Advanced Process Dynamics
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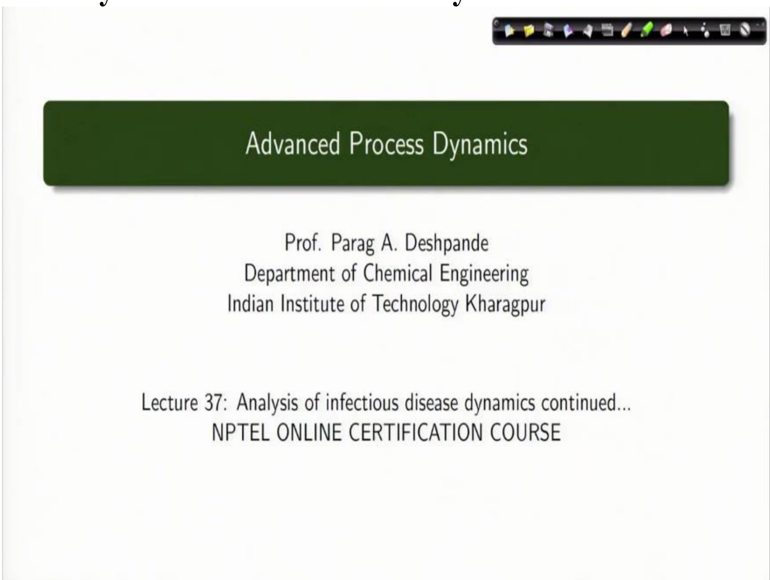
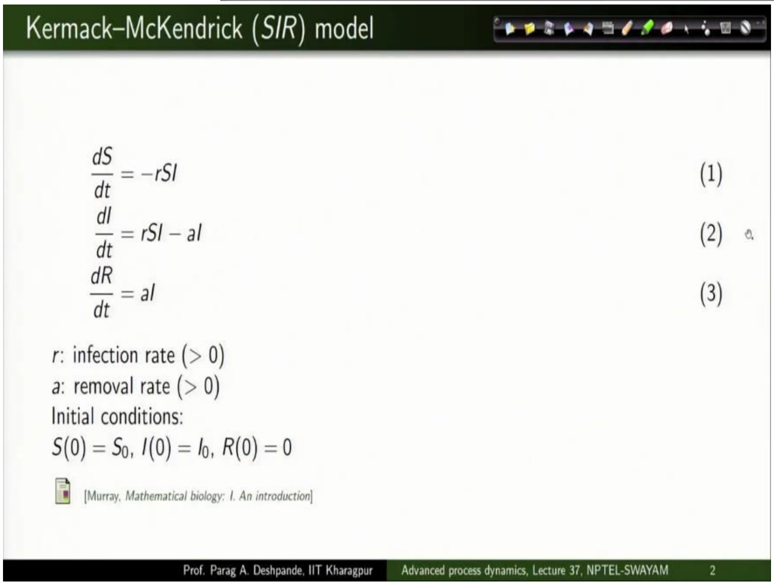
Lecture 37
Analysis of infectious disease dynamics continued…





Kermack–McKendrick (*SIR*) model

Key questions?

1. Given r, a, S_0 and the initial number of infectives I_0, whether the infection will spread or not?
2. If the infection does spread, how does it develop with time?
3. When will it start to decline?
4. When do you declare the spread of an infectious disease an "epidemic"

Kermack–McKendrick (*SIR*) model

$$\frac{dS}{dt} = -rSI \quad -(1)$$

$$\frac{dI}{dt} = rSI - aI \quad -(2)$$

$$\frac{dR}{dt} = aI \quad -(3)$$

Dynamical variable $= [S \quad I \quad R]^T$

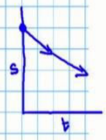
Order $= 3$

Non-linear

Autonomous system

$$\frac{dS}{dt} = -rSI$$

$$\left.\frac{dS}{dt}\right|_{t=0} = -rS_0I_0$$



$r > 0$, $S_0 > 0$, $I_0 > 0$

$$\Rightarrow \left.\frac{dS}{dt}\right|_{t=0} < 0$$

$$\left.\frac{dI}{dt}\right|_{t=0} = rS_0I_0 - aI_0$$

$$= (rS_0 - a)I_0 \quad \Big\} > 0$$

If $rS_0 - a > 0$, $\frac{dI}{dt}|$

$rS_0 > a$

$$\Rightarrow \frac{rS_0}{a} > 1$$

Kermack–McKendrick (*SIR*) model

$$\frac{dS}{dt} = -rSI \quad -(1)$$

$$\frac{dI}{dt} = rSI - aI \quad -(2)$$

$$\frac{dR}{dt} = aI \quad -(3)$$

$$\frac{dS}{dt} = -rSI$$

$$\Rightarrow I = \left(-\frac{1}{r}\right)\frac{1}{S}\frac{dS}{dt}$$

$$\Rightarrow I = \left(-\frac{1}{r}\right)\frac{d}{dt}(\ln S) \quad -(4)$$

$$\frac{dR}{dt} = aI$$

$$\frac{dR}{dt} = \left(-\frac{a}{r}\right)\frac{d}{dt}(\ln S)$$

$$\Rightarrow R = \left(-\frac{a}{r}\right)(\ln S) \quad -(5)$$

$S + R + I = N$ (= total population)

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}(S + I + R) = 0$$

$$\Rightarrow S + I + R = C$$

$$\Rightarrow S + I + R = N \quad -(6)$$

$$S + \left(-\frac{1}{r}\right)\frac{d}{dt}(\ln S) + \left(-\frac{a}{r}\right)\ln S = N \quad -(7)$$

