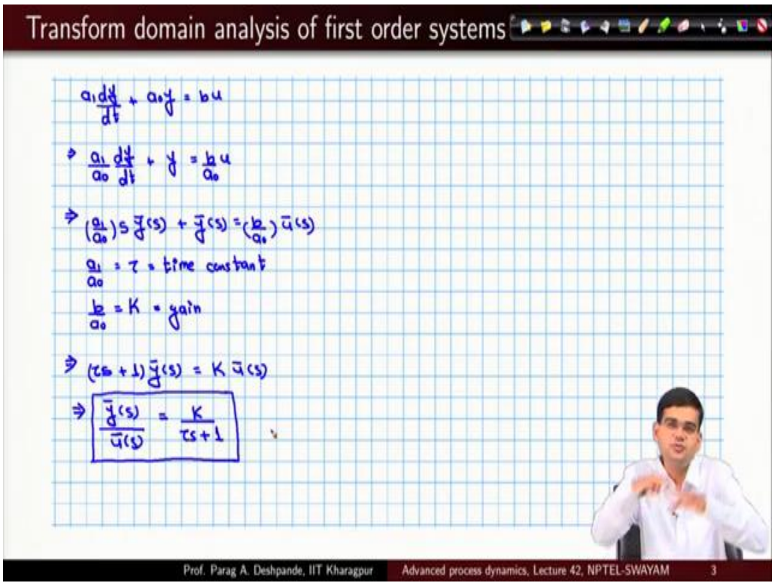
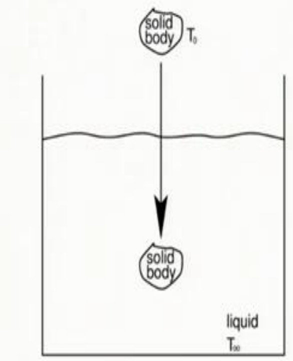
Lecture 42

Analysis of first order system subjected to ideal forcing functions







$$\frac{dT}{dt} = \frac{-hA_s}{\rho V c}(T - T_\infty) \qquad (1)$$

h = heat transfer coefficient

A_s = surface area of the solid body

ρ = density of the solid body

V = volume of the solid body

c = specific heat of the solid body

T = instantaneous temperatur[e of the] solid body

[Incropera and DeWitt, *Fundamentals of Heat and Mass Transfer*]

$$\frac{dT}{dt} = -\frac{hA_s}{\rho V c}(T - T_\infty) \quad -(1)$$

$$\Rightarrow \frac{dT}{dt} + \left(\frac{hA_s}{\rho V c}\right)T = \left(\frac{hA_s}{\rho V c}\right)T_\infty \quad -(2)$$

$$\frac{dT_s}{dt} + \left(\frac{hA_s}{\rho V c}\right)T_s = \left(\frac{hA_s}{\rho V c}\right)T_{\infty s} \quad -(3)$$

$$\Rightarrow \frac{d}{dt}(T - T_s) + \left(\frac{hA_s}{\rho V c}\right)(T - T_s) = \left(\frac{hA_s}{\rho V c}\right)(T_\infty - T_{\infty s})$$

Let $T - T_s = y$ and $T_\infty - T_{\infty s} = u$

$$\frac{dy}{dt} + \left(\frac{hA_s}{\rho V c}\right)y = \left(\frac{hA_s}{\rho V c}\right)u$$

$$\Rightarrow \left(\frac{\rho V c}{hA_s}\right)\frac{dy}{dt} + y = u$$

$K = 1; \ \tau = \frac{\rho V c}{hA_s}$

$$\Rightarrow \tau\frac{dy}{dt} + y = u$$

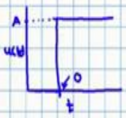
$$\Rightarrow (\tau s + 1)\bar{y}(s) = \bar{u}(s)$$

$$\Rightarrow \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{1}{\tau s + 1}$$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{\tau s + 1}$$

