Advanced Process Dynamics

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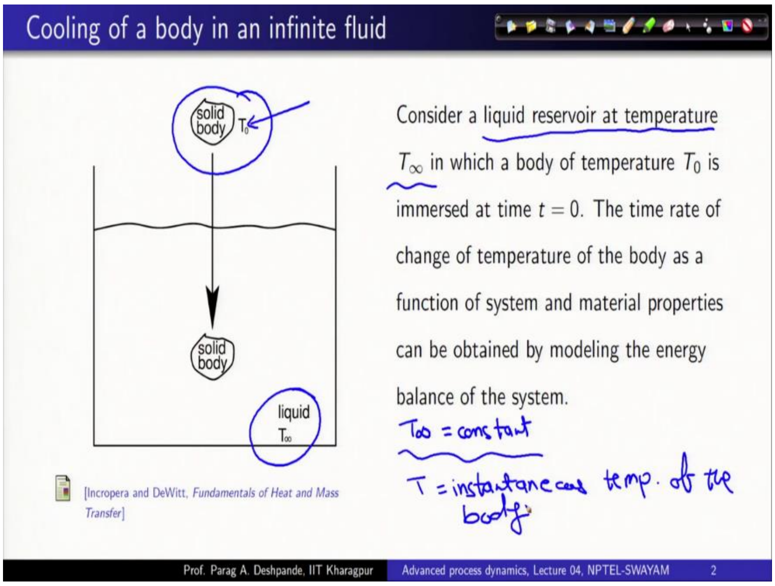
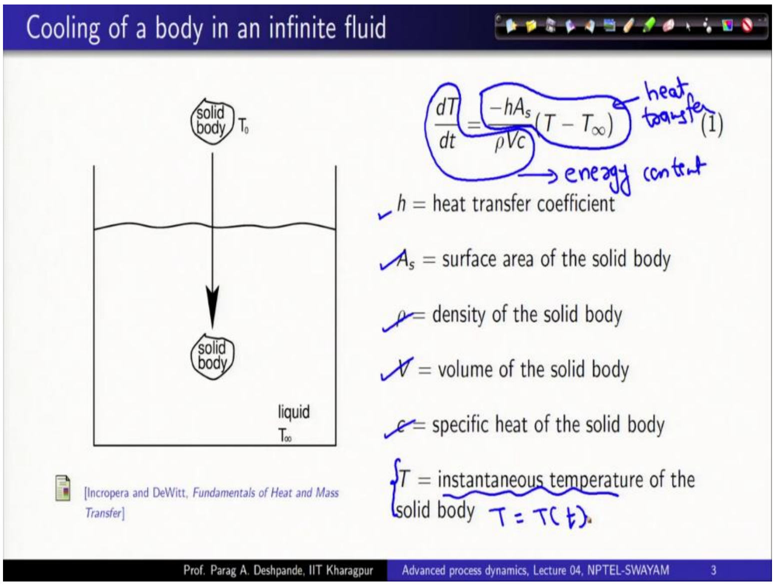
Lecture 04

Lumped Parameter Analysis of Cooling of a Body





Cooling of a body in an infinite fluid

- What is/are the equilibrium solution(s) of the system?
- Solve the model equation analytically to determine the time evolution of the system.
- Develop the phase portrait for the system.