Let $\ln S = x$

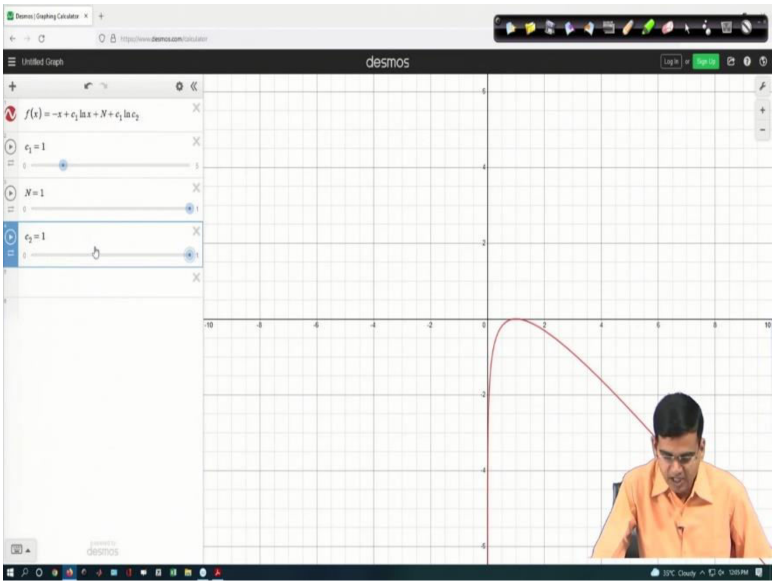
$$\Rightarrow S = e^x \quad -(8)$$

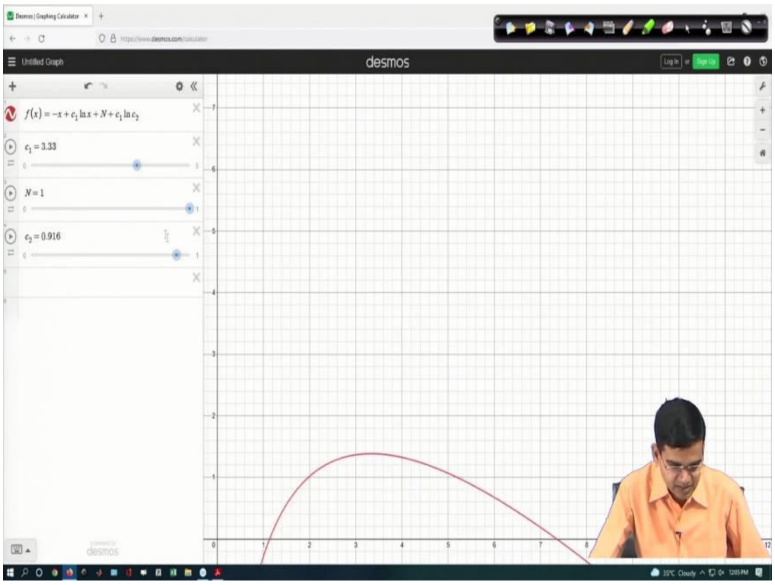
$$\Rightarrow e^x - \frac{1}{r}\frac{dx}{dt} - \frac{a}{r}x = N \quad -(9)$$