Step function of magnitude A

$$u(t) = \begin{cases} 0 & t < 0 \\ A & t > 0 \end{cases}$$

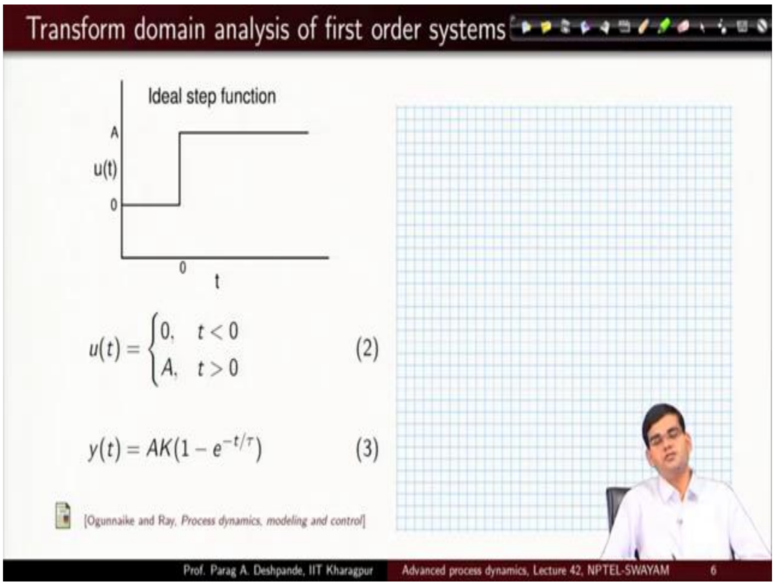


$$\bar{u}(s) = \frac{A}{s}$$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{\tau s + 1}$$

$$\Rightarrow \bar{y}(s) = \left(\frac{K}{\tau s + 1}\right)\left(\frac{A}{s}\right)$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{AK}{s(\tau s + 1)}\right\}$$



