Advanced System Dynamics
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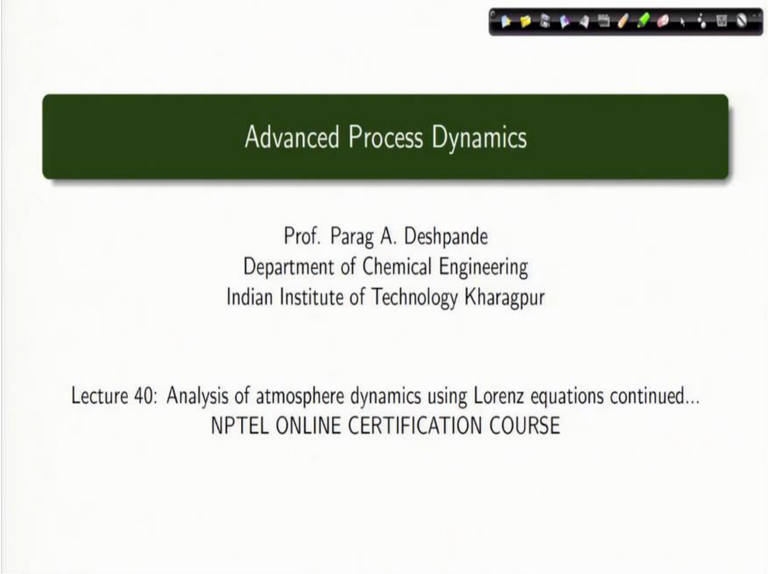
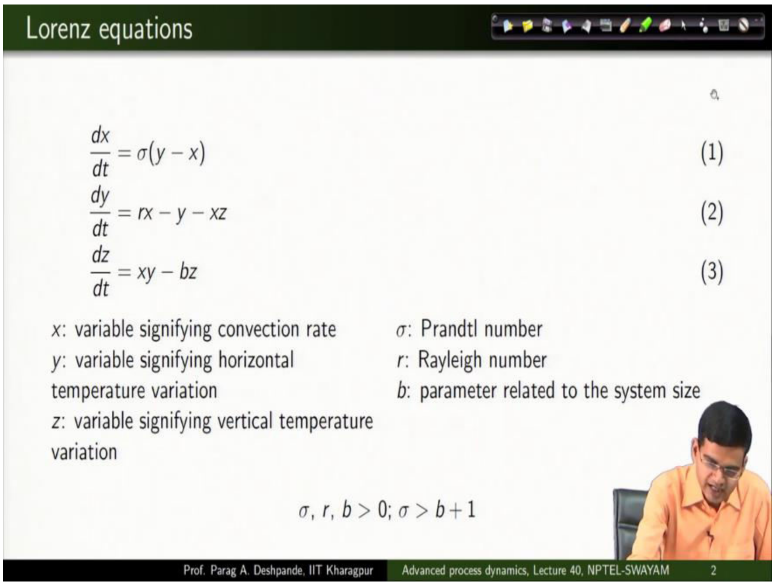
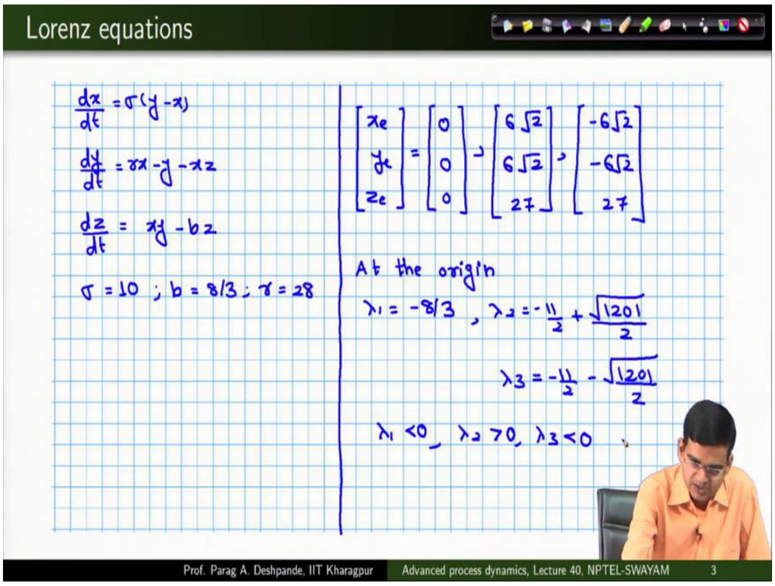
Lecture 40
Analysis of Atmosphere Dynamics Using Lorenz Equations Continued…