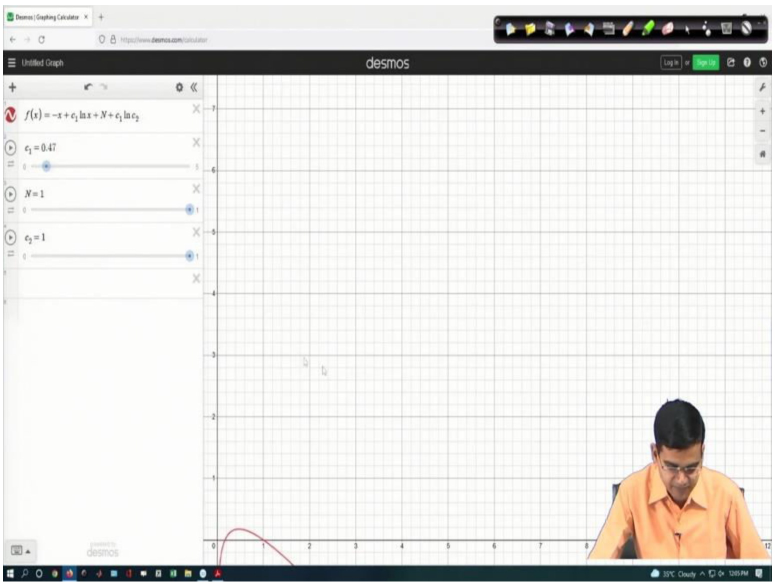
$$\Rightarrow \frac{dx}{dt} = f(x)$$

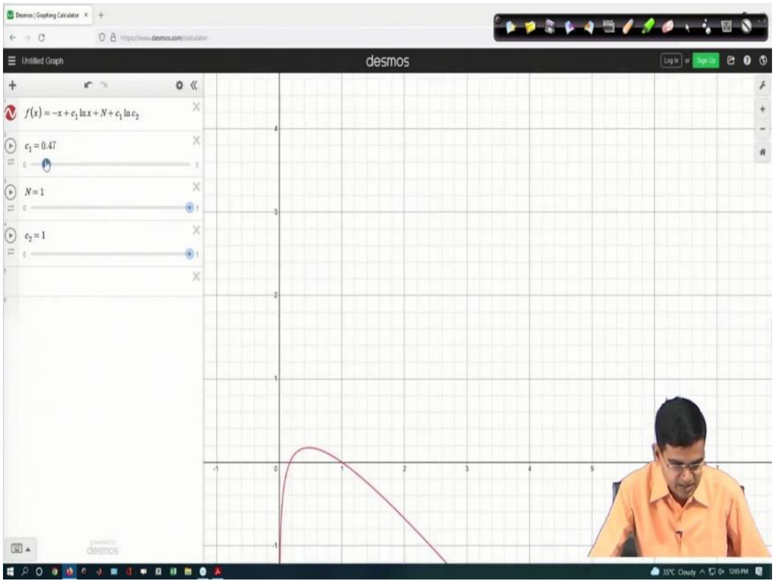
$\downarrow$

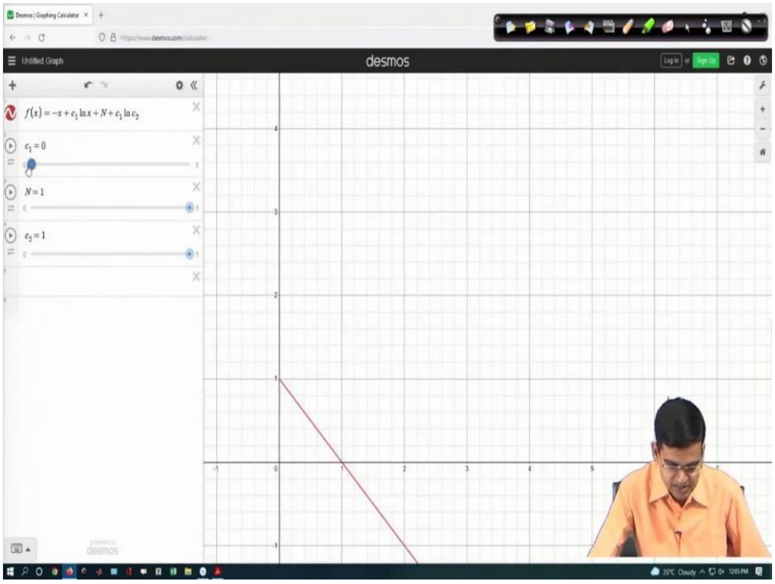
parameters a, r, N

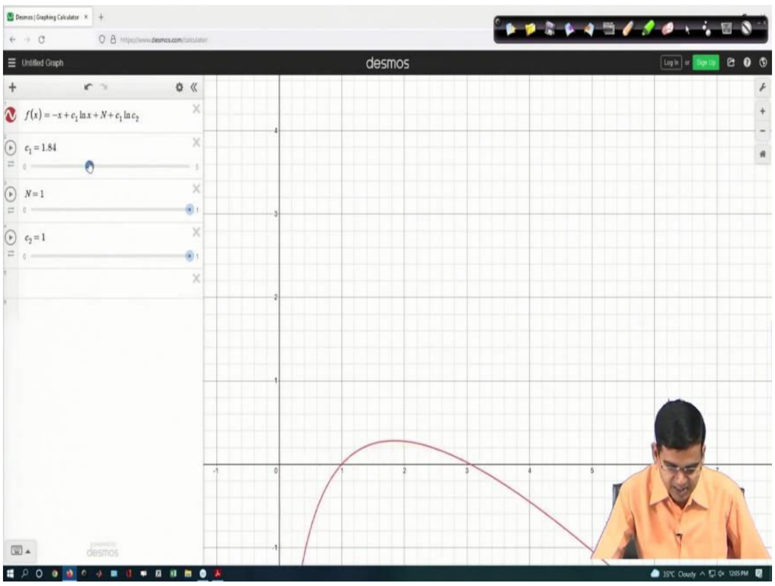

Untitled Graph
desmos
$f(x) = -x + c_1 \ln x + N + c_1 \ln c_2$
$c_1 = 1$
$N = 1$
$c_2 = 1$

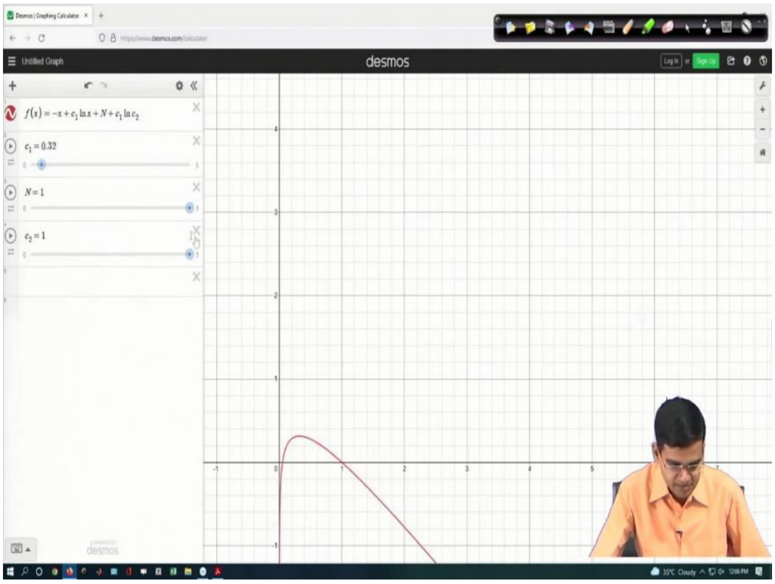

Untitled Graph
desmos
$f(x) = -x + c_1 \ln x + N + c_1 \ln c_2$
$c_1 = 3.33$
$N = 1$
$c_2 = 0.916$

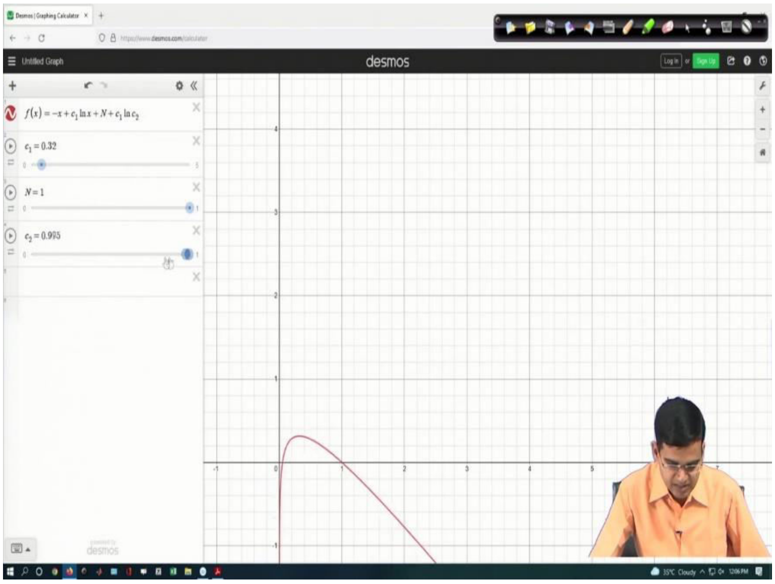

Untitled Graph
desmos
$f(x) = -x + c_1 \ln x + N + c_1 \ln c_2$
$c_1 = 0.47$
$N = 1$
$c_2 = 1$

