Advanced Process Dynamics

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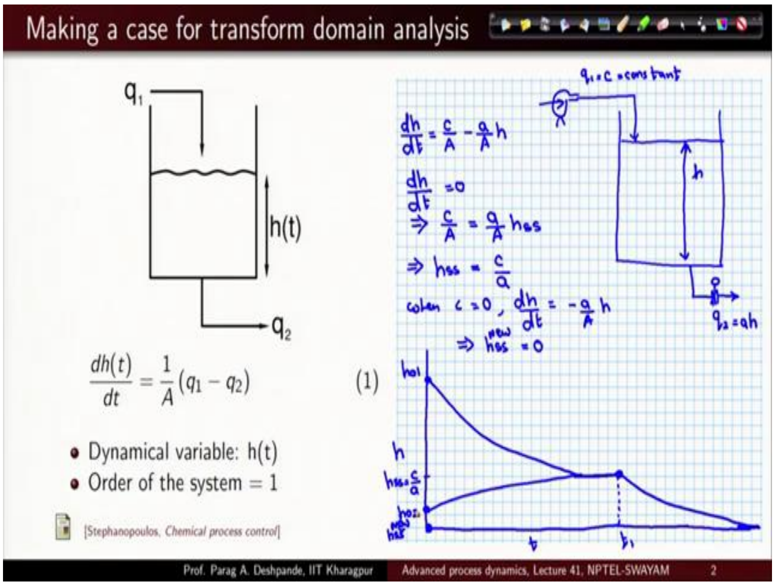
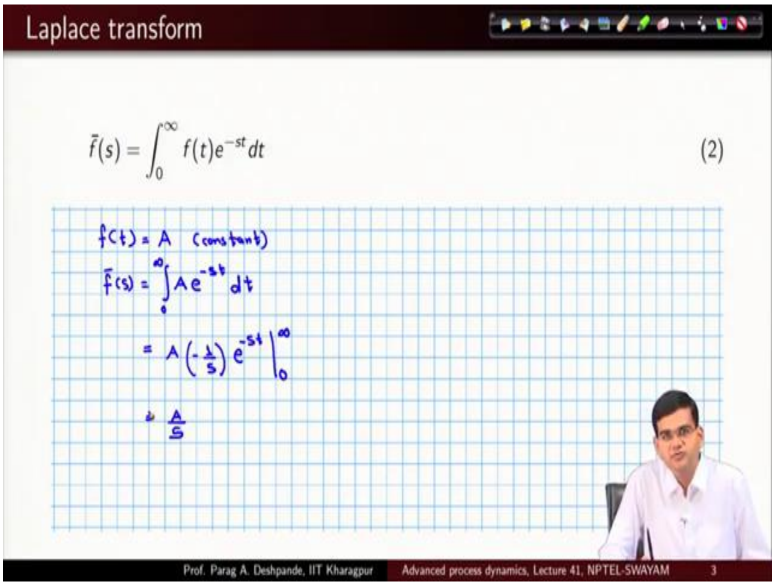
Lecture 41

Analysis of system dynamics in transform domain





Laplace transform

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2}\frac{df}{dt}(0) - s^{n-3}\frac{d^2 f}{dt^2}(0)\cdots \qquad (3)$$

$$\frac{dx}{dt} = ax + bu \qquad -(4)$$

$$\Rightarrow s\bar{x}(s) - x(0) = a\bar{x}(s) + b\bar{u}(s)$$

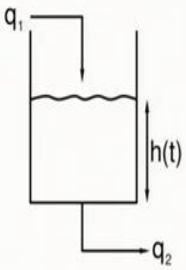
$$\Rightarrow (s-a)\bar{x}(s) - x(0) = b\bar{u}(s)$$

$$\Rightarrow (s-a)\bar{x}(s) = b\bar{u}(s) + x(0)$$

$$\Rightarrow \bar{x}(s) = \left(\frac{b}{s-a}\right)\bar{u}(s) + \frac{x(0)}{s-a}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[\left(\frac{b}{s-a}\right)\bar{u}(s) + \boxed{\frac{x(0)}{s-a}}\right]$$