$$\frac{dT}{dt} = \frac{-hA_s}{\rho Vc}(T - T_\infty) \qquad (1)$$

$$\frac{dT}{dt} = -\frac{hA_s}{\rho Vc}(T - T_\infty) \quad -(1)$$

$h, A_s, \rho, V, c \rightarrow$ constants

$$\frac{hA_s}{\rho Vc} = a \text{ (say)} \quad -(2)$$

$$\frac{dT}{dt} = -a(T - T_\infty) \quad -(3)$$

$$\frac{dx}{dt} = f(x) = 0 \leftarrow x_e$$

$$-a(T - T_\infty) = 0$$

$$\Rightarrow T_e = T_\infty$$

T_0 = initial temp

T_∞ = constant

$$\boxed{T_e = T_\infty}$$

Cooling of a body in an infinite fluid

$$\frac{dT}{dt} = -a(T - T_\infty) \quad -(1)$$

$$T(0) = T_0$$

$$T^* = T - T_\infty \text{ (say)} \quad -(2)$$

Since T_∞ = constant

$$dT^* = dT \quad -(3)$$

$$\frac{dT^*}{dt} = -aT^* \quad -(4)$$

$$\frac{dx}{dt} = ax \quad -(5)$$

$$T^* = Ce^{-at} \quad -(6)$$

$$T - T_\infty = Ce^{-at} \quad -(7)$$

At $t = 0$, $T = T_0$

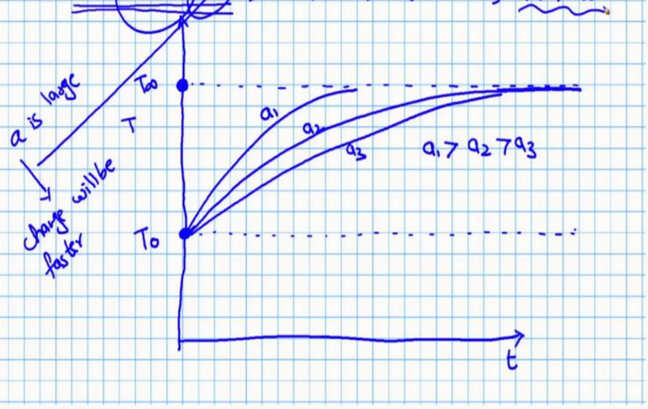
$$T_0 - T_\infty = C$$

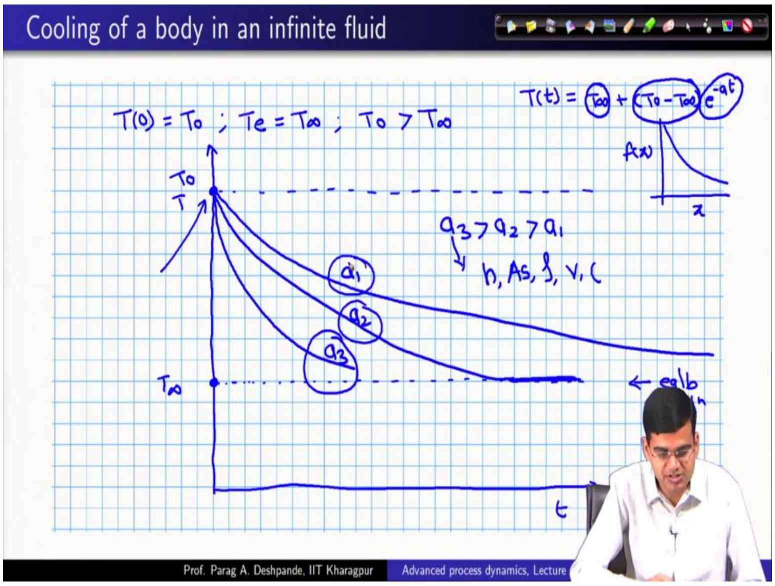
$$\Rightarrow T - T_\infty = (T_0 - T_\infty)e^{-at}$$

$$\boxed{T(t) = T_\infty + (T_0 - T_\infty)e^{-at}} \quad -(8)$$

Cooling of a body in an infinite fluid

$T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$; $T(0) = T_0$; $T_e = T_\infty$; $T_\infty > T_0$