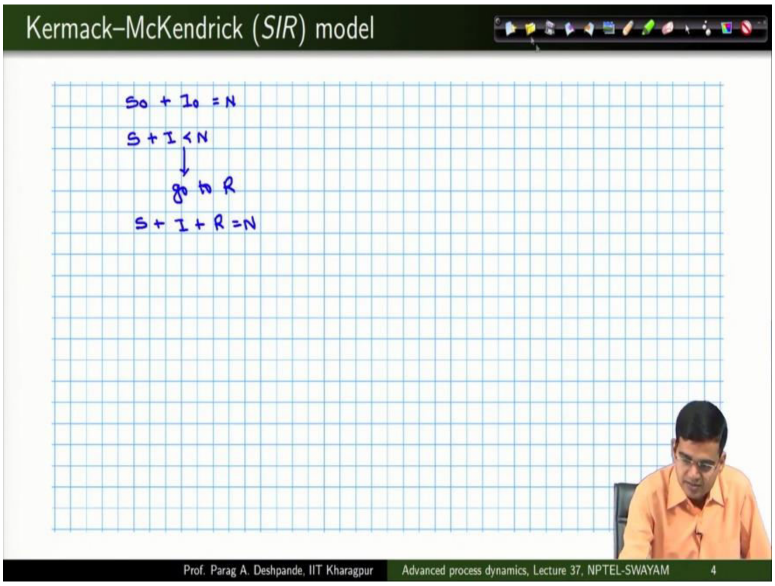

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = -x + c_1 \ln x + N + c_1 \ln c_2$
$c_1 = 0.47$
$N = 1$
$c_2 = 1$


Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = -x + c_1 \ln x + N + c_1 \ln c_2$
$c_1 = 0$
$N = 1$
$c_2 = 1$


Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = -x + c_1 \ln x + N + c_1 \ln c_2$
$c_1 = 1.84$
$N = 1$
$c_2 = 1$


desmos
Untitled Graph
f(x) = −x + c1 ln x + N + c1 ln c2
c1 = 0.32
N = 1
c2 = 1


desmos
Untitled Graph
f(x) = −x + c1 ln x + N + c1 ln c2
c1 = 0.32
N = 1
c2 = 0.995