Lorenz equations
dx/dt = σ(y - x)
dy/dt = rx - y - xz
dz/dt = xy - bz
σ = 10 ; b = 8/3 ; r = 28
[xe, ye, ze] = [0, 0, 0], [6√2, 6√2, 27], [-6√2, -6√2, 27]
At the origin
λ1 = -8/3 , λ2 = -11/2 + √1201/2
λ3 = -11/2 - √1201/2
λ1 < 0, λ2 > 0, λ3 < 0
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Advanced process dynamics, Lecture 40, NPTEL-SWAYAM
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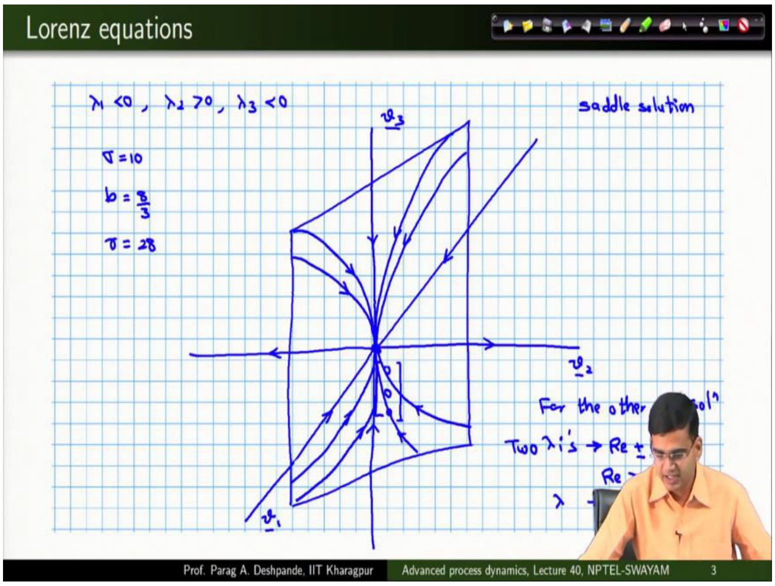
Lorenz equations
λ1 < 0 , λ2 > 0 , λ3 < 0
saddle solution
σ = 10
b = 8/3
r = 28
v3
v2
v1
For the other
Two λ's → Re
Re
λ
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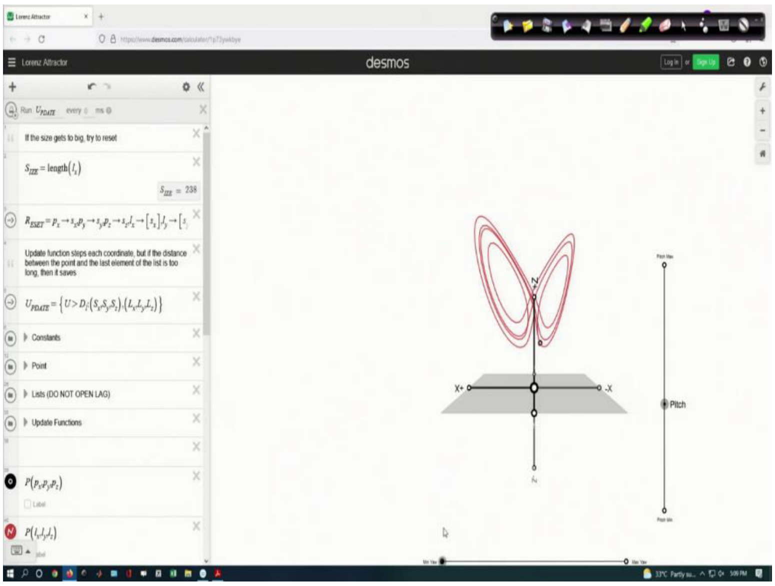
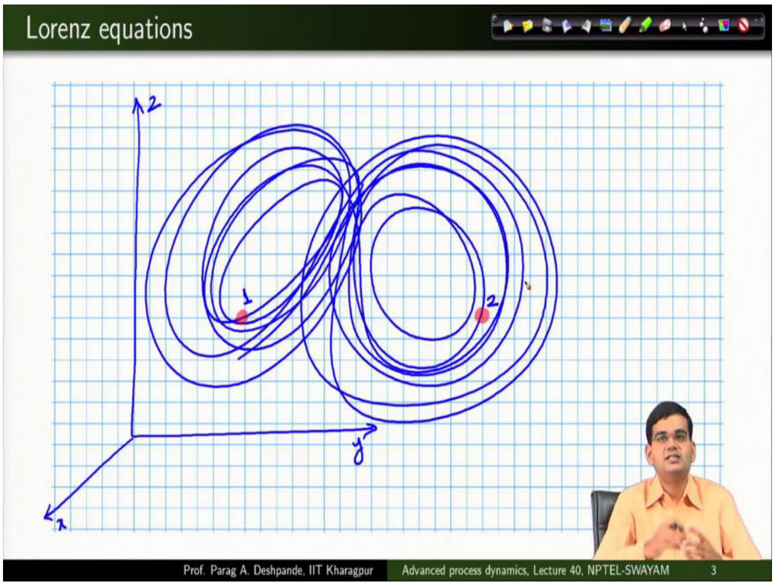