Cooling of a body in an infinite fluid
T(0) = To ; Te = T∞ ; To > T∞
T(t) = T∞ + (To − T∞) e^{-at}
f(x)
x
To
T
a3 > a2 > a1
h, As, ρ, V, C
a1
a2
a3
T∞
← eqlb
t
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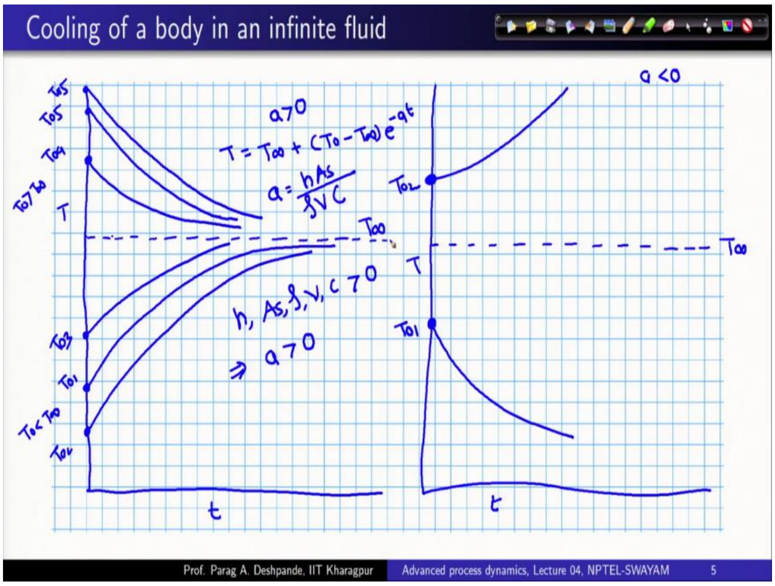
Cooling of a body in an infinite fluid
a > 0
T = T∞ + (To − T∞) e^{-at}
a = hAs / ρVC
h, As, ρ, V, C > 0
⇒ a > 0
To > T∞
To < T∞
T
T∞
t
a < 0
To2
To1
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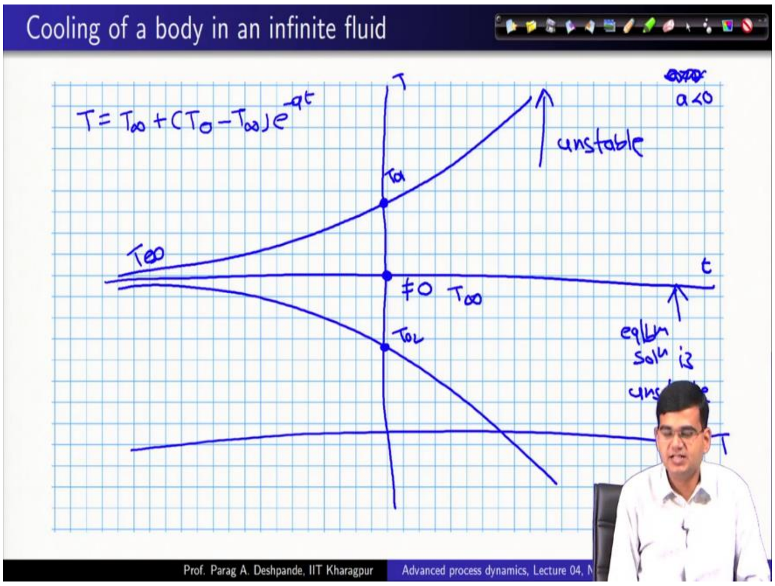
Cooling of a body in an infinite fluid
T = T∞ + (To − T∞) e^{-at}
a < 0
unstable
To1
To2
T∞
t
eqlbm soln is
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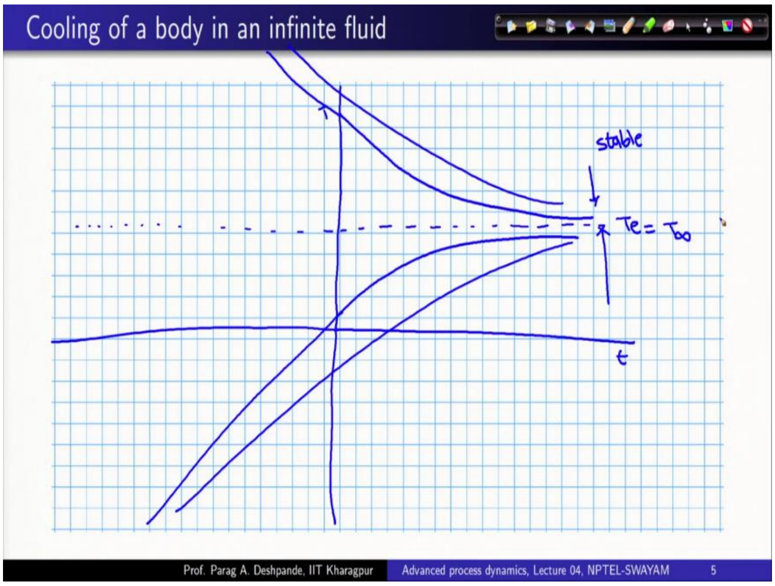