Kermack–McKendrick (SIR) model
S0 + I0 = N
S + I < N
go to R
S + I + R = N
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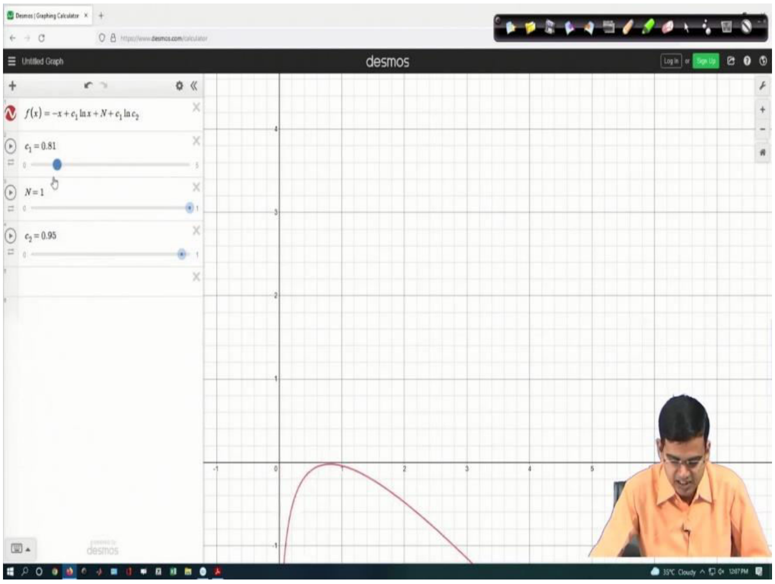
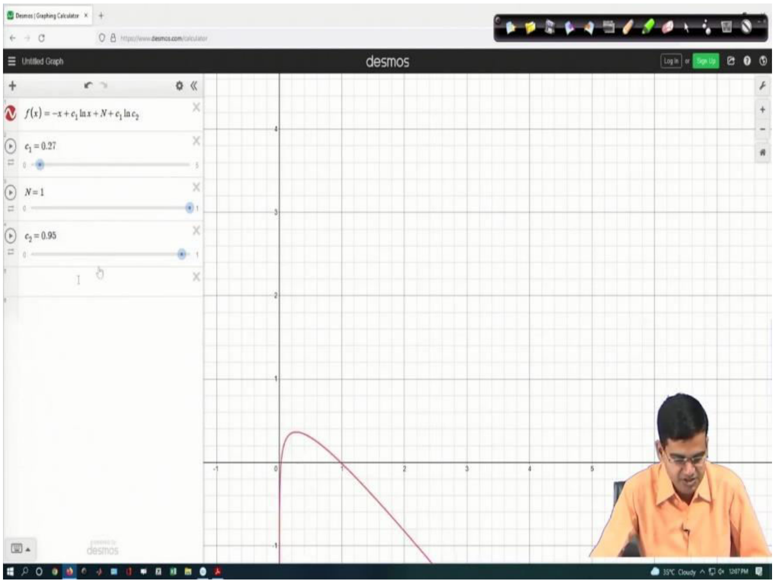
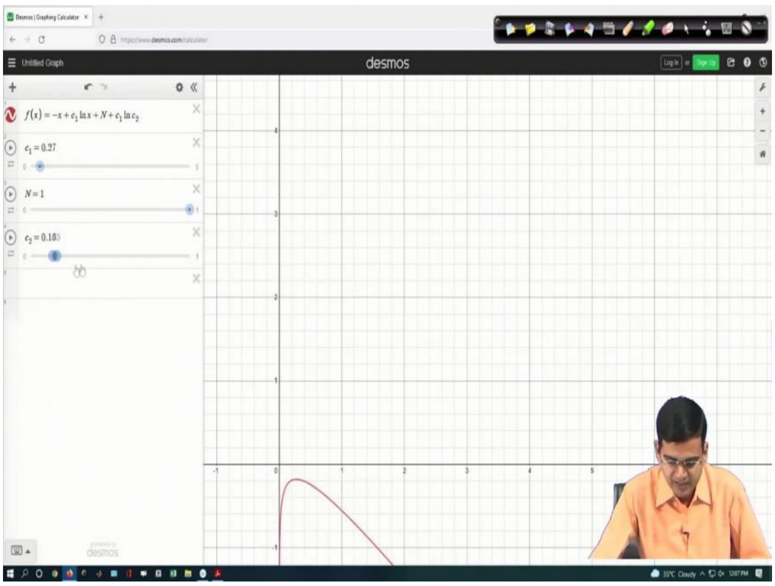
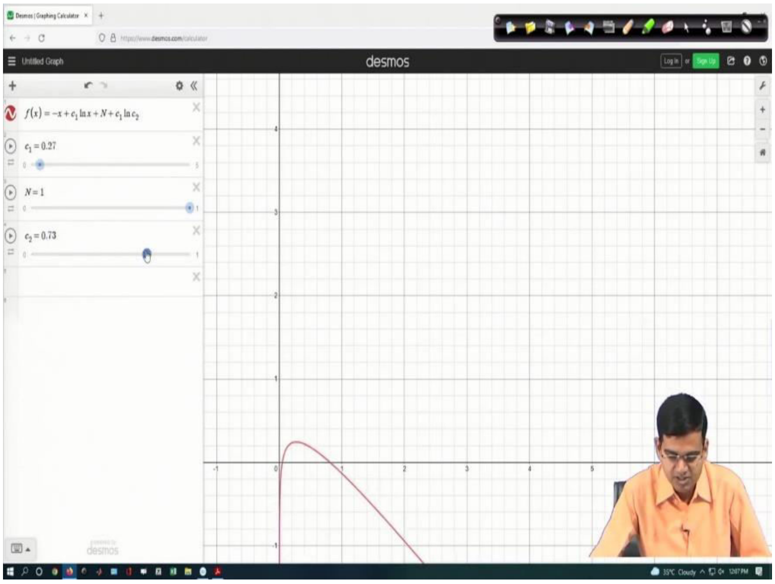
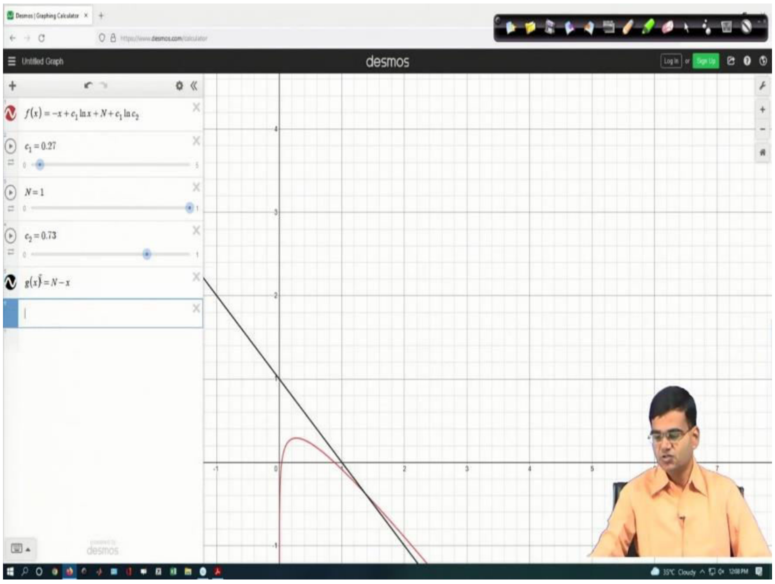
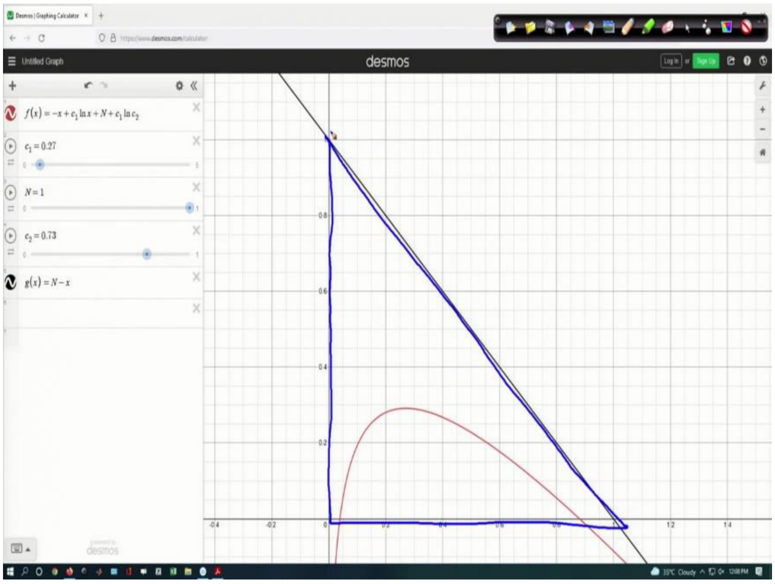

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = -x + c_1 \ln x + N + c_1 \ln c_2$
$c_1 = 0.81$
$N = 1$
$c_2 = 0.95$

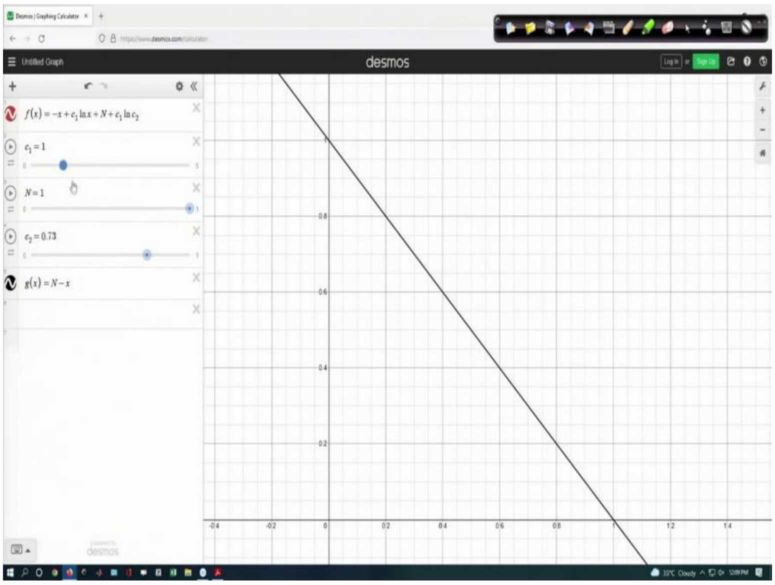

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = -x + c_1 \ln x + N + c_1 \ln c_2$
$c_1 = 0.27$
$N = 1$
$c_2 = 0.95$