Lorenz Attractor
desmos
If the size gets to big, try to reset
Update function steps each coordinate, but if the distance between the point and the last element of the list is too long, then it saves
Constants
Point
Lists (DO NOT OPEN LAG)
Update Functions
Pitch


Lorenz equations
z
1
2
y
x
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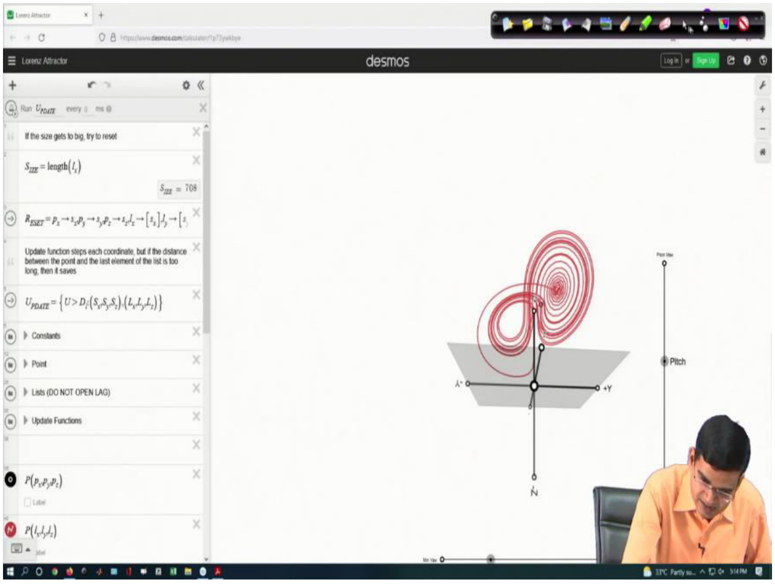
Lorenz Attractor
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If the size gets to big, try to reset
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Constants
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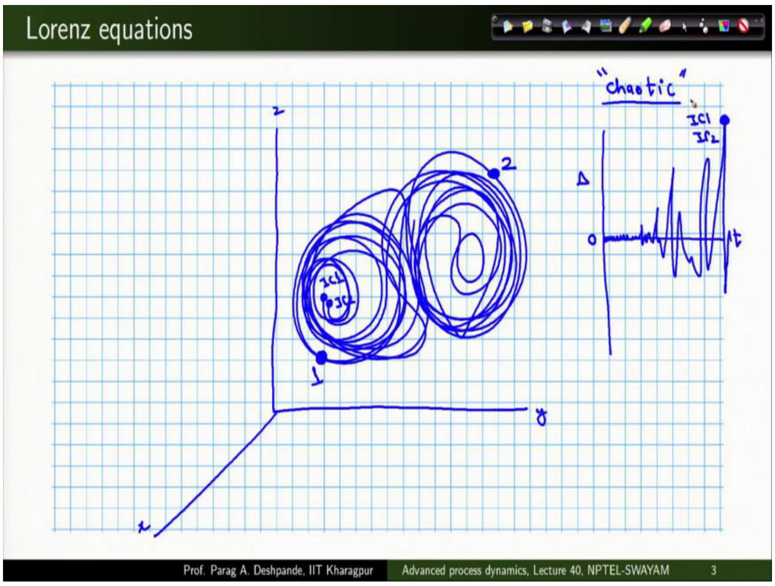
Lorenz equations
"chaotic"
IC1
IC2
Δ
0
t
z
2
1
y
x
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So, we were analysing arguably the most dynamical system that we have around us. And I said that atmosphere is that particular dynamical system. We were trying to model the dynamics using a very, very, very simplified version, which comes from momentum and energy balance equations, considering a two-dimensional system as our atmosphere consisting of a single fluid, which is heated from the bottom, cooled from the top.