So, we meet again to continue our discussion on linear first order autonomous systems. We will take an example from process industries today. And see how we can actually use these concepts for understanding process industry and allied systems. So, before we go into the details let us have a very quick recapitulation of what we learned from the first lecture till the third lecture.

We were interested in understanding the dynamics of first order linear autonomous systems. So, the definition of first order was that they are the systems in which there is one and only one governing dynamical equation and the equation is of first order and the equation is going to be an ordinary differential equation.

The system was defined as linear if the corresponding operator was a linear operator and the condition for linearity was that it has to satisfy two properties if u and v are the vectors in the linear vector space for which the operator was defined then the according to the first property

$$\hat{L}(u+v) = \hat{L}(u) + \hat{L}(v)$$

And if α is a member of the field of which the linear vector space is defined then

$$\hat{L}(\alpha u) = \alpha\, \hat{L}(u)$$

These are the two conditions which must be satisfied for a system to be linear. It is important to note that the solution functions themselves do not need to be linear. Which means that the solutions can be sinusoidal and the solutions can be exponential or it can be any non-linear function but the corresponding operator which governs the dynamics of the system must be a linear operator.

Finally, we learnt about the definition of autonomous system. So, if you have a linear first order ODE and you are in a position to rearrange the ODE such that

$\frac{dx}{dt}$ is given by some function of x and t then the system would be called autonomous if and only if the right-hand side f is going to be only a function of x.

Which means if you can express the system as $\frac{dx}{dt} = f(x),$ such that f is not an explicit function of time then the system is called autonomous. Now all of these concepts can be applied to examples which come from process industries and we are going to take one such example.

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So, let us look into the example today what we have in front of us is the schematic for the system we have a solid body which is maintained at a temperature T_0. So, the solid body has a temperature T_0 and the body is dropped in a reservoir which has a temperature of T_∞. So, liquid reservoir which is at a temperature T_∞ reservoir by definition means that the temperature of the reservoir is not going to change which means T_∞ is going to be a constant in your in our entire analysis T_∞ is going to be considered constant.

So, now when you drop a body which is at temperature of T_0 into a liquid which is at a temperature of T_∞ such that T_∞ is never going to change then what is going to happen to this system physically. What is going to happen is that the temperature T which is the instantaneous temperature instantaneous temperature of the body is going to change.

So, the instantaneous temperature of the body is going to change and what is the governing equation that it would follow the governing equation can be determined quite simply using energy balance.

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So, the governing equation is in front of you where $\frac{dT}{dt}$ the time rate of change of temperature of the body at any instant of time so T is the instantaneous temperature of the solid body is given as

$$\frac{dT}{dt} = -\frac{hAs}{\rho Vc}(T - T_\infty)$$

and it is not very difficult to see how this equation comes so you have

$$\frac{dT}{dt}\rho Vc = \frac{d}{dt}mc_P T$$

which is the energy content of the body.

So, the rate at which the energy content of the body changes should be equal to the heat transfer. Heat transfer to the reservoir of from there is a reservoir depending upon the temperature of the reservoir and the temperature of the body. So, you equate these two terms and get this model equation.