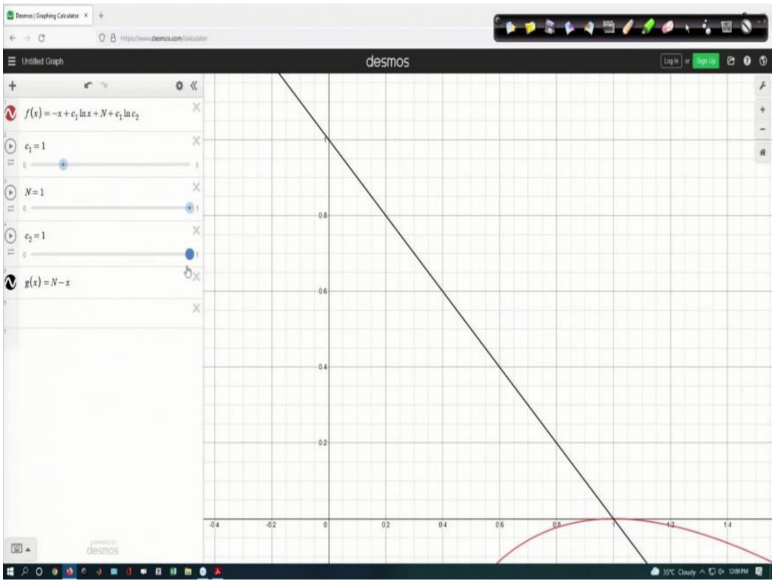

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = -x + c_1 \ln x + N + c_1 \ln c_2$
$c_1 = 0.27$
$N = 1$
$c_2 = 0.185$

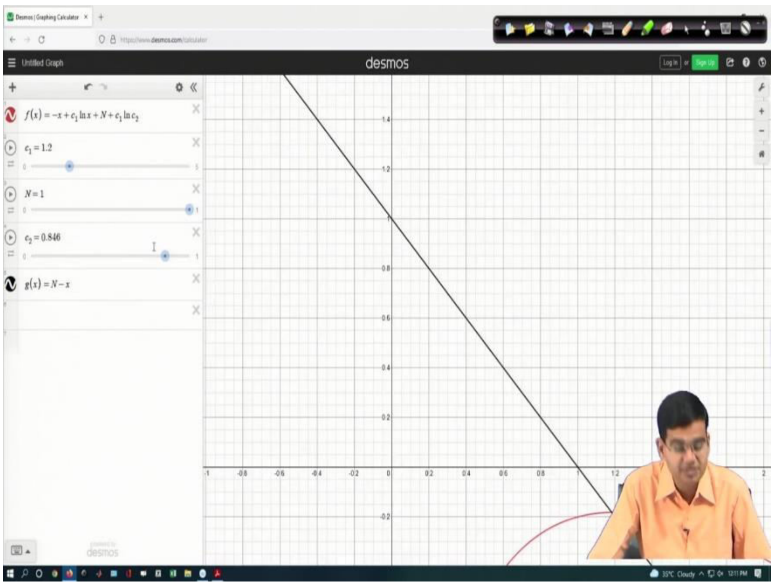

Desmos | Graphing Calculator
Untitled Graph
desmos
f(x) = −x + c1 ln x + N + c1 ln c2
c1 = 0.27
N = 1
c2 = 0.73

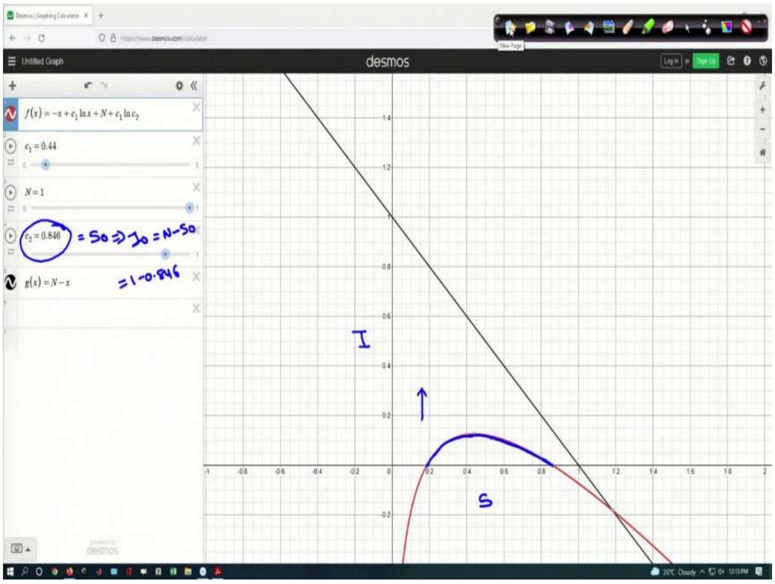

Desmos | Graphing Calculator
Untitled Graph
desmos
f(x) = −x + c1 ln x + N + c1 ln c2
c1 = 0.27
N = 1
c2 = 0.73
g(x) = N − x

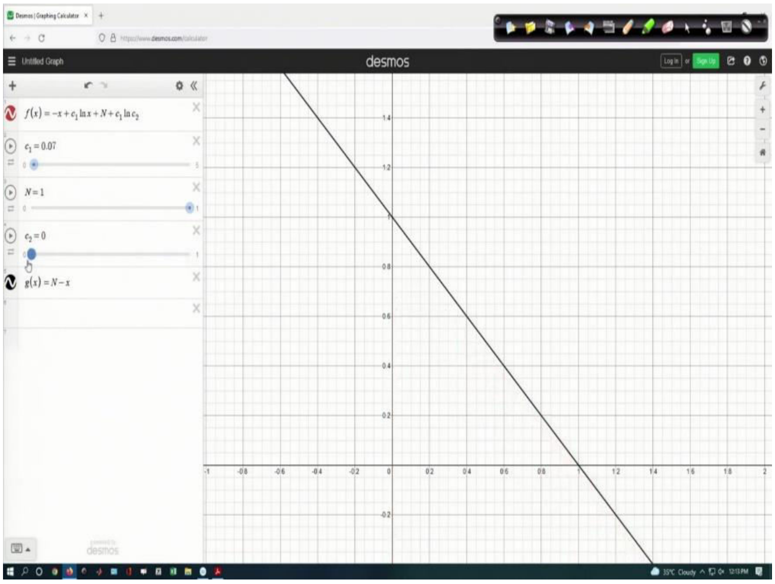

Desmos | Graphing Calculator
Untitled Graph
desmos
f(x) = −x + c1 ln x + N + c1 ln c2
c1 = 0.27
N = 1
c2 = 0.73
g(x) = N − x