Transform domain analysis of first order systems
Ideal step function
A
u(t)
0
0
t
$u(t) = \begin{cases} 0, & t < 0 \\ A, & t > 0 \end{cases}$ (2)
$y(t) = AK(1 - e^{-t/\tau})$ (3)
[Ogunnaike and Ray, Process dynamics, modeling and control]
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Advanced process dynamics, Lecture 42, NPTEL-SWAYAM
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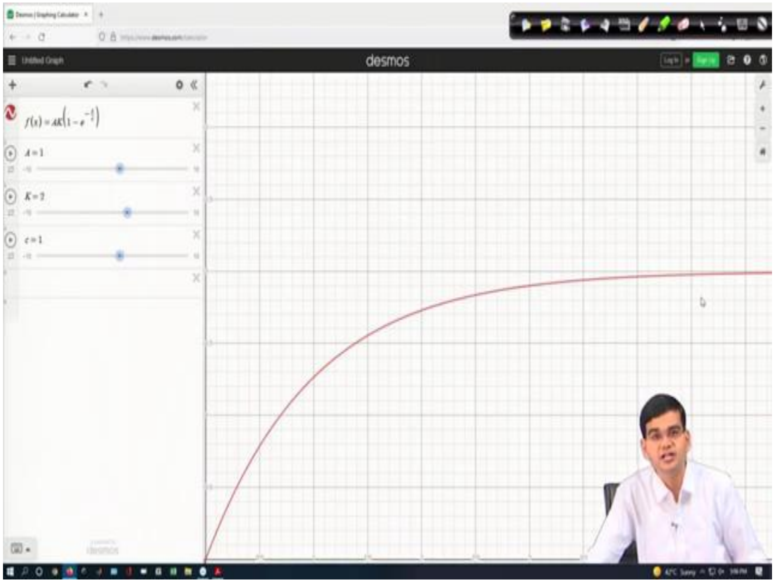
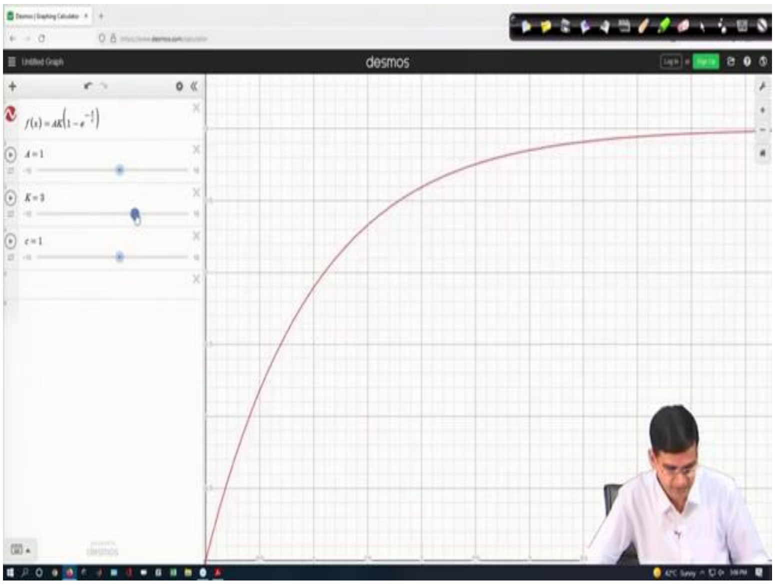
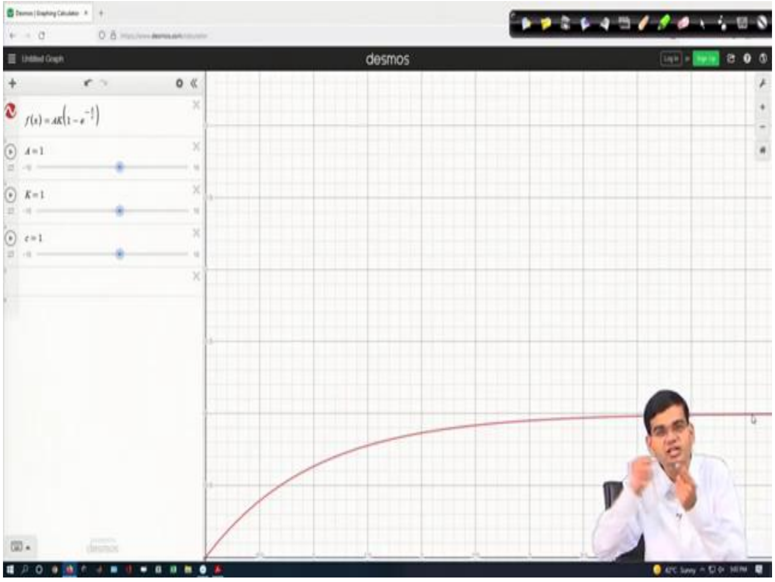
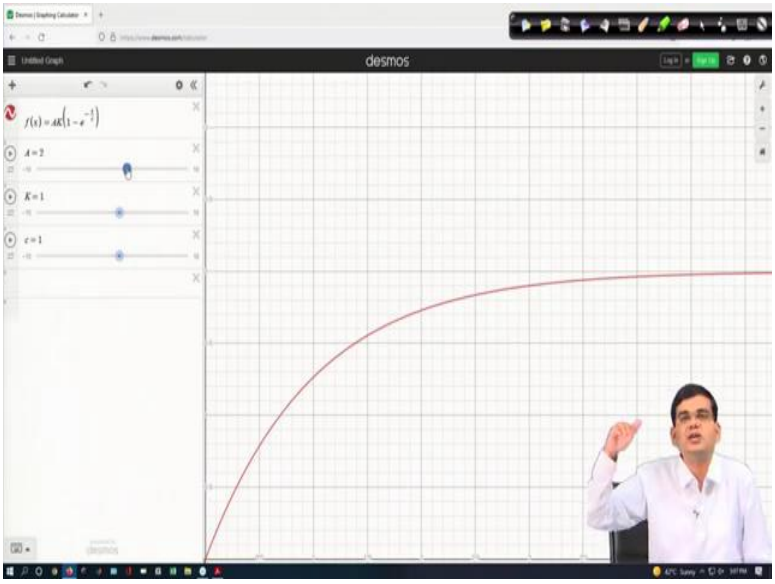