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And that resulted in a simple third order equation which is in front of you,

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - rz$$

$$\frac{dz}{dt} = xy - bz$$

We do not have the variables from the atmosphere directly, but the variables, which in one way or the other indicate the variables which are present in that atmosphere. So, x was the variable signifying the convection rate, y was the variable signifying the horizontal temperature variation, and z was the variable signifying the vertical temperature variation.

The three parameters which appeared in the system were Prandtl number, Rayleigh number and a parameter b, which indicated the size of the system. The constraints, which you must impose on the system are given here. We solved for a general system of equations, which is in front of you in the previous lecture. We linearized it, tried to determine the strategy for the nature of this equilibrium solutions. And then today, we will take one specific set of parameters σ r and b which was used directly by Prandtl.

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So, let us see let us write first, the equations that we have in front first, the equations are

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - rz$$

$$\frac{dz}{dt} = xy - bz$$

These are the questions. And then what Lorenz did was he chose a specific set of parameters. He considered σ as 10, b as 8 by 3 and r as 28. These were the set of parameters, which were identified by him.

Now, if that is the case, then by substituting these values in the equation and following the same procedure which we did in the previous lecture, we get three equilibrium solutions. In fact, we saw that for r < 1, you have only one solution, which is the origin. For greater than 1, you have three solutions. And r here is 28, so, therefore we would expect three solutions. So, x equilibrium, y equilibrium, z equilibrium for the current set of parameters will become [0 0 0] a solution.

What is the other, what are the other two solutions? They are in front of me. $6\sqrt{2}$, x_e y_e were always equal, so, I will again have $6\sqrt{2}$. And z_e was r - 1, so, therefore I have 27, correct. And if I have this solution, it is not very difficult to write the other solution. This will become $\begin{bmatrix} -6\sqrt{2} \\ -6\sqrt{2} \\ 27 \end{bmatrix}$. These are other equilibrium solutions.

Where did we use these equilibrium solutions? We used these equilibrium solutions in the Jacobian matrix. We determine the eigen values of the Jacobian matrix at these locations. You know the procedure, I am writing the final eigen values what you should get for this particular case, so, for at the origin. At the origin what is going to happen? λ_1 is going to be equal to -8 /3, λ_2 is going to be equal to $\frac{-11}{2} + \frac{\sqrt{1201}}{2}$

And λ_3 is going to be equal to $\frac{-11}{2} - \frac{\sqrt{1201}}{2}$. So, now, as I made a mention before in the previous lecture towards the end, we saw that the origin would be a sink solution subject to the condition that the value r is between 0 and 1.

For the other cases we could not guarantee it. You can in fact, see here that $\lambda_1 < 0$, $\lambda_2 > 0$ and $\lambda_3 < 0$. You have a saddle solution.