So, h is the heat transfer coefficient As is the surface area of the body ρ is the density of the body V is the volume of the body c is the specific heat of the body and it is important to know that T is the instantaneous temperature so the dynamical variable of interest in our such case is T. So, T is a function of small t which is time. So, now let us see what all can we analyze in this case.

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the first question which we need to addresses that what is or are the equilibrium solutions of the system. So, let us look into the model equation the model equation is

$$\frac{dT}{dt} = -\frac{hAs}{\rho Vc}(T - T_\infty)$$

Now what I see is that under certain approximations h, surface area of the body, ρ, V, c they all can be considered constants.

I am saying that under certain approximations because over a very wide range of temperatures the specific heat of the body may change and so on and this is true for all of the properties which have been mentioned here. But if we assume that all of these properties remain constant over a temperature range for which we are doing our analysis then what I see is that

$$\frac{hAs}{\rho Vc} = a$$

So, if I know the relationship between these constants and a then I can say that fine for example if heat transfer coefficient increases the value of my constant a will increase if the density increases the value of my constant a will decrease and so on. So, therefore for the sake of mathematical convenience I will replace these constants with a simple constant a making my model equation

$$\frac{dT}{dt} = -a(T - T_\infty)$$

This has been done just for the sake of mathematical ease. It is not going to change any of our results. So, my first question is what is or are the equilibrium solution of solutions of the system. In our previous case we saw that $\frac{dx}{dt} = f(x)$ if this is the dynamical equation then if I equate it with zero, then I get the equilibrium solutions x_e as the solution of f(x) = 0.

So, therefore in this case I can write

$$-a(T - T_\infty) = 0$$

which gives me

$$T_e = T_\infty$$

Now does this make sense? So, let us analyze what is going on you have a body and you have dropped that body in the reservoir that reservoir temperature is maintained constant and you allow the temperature of the body to change.

So, the temperature of the body changes from T_0 which is the initial temperature of the body to some temperature. So, when equilibration takes place which means nothing will further change with time and T_∞ remains constant the only way that the gradients in the system would vanish is when your T becomes T_∞ this is quite a trivial physics which can be extracted out of this analysis.

And this analysis seems to be consistent with what we know either from training or from experience so therefore our equilibrium solution is that $T_e = T_\infty$. So, now what we need to do is we need to solve the model equation analytically solve the model equation analytically to determine the time evolution how can we know the time evolution of the system and then correspondingly develop the phase portrait for the system. So, let us see how this can be done.

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So, our model equation let us write the model equation our model equation was

$$\frac{dT}{dt} = -a(T - T_\infty) \quad \ldots\ldots\ldots\ldots (1)$$

This is our model equation and the initial condition which has been given to us is

$$T(0) = T_0$$

This is the initial condition which has been given to us.

Now to solve equation (1) what I will do is I will introduce a new variable and I will say that

$$T^* = T - T_\infty \quad \ldots\ldots\ldots\ldots (2)$$

So, let us say that I have a new variable T* which is equal to $T - T_\infty$ and since T_∞ is a constant I can write

$$dT^* = dT \quad \ldots\ldots\ldots\ldots (3)$$

So, from equations (2) and (3) and substituting them in equation (1) what I get is

$$\frac{dT^*}{dt} = -aT^* \quad \ldots\ldots\ldots.. (4)$$

And does this equation look familiar indeed because in our previous lecture what we saw was that we had the equation $\frac{dx}{dt} = ax.$ So, it is basically the same equation.

Which means that the entire analysis that we did in our previous lecture holds true for this system as well in fact that is the basic motivation behind introduction of any mathematical technique so that if you understand the technique well you can then use that technique for a say for a series of different situations.