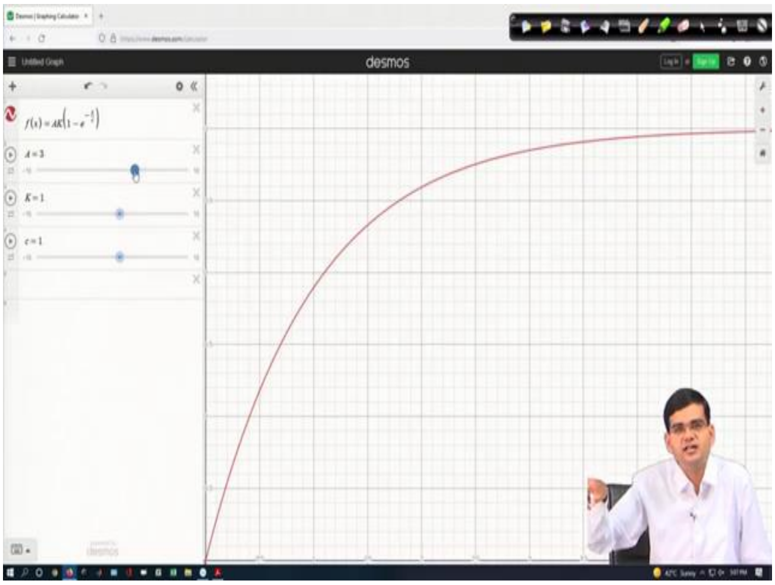
desmos
A = 1
K = 2
c = 1

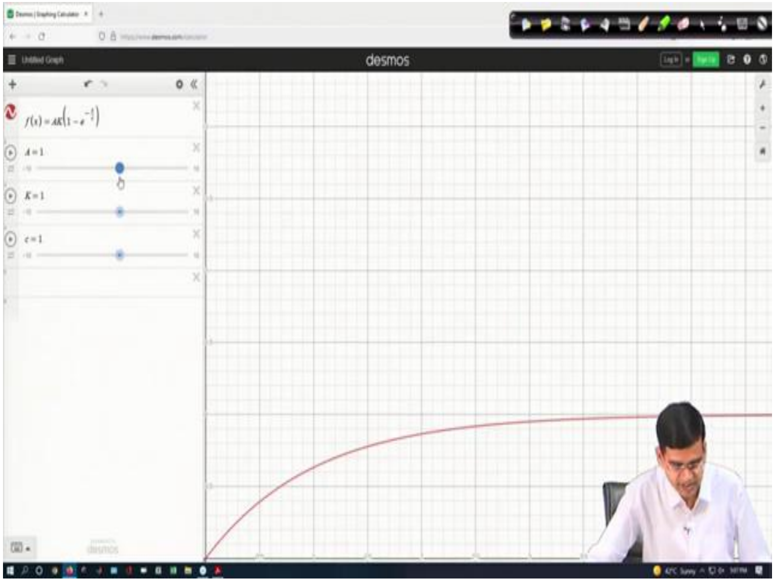


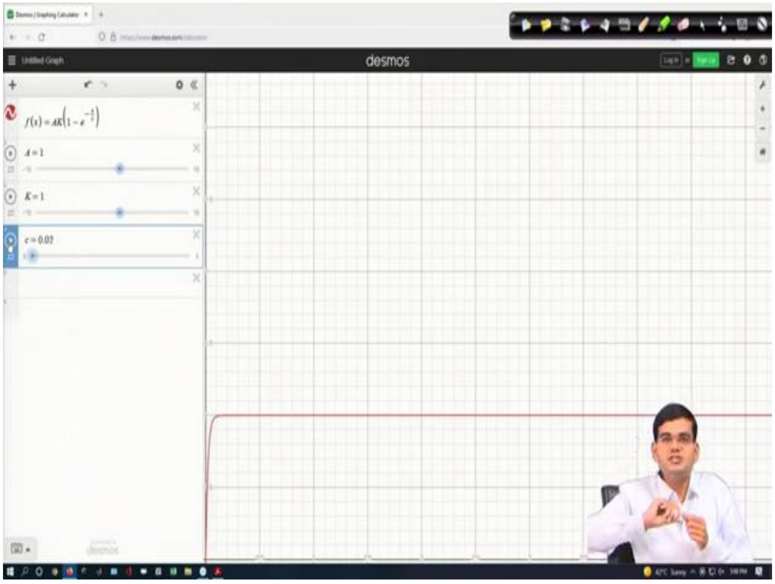
desmos
A = 1
K = 3
c = 1

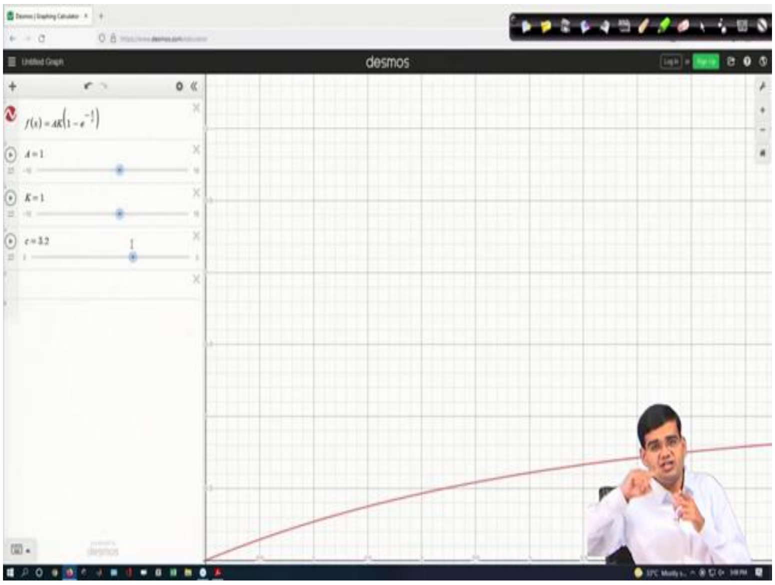

desmos
Untitled Graph
A = 1
K = 1
c = 1

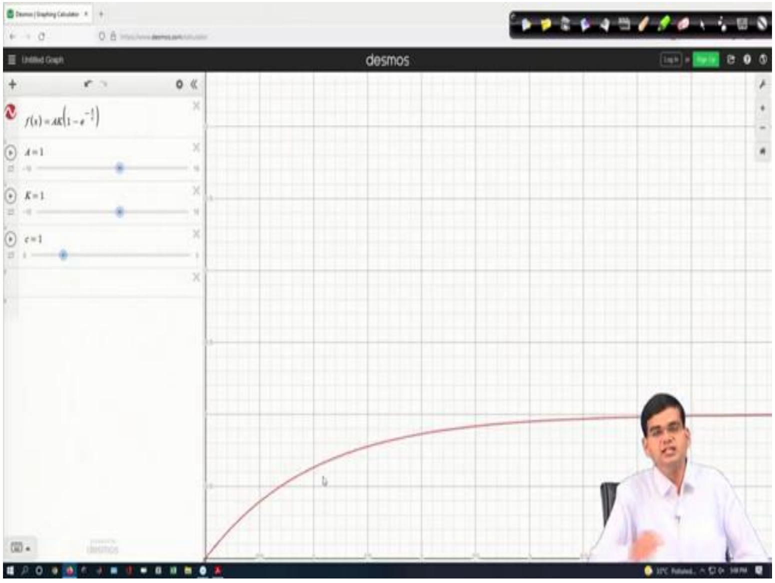

desmos
Untitled Graph
A = 2
K = 1
c = 1

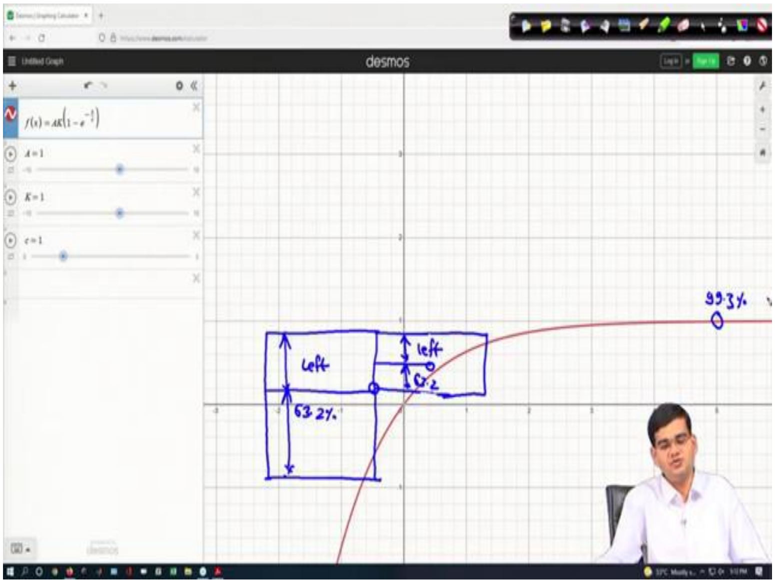

desmos
Untitled Graph
A = 3
K = 1
c = 1


Untitled Graph
desmos
$f(x) = AK\left(1 - e^{-\frac{t}{c}}\right)$
A = 1
K = 1
c = 1


Untitled Graph
desmos
$f(x) = AK\left(1 - e^{-\frac{t}{c}}\right)$
A = 1
K = 1
c = 0.02


Untitled Graph
desmos
$f(x) = AK\left(1 - e^{-\frac{t}{c}}\right)$
A = 1
K = 1
c = 3.2





In the previous lecture we invoked the idea of transfer function, the importance of transfer function is that if we know the input function or the forcing function which is applied to the system then with the help of the transfer function and the laplace transform of the input function or the forcing function we can determine the dynamics of the evolution of the system as a response to the forcing function, so what are the different types of idealized forcing functions that we can have?