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So, can I draw the phase portrait for this particular system, which confirms to the condition? So, let me write here, $\lambda_1 < 0$, $\lambda_2 > 0$ and $\lambda_3 < 0$. So, let me draw the qualitative behaviour of the phase portrait. So, I have the three axes. The three axes now will not be the x, y, z coordinate. These would be the eigen vectors because eigen vectors are the solutions are along the eigen vectors.

So, I can try this as a V_1, the eigen corresponding to λ_1, I can write to V_2, the eigen vector corresponding to V_2, I can write V_3 the eigen vector corresponding to V_3. Now I see that $\lambda_1 < 0$. So, therefore the entire line around V_1 is solution, and it is going to be a sink, it is going to be a stable solution. So, therefore, let me make this. You should imagine that the line is going inside the plane. [0 0 0] is a solution.

And now I am going to draw the arrows since I have the eigen value, which is negative. You will draw the arrows like this. Settled. Now I have λ_3, which is also less than 0, so the solutions along the vector V_3 would also be stable. So, let me draw this. This is going to be coming in, this is going to be coming in, but the solutions along vector V_2 are going, are going to be unstable because λ_2 is greater than 0. And therefore, I can draw the arrows like this.

Now, if you remember what we would do after this, it is not very difficult to see that we would try to identify a plane, which would correspond to the subspace, which would exhibit absolute stability. See, I have a saddle solution, so therefore I would have stable as well as unstable natures of the solutions. So, since I have three-dimensional plot, I have a third order system and two of the axes are stable axes.

I can identify a stable subspace. That would be the subspace, which would be going through the eigen vector V_1, the eigen vector V_3 and the [0 0 0]. So, let us imagine that that plane looks something like this. So, let us imagine that that plane looks like this. So, on this plane, every single point, every single line, phase line would pass through [0 0 0] and would exhibit sink solution. So, let me try to draw various phase lines. And then what would be the direction of the arrows?

Not very difficult to see that the arrows would be pointing inward because I am having a stable subspace. So, therefore I have a saddle solution at [0 0 0]. At [0 0 0] the origin is no more in this case, sink solution, but for this set of parameters, you have a saddle solution, saddle solution. Let us recapitulate what we did. We wrote the equations, we wrote specific set of parameters, the set of parameters that we have, let us write here down.

$\sigma = 10$, $b = 8 / 3$, and $r = 28$. We wrote σ being our equation, linearized, determined the Jacobian at [0 0 0] determined the eigen values. Eigen values came to be such that two eigen values were positive, were negative, one was positive, and therefore we had a saddle solution with a stable subspace and a straight line, which is unstable. What about the same analysis for the other two equilibrium solutions?

It would turn out, I am not writing, giving those details, it would turn out that for the other two solutions, for other two solutions, what will happen is that λ_i's, the eigen values, λ_i's, the eigen values would be such that you, they would be of the form real plus i times imaginary, which means that the eigen values would be complex numbers, such that real numbers would be greater than 0, two of them.

Obviously, if you have three eigen values, only two of them can be complex numbers. So, two eigen values would be such that they would be r_e, let me write plus or minus, i m such that the real part would be greater than 0. The real part would be greater than 0 and one λ would be negative. One λ would be negative, the other two eigen values would be complex numbers with positive real part.

So, do we remember what kind of behaviour does an imaginary part to an eigen value Imparts? It imparts a oscillatory or it imparts behaviour such that there are repetitions in your system, which means there are circulations. Specifically, if you have the eigen value of the form $r_e \pm im$, with $r_e > 0$, then you have a spiral source solution. For $r_e < 0$, you have a spiral sink solution. This is what we saw long back during our analysis of planar systems.

So, therefore now I will have my system such that one of the eigen values is negative and two eigen values will have these spirals. These are, this particular behaviour with these two spirals is called the famous Lorenz attractor. So, let us see how does the system evolve? How do the phase lines look like and what is the behaviour of the system? So, because Lorenz attractors are so famous, these have already been encoded in Desmos calculator.

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So, let us simply run the code to see the behaviour of these. So, let me run this. So, the point which you can see is the evolution in time. So, there are two spirals, which you can already see. The point is going around. Sometimes it spirals around in the same spiral, sometimes it goes to the other one. And what basically is going on is that you have diverging or spiralling our behaviour.