The general scheme



$$\frac{dh(t)}{dt} = \frac{1}{A}(q_1 - q_2) \qquad (4)$$

- Dynamical variable: h(t)
- Order of the system = 1

[Stephanopoulos, *Chemical process control*]

$$\frac{dh}{dt} = \frac{1}{A}q_1 - \frac{1}{A}q_2 ; \; q_1 = c ; \; q_2 = ah$$

$$\Rightarrow \frac{dh}{dt} = \frac{c}{A} - \frac{a}{A}h \qquad -(5)$$

$$\frac{dh_{ss}}{dt} = \frac{c_{ss}}{A} - \frac{a}{A}h_{ss} \qquad -(6)$$

$$\frac{d}{dt}(h - h_{ss}) = \frac{1}{A}(c - c_{ss}) - \frac{a}{A}(h - h_{ss}) \qquad -(7)$$

$$\left.\begin{aligned} h - h_{ss} &= y \\ c - c_{ss} &= u \end{aligned}\right\} \text{- deviation variables}$$

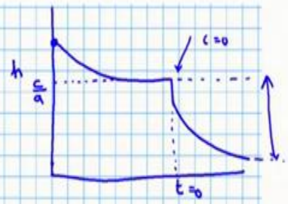
$$\frac{dy}{dt} = \frac{1}{A}u - \frac{a}{A}y$$

The general scheme

$$\frac{dy}{dt} = \frac{1}{A}u - \frac{a}{A}y$$

$$\Rightarrow \frac{dy}{dt} + \frac{a}{A}y = \frac{1}{A}u$$

$$\Rightarrow s\bar{y}(s) - \underbrace{y(0)}_{\{\text{deviation at } t=0\} = 0} + \frac{a}{A}\bar{y}(s) = \frac{1}{A}\bar{u}(s)$$



$$\Rightarrow \left(s + \frac{a}{A}\right)\bar{y}(s) = \frac{1}{A}\bar{u}(s)$$

$$\Rightarrow \boxed{\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{1/A}{s + a/A}} \leftarrow \text{transfer function}$$

$$y(t) = \mathcal{L}^{-1}\left\{\left(\frac{1/A}{s + a/A}\right)\bar{u}(s)\right\}$$

Hello and welcome back to this course on advanced process dynamics, as we enter this 9th week of instruction of this course. Let us now start analyzing the system dynamics in transform domain. So, before we start our analysis formally, we need to understand why we need such an approach, we have been doing the analysis in state space domain, we saw that a lot of dynamical features of the system can be extracted when you do the analysis in state space domain, but when exactly do you need the transform domain modeling.

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So, let us have a look into this particular aspect with the help of this problem that we have been taking which we have been studying right from the beginning, this is a liquid level problem. So, we have a tank in front of us with two streams, one inlet stream with volumetric flow rate q_1 and an outlet stream with volumetric flow rate q_2. The level of the liquid inside the tank is our dynamical variable which is denoted as h(t); the governing equation is in front of us and we will take one specific case for motivating ourselves as why we need transform domain analysis.

So, let us consider one case in which we have the tank in which the inlet stream is equipped with a pump, so q_1 is the volumetric flow rate coming through the pump and we can set this pump such that q_1 becomes a function of time now, one specific case which we can use is that

$$q_1 = c$$

c is a constant. So, this is the case which we are considering and then I have the outlet which is fitted with a valve such that q_2 is a linear function of the liquid level. So,

$$q_2 = ah$$

The level of the liquid in the tank is h. So, if this is my system description the model equation which you can see on the left-hand side can now be written as

$$\frac{dh}{dt} = \frac{c}{A} - \frac{a}{A}h$$

So, this is my governing dynamical equation and one specific quantity which I would like to know is the steady state level of the liquid in the tank. So, in transform domain analysis which we have been doing till now we used to use the term equilibrium solution, now onwards let us use the term steady state.