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So, let us look into five different types of forcing functions which we can imagine. The plots are in front of you, the first one is the ideal step function. In the previous lecture, we took the example of the liquid level system and we said that imagine that there is a situation where suddenly in input flow stops. So, within no time you have changed your quantity which was at steady state to a new steady state quantity. Now, you can have a positive magnitude or you can have a negative magnitude, previous example the magnitude was negative but here what you can see in the plot is an ideal step function of magnitude A. So, we always start our analysis at time t = 0 or whenever this change happens you assign that particular time as 0 and before 0 my u here is in deviation variable form, so there was no deviation before the time 0 so, the quantity of the forcing function was 0 and then suddenly at t = 0 you increase your forcing function to magnitude A and then you hold it there, the forcing function continues to have that value at A, so this is the ideal step function.

Instead of holding it forever at a you may bring it back again to 0, so what is going on at time t = 0 before t = 0 and physically there is nothing called t < 0 so, whenever the disturbance occurs you call that time as 0 and that is possible as we discussed in the previous lecture. So, at time t = 0 you give a positive step of A, you hold it for time say b and then again you give A negative a step from considering the A as 0 you give -A, so that your system comes at 0 and then again you hold it, so that gives rise to a ideal rectangular pulse function. So, here you see that this is A, so this is +A that you have given here up and from here you have given -A and you come to 0, and this happens at time b, so you have two times here 0 and b, ideal rectangular pulse function.

Ideal impulse function, within no time you give a very large magnitude disturbance and then the disturbance again comes back to 0, such that the area under this disturbance is unity, so that is called the ideal impulse function. So, this width of the peak is 0, but the area is unity, so that is the ideal impulse function.

What is ideal ramp function? You had a steady state and then you start giving disturbance that at a constant rate your forcing function keeps on increasing, so therefore an ideal ramp function is characterized by this gradient and this gradient is constant, it does not change with time. Finally, an ideal sine function, so at time t = 0 you start this input function to have these oscillations and these oscillations would be characterized by the amplitudes of oscillation say this is A, so your formula would be

$$u(t) = A \sin \omega t$$

So, these are the various ways and there can be other ways as well, the popular ones have been shown here. So, subject to all of these input functions we should be in a position to know the response of the system. So, let us look into the response of the system, if the system is first order.

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So, if I have a first order system how can I, can I write a general transfer function and from the general transfer function how can I use the previous forcing functions to determine the dynamics? So, we said that our general first order equation in transform domain can be written as

$$\frac{dx}{dt} = ax + bu$$

Now what we will do is to analyze the system in transform domain I will need to rearrange this equation and put the equation in a specific form and secondly, we would assume that the governing equation is in deviation variable form, so that the life becomes easy to do transformations.

So, imagine that my system is given as

$$a_1\frac{dy}{dt} + a_0y = bu$$

so if this is the governing equation then what I do is I write this as

$$\frac{a_1}{a_0}\frac{dy}{dt} + y = \frac{b}{a_0}u$$

It is not very difficult to see that u is the forcing function here, so how would I determine the transfer function? I will take the laplace transformation and assuming that y and u are in deviation variable form I can do the laplace transformation of this equation simply as

$$\frac{a_1}{a_0}s\bar{y}(s) + \bar{y}(s) = \frac{\mathrm{b}}{a_0}\bar{u}(s)$$

Now appreciating the physical significance of these quantities which have been clubbed, I will assign some names to them. So,

$$\frac{a_1}{a_0} = \tau$$

and this is called the time constant of the system and

$$\frac{b}{a_0} = K$$

and this is called the gain of the system. So, therefore I can write this as

$$(\tau s + 1)\bar{y}(s) = K\bar{u}(s)$$

from where the general transfer function for a first order system can be written as

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{(\tau s + 1)}$$

It is important to remember that K is the gain of the system and τ is the time constant of the system, we will understand the physical significance of these two quantities a little later.