So, you have two of these collections of curves. So, I can change for example, the orientation. And I can change the orientation like this. You see, there are two spirals here. These correspond to, and these two would correspond to two different states. When two points on the two curves would correspond to two states of your system. So, I will let this system evolve and we will come back to this animation again.

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But let us see what we can understand from here. So, what I have is a behaviour, which looks quite a lot like this. And if I take two points on this particular phase portrait, say point 1, which is here, and point 2, which is here, and this is in three dimensions. So, what do these two points, these two points have been sampled in time.

So, you can find out a path which joins these two points. So, therefore these two points exist in, as two states of the system in time. So, let me draw the axis as well. These are in three dimensions. So, x, y, z. So, at point 1, there would be some value of x, y, z, at point 2, there would be some other value of x, y, z.

So, what does this mean? Let us say I am at some point of time, I am at 1, point number 1, which means I have some value of temperature variation along x axis, along the vertical height, temperature gradient, along the horizontal plane, the velocity field and so on, some point of time. And then the system evolves to point 2.

Say, point 1 is in the morning, point 2 is in the evening, two very different states. And this is quite usual observation in weather. You may have two very different weathers at two different points of times. In fact, you may go from one point in one of these cluster of lines to another point in the cluster of lines directly by a single curve. Many times the curves, the solutions circle around one, and then they move on to another other.

But when you cross over from one region to the other region, within that short period of time, your state of the system is changing drastically, which physically means that when you go from point 1 to point 2, and if there is a simple shortest curve which exists between point number, point 1 and point 2, then what this physically means that within a short period of time, you can see a drastic change in that atmosphere, in the condition of that atmosphere.

Or you can see a drastic change in the weather. You see bright sunlight, and then in very short interval of time, you see it starts raining and the rain again goes away, and the weather changes. Precisely, given by this. Let us go back and see what has happened to our evolution.

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So, the system is evolving. The system is evolving and there are a lot of spirals. And what you basically would see is that there are two, you can see say that there are two attractors and then the points are moving away from them because the eigen values are complex numbers. So, therefore there are spiralling behaviours. The spiralling behaviour is because the eigen values are complex numbers.

They are spiralling out. They are not converging at this point of, right at this point of time, you see here, what is going on in your evolution here? Your points are going away. So, it is called an attractor because there are two points, but the solutions are moving out. So, when the solutions are moving out, you have two very, very different states of your system.

And what you basically see, see, you can see the system to spiralling out, very different behaviour, and then what you see is something which is also in correspondence to it, correspondence to the physical observation that the state of the atmosphere, in other words, the weather can change over a short period of time. But this is not what is the most interesting observation which came out of the theory of Lorenz.

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What Lorenz told was that this behaviour, which you see here. So, I can, again, draw this x, y, z, the behaviour which you see here is chaotic, which means what? Let us say, I start from this point. This is my initial condition. If this is my initial condition, I start spiralling around and get these two parts of my spirals. Let us say I get this. So, what would you expect? You would expect that this is an average behaviour of my system. This is the average behaviour.

So, now this is my initial condition 1. I had some velocity distribution, velocity field, some temperature field. And then I am, why am I doing this? I am doing this because I want to make predictions about weather. What is the meaning of, what is the purpose of doing all of this business? Because you want to do prediction of your weather. You want to do weather forecast. So, I have supplied the system with some initial condition, and now what I do is I make a very, very, very small change in my initial condition. This is my initial condition 2.

And then from initial condition 2, I start determining the spirals again. What Lorenz observed was that if you do this, when you do this, and if I can draw the departure, so, this is say time, so, one condition is IC_1, another condition is IC_2, and this is the departure function, the difference between the path followed by the two spirals, which means the difference between the condition of the atmosphere, or in other words, what is the weather, at some point of time in future, what is going to happen?

So, what we will see here is that since the initial conditions are very close, the delta would be close to 0. But after some time, you will see that there are lot of fluctuations which means that my system is now evolving in a completely different direction than what I would otherwise have expected.