The current situation being the analysis of dynamics of change of temperature of a body but if analogous equations arise in other situations as well the overall dynamics the features of the dynamics will remain the same. So, let us see how we can solve the equation as we solve, yesterday we will have

$$T^* = ce^{-at} \text{ .... (6)}$$

And I need to determine the integration constant that can be done with the help of the initial condition so let me plug in the original definition T* from equation (2).

$$T - T_\infty = ce^{-at} \quad \text{........... (7)}$$

At t = 0, $T = T_0$

So. I will substitute it in equation number (7). So, I have

$$T_0 - T_\infty = c$$

In other words, I have

$$T - T_\infty = (T_0 - T_\infty)e^{-at}$$

or finally I can write my time evolution so let me make it explicit that T is a function of small t as

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at} \text{ .... (8)}$$

So, the exponential has -at so this is my final solution to the model equation.

And this now puts me in a position to draw the phase portraits for the system. So, let us see if we can draw the phase portraits for this particular system.

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So, the model equation is

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$$

So, how can I draw the phase portrait for the system using this model equation? Now as I see that so let me draw the coordinates first this is going to be time this is going to be temperature. So, my initial temperature is T_0.

So, T_0 and my equilibrium temperature is t infinity which means that I am going to start my system with temperature is equal to T_0 and it should ultimately end up at T_∞ so as time T tends to infinity my temperature capital T should tend to T_∞. So, let me have a condition an initial condition which is given as T_0.

So, this is the initial condition of the system and finally the system has to reach the temperature T_∞. Assuming $T_\infty > T_0$. So, you have a water bath which is maintained at say 70^0C and then you drop the sphere or body from the top at room temperature.

So, this is the typical condition which it would satisfy so let me make a dotted line and another dotted line. So, as the temperature reaches t infinity the gradients would become zero. And the initial condition is T_0. So, therefore your entire system would be confined within these two limits you will have T_∞ at the top and T_0 at the bottom.

So, now how would you draw lines which correspond to this equation such that the bottom is T_0 and the top is T_∞ you see the function here this is an exponential function an exponentially decaying function in fact. So, what is going to happen you will start with T_0 and your ultimate value T_∞ is reached by multiplication of T - T_∞ by an exponentially decaying function this is how we understand the equation.

Well, you can understand the equation in other terms you start with T_∞ and then you keep on adding the difference but the difference itself keeps on reducing exponential with exponential with time. So, therefore the way you can look at this is that as at long time intervals the system is coming here asymptotically.

And then it has to start from here so the way we can join these two is like this this is one of the phase lines. As I change a as I change a, what can happen is if you start with the same initial condition depending upon the system properties you may reach quickly or you may reach slowly but asymptotically what is going to happen is that you are going to reach temperature T is equal to T_∞.

So, what do we understand from this solution is that if you have $T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$ the exponential part of your function will give the curve to the system. What is the importance of this parameter a? If a is large if a is large the change will be faster.

So, therefore if I have a_1, a_2, a_3 then I can write $a_1 > a_2 > a_3$ in magnitude. So, this is the phase portrait for the system but this corresponded to a situation where $T_\infty > T_0$ was which means that you dropped your body in the oil bath or a reservoir which was at a temperature which was greater than the initial temperature of your body. Let us see if we can do this in a reverse manner.

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So, our new situation is like this that you have $T(0) = T_0$; $T_e = T_\infty$; $T_0 > T_\infty$. So again, I will draw the axis this is time this is temperature again my system has two extremities one extremity is T_0 the other extremity is T_∞.

In the previous case $T_\infty > T_0$. Now I have reversed the situation T_∞ is now small and T_0 is large which means that say you have a hot plate a hot steel plate and you quench it in an oil bath to reduce its temperature quickly that is what you want to do that is a metallurgical operation which is done.

So, in this case again I have two extremities so let me say that I have T_0 here and I have T_∞ here and these are the extremities. So, all of my phase lines will lie between this these two extremities only. Fine and then what I see is that since T_e is T_∞. So, this is my equilibrium temperature this is my equilibrium solution rather so as $T \rightarrow \infty$, I should reach here.