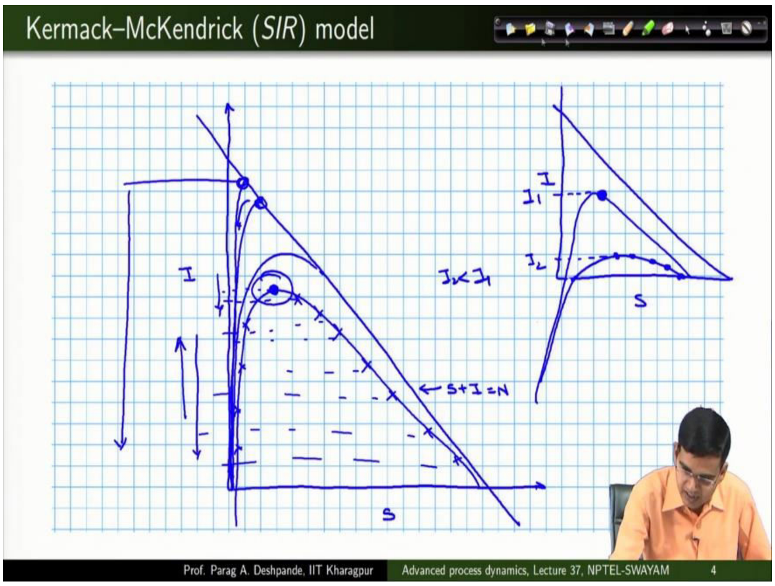

Untitled Graph
desmos
f(x) = −x + c_1 ln x + N + c_1 ln c_2
c_1 = 1
N = 1
c_2 = 0.73
g(x) = N − x


Untitled Graph
desmos
f(x) = −x + c_1 ln x + N + c_1 ln c_2
c_1 = 1
N = 1
c_2 = 1
g(x) = N − x


Untitled Graph
desmos
f(x) = −x + c_1 ln x + N + c_1 ln c_2
c_1 = 1.2
N = 1
c_2 = 0.846
g(x) = N − x


desmos
Untitled Graph
f(x) = −x + c1 ln x + N + c1 ln c2
c1 = 0.44
N = 1
c2 = 0.846
= So ⇒ Io = N−So
= 1−0.846
g(x) = N − x
I
S


desmos
Untitled Graph
f(x) = −x + c1 ln x + N + c1 ln c2
c1 = 0.07
N = 1
c2 = 0
g(x) = N − x


Kermack–McKendrick (SIR) model
I
S
S+I=N
I2<I1
I1
I2
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Welcome back. So, continue our discussion on analysis of dynamics of infectious diseases.

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The model that we had in front of us was a third order nonlinear autonomous dynamical system, where the dynamical variable was S, I, R corresponding to the population of susceptible, infectives and removed. The three equations are in front of you. And with the help of these three equations and various assumptions that we made a mention of in the previous lecture, we were trying to answer several questions.

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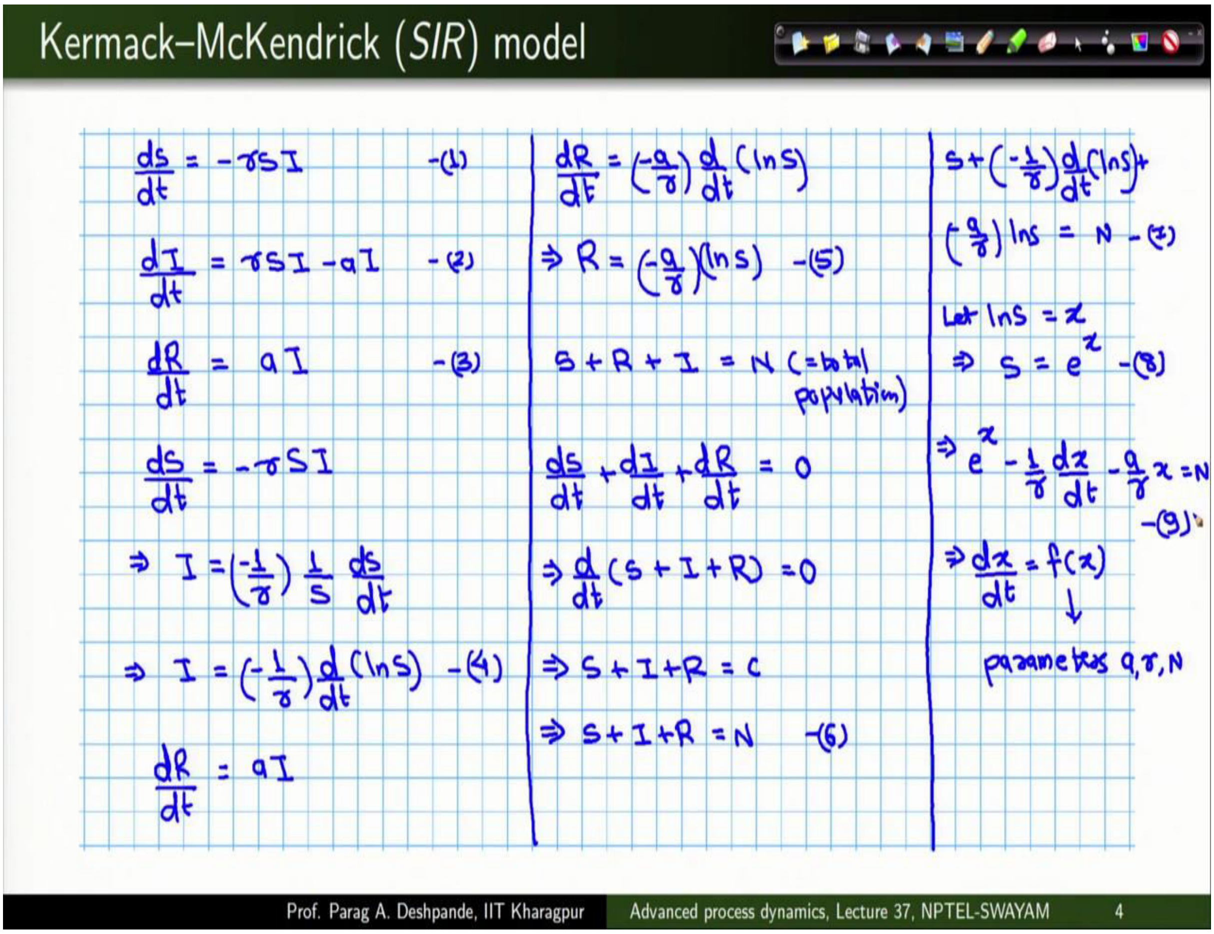
Like if we know the parameters of the system, can we tell whether the infection would spread or not. We came across the term reproduction number in the previous lecture, which I said that would give an idea that when the reproduction number R_0 was greater than 1, then the infection would in fact spread.