Let me put it in different words. If I determine my departure, variable delta, and its value is 0, then it simply means that I start with two different initial conditions, which defer by us a very small value. Then, if I am expecting rain in the evening, it would rain in the evening, whether I use initial condition one or initial condition 2.

That does not happen in reality, because again, I can use two points, one corresponding to spiral evolutions, following initial condition one, which ends up here, point 1. And then if you see the departure variable on the right-hand side, there is a large difference. At this point, there is a large difference here. And therefore, I may end up following initial condition 2 here, point number 2.

And you see x, y, z for point number 1 and point number 2 are very different, which means that the velocity field at point 2 will be very different compared velocity field at point 1. The temperature differences would be very different at point number 2, compared to point number 1. In other words, the weather at point number 2 would be very different from the weather, which you get from point number 1.

And why does this happen? This happens because your system is chaotic. And there was in fact, a huge turmoil in the scientific community when Lorenz introduced this idea, that even if you introduce a very small change in the initial condition of your system, then also it is inherently the nature of the system of equation that model your system, that you may get very different behaviours.

Now, what are the repercussions of this? The repercussion is that, that when you say that, you make a statement about whether in the last, in the next few weeks, for example, that it is going to be rainy all these days. Let us say rain does not happen it is all dry. Why would this happen? Well, you may say that the models are very simple, and therefore, in reality, the atmosphere is incredibly complex, and therefore there is a problem with the model.

This can be one logic, and it is quite acceptable logic. So, let us say you make a complex model, then you are, and then also suppose your predictions fail. Then what logic do you give? The argument that you give is that the mathematical models are incredibly complex, and therefore you are not able to give a deterministic future using these set of models. But that is not what the point of Lorenz was.

Lorenz said that see these models are very, very simple, and it is not, the failure of weather forecast is not because of essentially the failure of models. It is because of the very nature of the models because the equations are chaotic and chaotic by definition means that they are highly sensitive to the initial conditions. So, therefore, even if you give very small, if there occurs a very small difference in the initial conditions, you may forecast a very different condition.

Now, this also means something which is very interesting. So, you start with an initial condition for your, for your model. You make a prediction, and if the prediction is correct, then corresponding to those set of say, velocity fields and temperature gradients you would expect some future weather in your region. But now different initial condition means, a small difference in the initial condition means, say a small difference in the wind pattern and this is what leads to the famous concept of Butterfly Effect.

A butterfly may start flapping its wings, well in the original discussion by Lorenz, he made a mention about a seagull, but today we famously call it as the butterfly effect. So, it is said that a butterfly may start flapping its wings in Brazil and according to the argument put forward by Lorenz, it may bring about a tornado in Texas or Florida, some Southern states in the U.S.

And that seems to be something very strange, but the argument of Lorenz was well, that in fact is the case because the system is chaotic and therefore, if you change your initial conditions, even by a small amount, the fate can be very different. And therefore, it is not so, much for the nice weather conditions that in India, but in Western, in European, as well as North American countries, where the weather conditions are much harsh, the dependence of day-to-day operations of the population there depends a lot upon the weather forecast.

They look on their mobile what is the weather going to be like, and then they decide upon a lot of their day-to-day activities. And therefore, weather forecast is very important for them. And therefore, making accurate weather forecast is also important for them, but it is difficult.

It is difficult because of the chaotic nature of the equations. And therefore, what is important is that you must provide the system with good initial condition.

And how would, how is the initial condition provided? Well, you need to give velocity field, you need to give temperature fields and so on. And for that you need really good instrumentation. And today, when the instrumentation is improving, you have much more number of sampling points all around the globe. You can now make better future predictions compared to what you could do in the past.

And therefore, you might be observing that the weather forecast these days are much accurate compared to the weather forecast, which used to happen, which used to be, which used to be, which used to terribly fail in the past. And all of this happens because you have an extremely dynamical system, which is governed by a chaotic set of equations. And the dynamics prediction of future dynamics is highly sensitive to the initial conditions.

So, we would stop here. We discussed in really great details, the analysis of linear non-linear continuous time, as well as discreet time domain systems in state space domain. From the next week onwards, I will switch gears and we will start analysing the system in transform domain. Till then, goodbye.