So, asymptotically I should have this solution. So, how do I start from here and reach here taking into account the exponential decay which is associated with my model equation. So, if we remember our model equation was

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$$

So, what do I do I start with T_0 and because of this exponential decay the typical behavior of an exponential decay f(x) vs x is this and then this would be augmented with T_∞ and T_0 - T_∞. So, the way the phase line would look like is this. This is one of the phase lines.

I can have same starting initial point and different phase lines all of them would asymptotically go to T_∞ and so on. And if this is a_1 this is a_2 and this is a_3 three different values for a then as I said the magnitude of a will give the quickness of the system to adjust to the new condition and therefore in this, I can write $a_3 > a_2 > a_1$ fine.

So, and all of these a's are functions of heat transfer coefficient, surface area, ρ, V, c and so on. So, now you can adjust the values of a, As, h, ρ, c, V and so on to get the value of a different values of a…. a_1, a_2, a_3 and so on and then you can determine how quickly or slowly will your system reach the equilibrium state or it would go to T_∞. These are the different solutions.

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Now, what I can do is I can make another set of phase portraits to give you an idea about the solution or the solution behavior. So, I can superimpose both of them I have T here, I have t here. So, if this is my T_∞, then when $T_0 < T_\infty$, what is going to happen, I am going to rise exponentially I am going to rise like this not exponentially I am going to rise like this and therefore for different T_i's…… T_{01}, T_{02} and so on my phase lines would look like this.

And when $T_0 > T_\infty$, then what I will see is that I will see exponential decay like this and all of them would go exponent go asymptotically to T_∞. But now what I see is that this is true when a > 0 because your equation was $T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$. This is true only for a > 0.

And a in fact was $a = \frac{hAs}{\rho Vc}$. Now, all of these quantities heat transfer coefficient, surface area, ρ, this is the density, volume, specific heat they are all greater than zero which means a would always be greater than zero had it not been the case. Then how would the phase portrait have looked like.

A physically this is not correct physically this is not correct let me emphasize this but for the sake of mathematical completeness if you want to draw the exact same system for a < 0, then how would the system look like. Well, you will have again the same equilibrium solution T_∞, so if you start with T_{01} start what is going to happen you will not reach here in fact you will go down.

And for $T_{02} > T_\infty$, you would rise here. Now this looks a little weird looks a little weird but this is in fact not mathematically inconsistent because what then you will need to do is you will need to go so let me draw it again.

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Now, I have considered negative time as well. So, this is the temperature I have considered the negative time as well so a is in fact greater than 0 this how the solutions would look like you start with this and this is going to be a solution. So, you start with T_0 here you diverge and similarly you go here this is T_{02} and this would be your T_∞.

So, you will need to shift the axis so this is not zero so this point is not zero this is not zero this is T_∞. You can draw the x-axis like this so this is T. So, now what do we see here? What we have seen here is that for a less than for so this has to be $a < 0$ there is a correction there a has to be less than zero.

So, mathematically if there is a condition such that for the model equation

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at} \quad ; \; a < 0$$

making the exponential positive this will be the phase portrait physically for the current situation this phase portrait will not hold true. But you may tomorrow come across some other situation some other system for which you may get similar analysis and then you may be ready to understand that the solution behavior would look like this.

And therefore, what you see is that there is a divergence so the system is unstable this line this equilibrium solution, solution is unstable.

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Whereas for the current system for the current system when you drew the phase portrait like this, this is the equilibrium solution temperature time the equilibrium solution is like this or it looks like this T equilibrium you will see that the equilibrium is stable. In fact, what I can do is I can make this particular portrait also mathematically complete we have considered time only positive what you can do is time temperature everything for the sake of mathematical completeness negative so what is going to happen is you can draw phase lines like this these would be the phase lines.

And all of them would asymptotically reach $T_e = T_\infty$ as time $t \rightarrow \infty$. So, we would stop here and continue the discussion on this particular topic and see whether we can actually see these details using an alternative method. Thank you.