So, steady state solution would also be obtained by setting

$$\frac{dh}{dt} = 0$$

From where I can I get

$$\frac{c}{A} = \frac{a}{A} h_{ss}$$

In other words, my steady state level of the liquid in the tank is

$$h_{ss} = \frac{c}{a}$$

So, what does this mean if I want to plot the variation of the liquid level with time t and h then I know that I will have a quantity $\frac{c}{A}$ which would be my steady state solution which is basically my equilibrium solution. As time $t \to \infty$, I know that I will reach this condition and therefore depending upon the initial level of the liquid in the tank it can be h_{01} or h_{02}.

My level of the liquid in the tank would reduce with time if I start with h_{01}, why would this happen, well the physics is very simple that $h_{01} > \frac{c}{a}$, therefore since my outlet flow rate is dependent upon h, I have large amount of flow out and that quantity is larger than the amount of liquid which is coming in and therefore I have already developed the phase portraits of equations of this type, so therefore I know that I will have a phase line which would look like this.

Conversely $h_{02} < \frac{c}{a}$, which means there is not sufficient amount of liquid which is going out of the tank and therefore the level of liquid would rise with time, and therefore you can draw a phase line which looks like this. So, now I have this analysis and I have done this analysis using the usual state space domain analysis which we have been studying till now.

Now at some time t = t_1, so let us say this is t = t_1. My pump suddenly stops working which means that my C = 0. So, in such a situation I have now so, when C = 0, the situation when suddenly at some point of time my pump stops working, I can write

$$\frac{dh}{dt} = -\frac{a}{A} h$$

from where I can write

$$h_{ss,new} = 0$$

This is going to be my new study state corresponding to the situation that the pump has stopped working at C = 0 will be equal to zero.

So, now I have the system at a state where you had achieved asymptotically the steady state solution which was equal to $\frac{c}{a}$ but now C has become zero and from t_1 onward I should again asymptotically reach this limit and again using my state state domain analysis I know that my solution one of the phase lines would look like this.

So, this is my $h_{ss,new}$, so what did I have to do for doing this particular analysis well I had to change my equation every time I had some changes in the conditions of my system I changed my system, I changed my governing equation corresponding to that new condition and then I solve for the equation again. Therefore, you can get one initial phase line, you get you settle to one equilibrium or steady state solution then suddenly your system is forced to undergo a change and then you start following another line.

So, I now imagine that I am interested in understanding the dynamics between two steady states, so if I have this dynamically equation in front of me, I can solve it, I can draw my phase portraits, I can basically know everything which can happen to my system, but imagine a situation where all I am interested in is the dynamics of my system between two steady states in this particular case

$$h_{ss} = \frac{c}{a}$$

and

$$h_{ss,new} = 0$$

How does the system evolve between these two steady states? When you have to answer this question? it is easier, it is more convenient to do this analysis in transform domain. I must emphasize that every analysis which you can do in state space domain can be done in transform domain and conversely every analysis which you can do in transform domain can be done in state space domain as well, but you should judiciously choose a particular domain depending upon the kind of answers you want and the situation you are in. If you know that I know my steady states and I know that there exist certain forces in my system which make my system undergo a change then corresponding to those changes which have been enforced upon the system if I want to know the dynamics between two steady states then I would resort to transform domain analysis.

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If I appreciate that this is how I am going to do what is the transform technique that I would be using, I must emphasize that we are using the models which are ordinary differential equations, and for handling systems which are governed by ordinary differential equations the transform which will be using is Laplace transform. So, the Laplace transform relationship is in front of you, what you have is

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt$$

So, its important to understand that what a transform does is it changes the system to go from t domain to s domain, so when you do this integration, your original dynamical variable would be in terms of t and you would be converting it in as a function of s.

So, its not very difficult to see how this integration can be done and therefore depending upon the kind of functions which you would repetitively come across in a for your case you can make a table, this is generally what is adopted in process dynamics and control that you make a table and you refer to that standard table over and again, but to get a very elementary idea I assume that you already have a very good idea of how to do these kinds of transformations.