So, now the whole problem of getting the dynamical response of the system gets reduced to identification of K, identification of τ and putting the forcing function into laplace transform so that I can have

$$\bar{y}(s) = \frac{K}{\tau s + 1}\bar{u}(s)$$

and then you take the inverse laplace, we will do this before that let us take an example.

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The example which is in front of you is the example which we solved long back in this course. I believe in the first week of this course in the state space domain analysis. So, you have a body, a solid body which was at a temperature T_0 and it suddenly dropped in a fluid of temperature to T_∞. So, now I would like to do the transform analysis of the system and what can be a situation which can happen, imagine that in the previous analysis we assume that T_∞ is a constant, please refer back to that particular lecture and you would find that we very emphatically said that T_∞ is a constant in this particular analysis.

Now imagine that I have certain way of changing T_∞, so I have one heater within this particular reservoir which slowly heats the reservoir fluid itself at a constant rate with time. So, T_∞ will start rising with time, so what would be the response of the system, if I want to know this kind of answers I would need to resort to this analysis.

So, let us try to pose this problem as a transform domain problem. So, my equation is

$$\frac{dT}{dt} = -\frac{hAs}{\rho Vc}(T - T_\infty) \text{ .... } (1)$$

This is the equation which has been given to me and let me rearrange this equation as

$$\frac{dT}{dt} + \frac{hAs}{\rho Vc}\ T = \frac{hAs}{\rho Vc} T_\infty \text{ .... } (2)$$

Now can we understand the significance of T_∞ and why would T_∞ act as a forcing function, from the state space domain analysis we saw that T_∞ is the equilibrium temperature of the system. So, if I keep on changing T_∞ my system would be forced to change the temperature because the system always has a tendency to catch up to T_∞ and T_∞ itself is changing, so therefore the system will keep on evolving depending upon how T_∞ is in evolving and therefore T_∞ here is acting as a forcing function, I put it on the right-hand side, so it is equivalent to u which I had said before.

So, let me write this equation in the deviation variable form, for doing that let me write this as

$$\frac{dT_s}{dt} + \frac{hAs}{\rho Vc}\ T_s = \frac{hAs}{\rho Vc} T_{\infty s} \text{ .... } (3)$$

I will subtract equation (3) from equation (2), I get

$$\frac{d(T - T_s)}{dt} + \frac{hAs}{\rho Vc}\ (T - T_s) = \frac{hAs}{\rho Vc}(T_\infty - T_{\infty s})$$

and let us use our standard notations. Let

$$T - T_s = y$$

and

$$T_\infty - T_{\infty s} = u$$

So, I can write the above equation as

$$\frac{dy}{dt} + \frac{hAs}{\rho Vc}\ y = \frac{hAs}{\rho Vc} u$$

and to determine the transfer function I changed this equation in the previous case to one particular form, I made the coefficients of y as unity. So, from here I will write

$$\left(\frac{\rho Vc}{hAs}\right)\frac{dy}{dt} + \ y = u$$

So, if this is the case then I can identify the gain of my system simply as unity, so this is a system which has

$$K = 1$$

and

$$\frac{\rho V c}{hAs} = \tau$$

from where I can write

$$\tau \frac{dy}{dt} + y = u$$

which means

$$(\tau s + 1)\bar{y}(s) = \bar{u}(s)$$

From where I get the transfer function for my system as

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{1}{(\tau s + 1)}$$

So, from this analysis we can say that the temperature variation in the solid body dipped in a reservoir with varying temperature of the reservoir is a first order system which has unity gain and which has a time constant given as $\frac{\rho V c}{hAs} = \tau$. The transfer function would follow the first order transfer function. So, now I can determine the time variation of the temperature of the system if I know the forcing function u(t). Let us see how you would do that it is not very difficult to see that you have in this general case.

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For example, you have

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{(\tau s + 1)}$$

and for a step function of magnitude A, what is u(t),

$$u(t) = \begin{cases} 0 & t < 0 \\ A & t > 0 \end{cases}$$

So, $t > 0$, let me draw it this is u(t), this is t, so this point would be t = 0 and this would be A, and we already derived the laplace transform of a constant function A in the previous lecture. We know that

$$\bar{u}(s) = \frac{A}{s}$$

so how would you determine the response of the system? Well

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{(\tau s + 1)}$$

which means

$$\bar{y}(s) = \frac{K}{(\tau s + 1)} \frac{A}{s}$$

which means

$$y(t) = L^{-1}\left(\frac{AK}{s(\tau s + 1)}\right)$$

I leave this as an exercise for you to determine y(t) by doing this laplace transformation, I give you a hint that for all of the problems of this sort you will have to do partial fractions and then invert the laplace transformation.

So, we will spend the time during these lectures on trying to understand the dynamical behavior rather than solving these trivial mathematical steps. So, the first thing which we can have look into is this response of the system to an ideal step function, so as I said the ideal step function goes like you have 0 in the deviation variable form but $t < 0$and at $t > 0$ you maintain a constant magnitude A. The functional form of u(t) is given as

$$u(t) = \begin{cases} 0 & t < 0 \\ A & t > 0 \end{cases}$$

and I told you the recipe of getting y(t). If you do it correctly you will get

$$y(t) = AK(1 - e^{-\frac{t}{\tau}})$$

So, now let us see how does this response look like, so let us plot.

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$$f(x) = AK(1 - e^{-\frac{t}{\tau}})$$

Let us denote τ by C. So, this is the response and since everything is in the deviation variable form the disturbance starts at t = 0. We would be bothered only about the first quadrant, the other quadrants do not matter. So, what is the meaning of gain K? The meaning of gain K is that if you provide the system with unit disturbance by how much does the final disturbance or new state of the system get amplified, and how would you know the ultimate fate, you will move towards $t \to \infty$. So, therefore this if you look at far right and this is the $t \to \infty$. So, if I change gain from 1 which you can see here to 2 you see your ultimate response has gone from 1 to 2. I will make it 3, you see as time $t \to \infty$. you have gone to 3, so that is the meaning of gain K.

Then let us maintain the unit gain, so if you provide unit disturbance your output ultimate would be amplified only one times, I mean it would stick to one time of your given forcing function. Amplitude A, you see here at the far right the ultimate response is unity, because that is the forcing function amplitudinal force function you you provide and therefore it is not very difficult to see the effect of the significance of this if I make it 2 it has to ultimately catch up to 2, if I make it three it would ultimately catch up to 3 as time $t \to \infty$, at far t for right side you the system is going towards that. So, let us again consider a unit step input with unit gain, now I will see the effect the importance of time constant and time constant is always a positive quantity.

So, let me maintain it between 0 and 5 for example So, let me animate it, so I am animating this and I am trying to see what happens as the response as the time constant increases or decreases. So, for a small time constant when the time constant is small the speed of your response is large; you see here I have put a very small time constant of 0.02 and the system has immediately jumped because what is the system trying to do, you have forced the system to be at a new steady state which is unity and when your time constant is very small the system jumps and tries to come to the new steady state as quickly as possible and if I increase your system will ultimately reach that value of unity, but it is reaching there slowly.

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What happens at time constant 1, what is the meaning of time constant you will see that in your system you have an exponential term $1 - e^{-\frac{x}{t}}$. So, the time required, so here on the x axis you have the time, so this time constant 1 corresponds to a change in 63.1 percent of or 63.2% of the applied forcing function. So, if you apply a unit forcing function in one time constant it would reach 63.2% of the total applied one, in the next time constant it would again cover 63.2 of the left one. So, in the in the first, time constant everything was left you start from 0, so you start from here so this much was left, in the first time constant this much was left and then when you reached here this was 63.2 % of this entire box, first time constant. Next time constant how much was left, only this much was left in the second time constant only this much was left and therefore this is the 63.2 and this is left, this is covered and this would be again 63.2 and this would be left and so on.

So, in every time constant your system would cover the 63.2 of the left one and therefore what happens at four, time constants? At four, time constants you will reach 98.2% t of the applied response and so on. So, within five time constant you can assume not that it will happen you can assume that, so here you will see that at five, time constants you have reached 99.3; this is this is 99.3% of the applied forcing function. So, what you will see is that you have reached 99.3% and since the system catches up exponentially you will actually require infinite amount of time to reach the absolute value of the applied forcing function. So, since in five for time constants you reach 99.3% of your applied step you assume that you have reached your new steady state.

So, this was for the case when you applied unit step function, we will further take up the other cases of the idealized; the first order system subjected to idealized forcing function in the lectures to come, till then good bye.