To get a very elementary idea we can show one very simple result that if f(t) is a constant say A, then

$$\bar{f}(s) = \int_0^\infty A\, e^{-st}dt$$

which would be equal to

$$\bar{f}(s) = \mathrm{A}\left(-\frac{1}{\mathrm{s}}\right)\mathrm{e}^{-\mathrm{st}}\Big|_0^\infty$$

So, this will give you

$$\bar{f}(s) = \frac{A}{s}$$

So, this is how you have done the transformation of a constant function A to the s domain via the Laplace transformation and your answer is A over s.

One very specific application that we would be coming across in this particular course is to use Laplace transformations for solving differential equations, obviously ordinary differential equations our dynamical systems are all governed by ODEs and therefore what we would like to do is we would like to have the Laplace transform of of our ODE.

How do you determine the Laplace transform of an n^{th} order derivative, its in front of you given as equation (3).

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2}\frac{df}{dt}(0) - s^{n-3}\frac{d^2 f}{dt^2}(0) \ldots\ldots\ldots \quad (3)$$

Again, you will see that you have gone from t domain to s domain. So, therefore again there would not be any time in your transform and therefore you will have transformed your equation to s domain.

So, one very specific example that we can see for example a first order ODE, so imagine that I have a first order ODE of a dynamical system we know that the general form is given as

$$\frac{dx}{dt} = ax + bu \ldots\ldots\ldots\ldots\ldots (4)$$

If you remember the notations x is your dynamical variable and u is your input function or forcing function, so how would I do the Laplace transform of this particular equation?

The way I would do Laplace transform of this equation is this that

$$s\bar{x}(s) - x(0) = a\bar{x}(s) + b\bar{u}(s)$$

from where I can do some rearrangements and I can write

$$(s-a)\bar{x}(s) - x(0) = b\bar{u}(s)$$

Now when I want to solve this equation (4), what basically I want to do I want x(t), but if I see the Laplace transformation what I have is $\bar{x}(s)$.

So, I can do some more rearrangements, I can write this as

$$(s-a)\bar{x}(s) = b\bar{u}(s) + x(0)$$

from where I can write

$$\bar{x}(s) = \frac{b}{s-a}\bar{u}(s) + \frac{x(0)}{s-a}$$

So, I have an expression for $\bar{x}(s)$ and the way I did the Laplace transformation which took me from t domain to s domain I can do an inverse Laplace transformation which would take me from s domain to t domain, so I will get x(t).

So, from here I can write

$$\bar{x}(s) = L^{-1}\left[\frac{b}{s-a}\bar{u}(s) + \frac{x(0)}{s-a}\right]$$

I can get an explicit expression for x(t). Since I have my equation (4) which is $\frac{dx}{dt} = ax + bu$. I would know u and from u(t) I would know the Laplace transform of u(t) that means $\bar{u}(s)$ and then I can do an inverse Laplace transformation. So, the initial condition also must be known to me and I will put that initial condition divided by s-a, take the inverse Laplace transformation and I must get the overall x(t).

So, this is the overall system which we would be following but we would not even need this especially this last part if you see we can take care of it to make it even simpler. How would we do this, we will worry about this a little later but the general scheme goes like this that you have the differential equation, you take the Laplace transformation, you rearrange the terms, you take the inverse Laplace transformation, so this is the general scheme which we would be following.

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So, if this be the case, let us try to develop these steps for the liquid level tank system that we have in front of us. So, we have

$$\frac{dh}{dt} = \frac{1}{A}q_1 - \frac{1}{A}q_2$$

and we said that

$$q_1 = c \ ; \ q_2 = ah$$

So, I can write

$$\frac{dh}{dt} = \frac{c}{A} - \frac{a}{A}h \text{ .... } (5)$$

Now, what I would like to do is, I would like to take the Laplace transformation, but as I made a mention a few moments back that there would occur a term x(t) and you would have to take inverse Laplace and so on and I would like to do some kind of a arrangement to take care of that complexity.

So, let me do one thing let me write this same expression at steady state. So, I can write this as

$$\frac{dh_{ss}}{dt} = \frac{c_{ss}}{A} - \frac{a}{A}h_{ss} \text{ .... } (6)$$

So, from equation (5) and (6), I can write

$$\frac{d(h-h_{ss})}{dt} = \frac{1}{A}(c - c_{ss}) - \frac{a}{A}(h - h_{ss}) \text{............. } (7)$$

What have I done? I have written my model equation. I have then written my model equation at steady state and I have subtracted one from the other. So, now when I declare

$$h - h_{ss} = y$$

$$c - c_{ss} = u$$

Then, I can write equation (7) as

$$\frac{dy}{dt} = \frac{1}{A}u - \frac{a}{A}y$$

Now, I have changed my original equation, the original equation was

$$\frac{dh}{dt} = \frac{c}{A} - \frac{a}{A}h$$

and I have converted it a little by changing the variable to deviation variables, so you can refer these two as deviation variables.

When this happens what can be done, now I will try to solve the equation in deviation variable form rather than the absolute variables and then I will see that there is one specific advantage that I get. So, my equation is

$$\frac{dy}{dt} = \frac{1}{A}u - \frac{a}{A}y$$

So, if I take this Laplace transformation, before that let me do this rearrangement

$$\frac{dy}{dt} + \frac{a}{A}y = \frac{1}{A}u$$

and now I will do a Laplace transformation, what I will do is I will write

$$s\bar{y}(s) - y(0) + \frac{a}{A}\bar{y}(s) = \frac{1}{A}\bar{u}(s)$$

Now the advantage which I had by taking deviation variable would be very apparent here that I have this quantity y(0) which means that the deviation at t = 0, so what is t = 0? Well, I have a linear system, so I can always translate my time or basically start my time at any point of time and declare the time as t = 0.

So, when would I do that, in my previous analysis you saw that you had this you have h, there was a reduction in liquid level such that you reached here $\frac{c}{a}$ and then at some time t you started another dynamics because here C = 0. So, what I will do is since I want to know the dynamics between these two extremities. I will declare this as t = 0, I am not concerned about anything which was going on as steady state before that time. So, at time t = 0 what is the value of the deviation variable this is simply zero, there was, so the system was at steady state, so there was no deviation from that steady state at time t = 0. So, therefore I can write this as

$$\left(s + \frac{a}{A}\right)\bar{y}(s) = \frac{1}{A}\bar{u}(s)$$

and this means that

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{\left(\frac{1}{A}\right)}{s + \frac{a}{A}}$$

So, I have written this quantity in a specific form which is $\frac{\bar{y}(s)}{\bar{u}(s)}$, this is called the transfer function of the system, what is the system, the liquid level in a tank.

So, this is the transfer function and how would I get the variation of liquid level from one steady state to the next steady state subjected to u which means $c - c_{ss} = u$. So, this is the forcing function for your system, so u(t) is the forcing function which acts on your system and subject to that forcing function now if you would like to know the response of the system then what will happen y(t) can simply be written as

$$y(t) = L^{-1}\left(\frac{\left(\frac{1}{A}\right)}{s + \frac{a}{A}}\bar{u}(s)\right)$$

When I solve this, I will simply get variation of y with time, but what is so special in this, the special thing which which is there in this particular case is that you will get y which means the deviation from the steady state from the moment the forcing function was applied which means that your system was at steady state and then there was a forcing function which was applied and the system reaches a new steady state what is the dynamical behavior between these two steady states?

The only thing which is required is I assume its not very difficult for you to determine the inverse Laplace transform, if you can do Laplace transformation you can do inverse Laplace transformation as well, the only thing which is left is to know the functional form of u(t). If you know u(t) you can take Laplace transformation and put it here, take the inverse Laplace you will get y(t). There are several functional forms which are called ideal forcing functions which may be applied, the example that you had the pump which was giving input at a steady state and then suddenly the flow rate stopped C = 0 is an example of one specific type of a forcing function. We will know the details of these specific forcing functions called ideal forcing functions in the next lecture. Goodbye.