But now, we will also like to know the exact details of the dynamics and what would be the maximum number of infectives that one can expect during the pandemic and so on. And if these predictions can be made during the early stages of your infection, so that one can in principle, be prepared to face that the pandemic using various preventive measures.



I will solve equation number 9 for S substituted in equation number 8 to get the value of S, which means, it would give me the time dependence of variation of susceptibles in the population. Then what I will do is I will then use that function S(t) in equation number 4 to give me the time variation of infectives in my population and equation number 5 to give me the time variation of removed population from my system.

So, in principle, I can follow this entire method and get the time evolution of S, I and R in my system. This in principle is possible, but the mathematical form that we would get will not be very convenient. So, I encourage you to solve equation 9. It is a nonlinear equation. So, it is a little tricky to solve it.

But, if you manage to get an analytical expression, then also you will see that getting an explicit expression of S in terms of t and subsequently S in terms of t is going to be a little difficult. So, therefore, can we do, can we take an alternative approach to do this analysis. So, let us see if we can do this.

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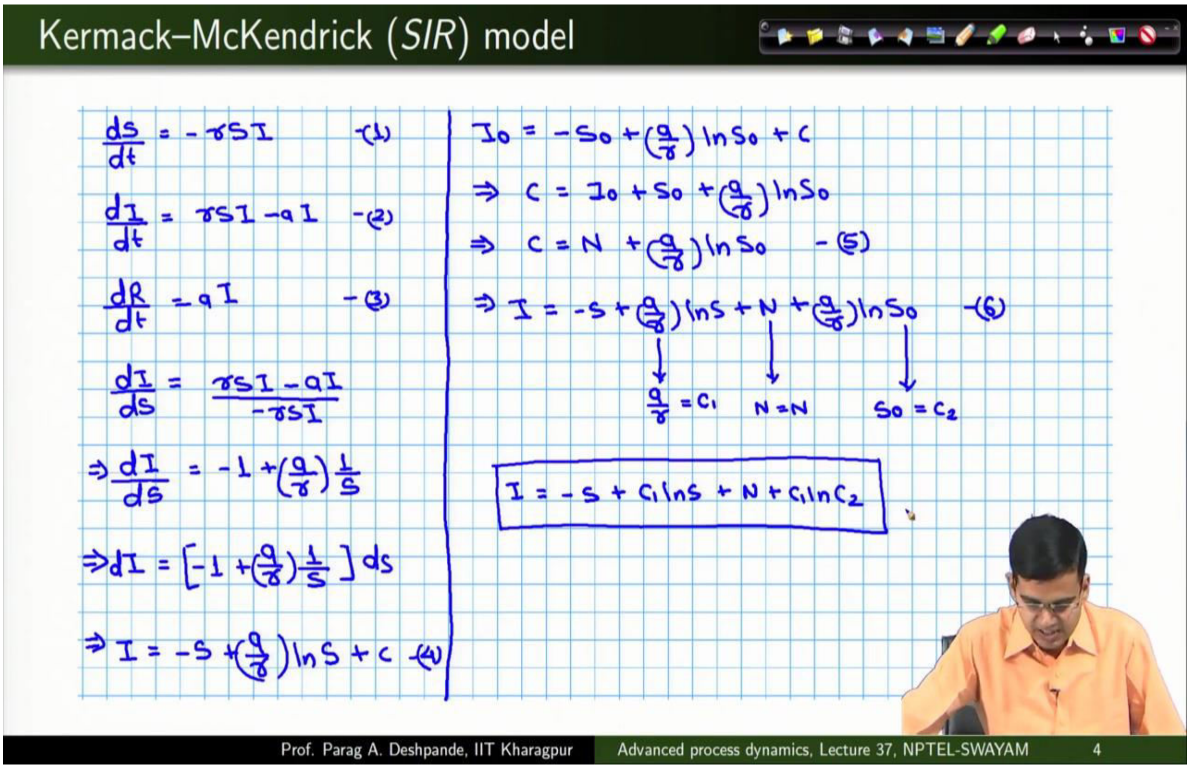

So, let me see if I can plot it using my online plotter.

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My equation is this, $f(x) = - S + C_1 \ lnS + N + c_1 \ lnC_2$. So, let me get rid of the text here. Now, what did we learn from our analysis N, capital N is the population, let me consider a normalized population between 0 and 1. The population will always be positive, but I am considering a normalized population which means the population varies from 0 and 1.

So, if I have to introduce I_0 it would be a fraction. So, 0.1 means 10 percent of the initial population is the infected persons. So, that is the meaning. So, let me make it from 0 to 1. Then C_1 corresponded to the ratio a / r. a is positive, r is positive. So, therefore, let me make this quantity say from 0 to 5, and C_2 is the initial population. So, C_2 is the initial population of the susceptible the range has to be between 0 and 1 because the maximum population is 1.

So, the range will be between 0 and 1. And therefore, now I will tweak around this to get the phase lines. So, let us see what happens. I changed the number of susceptibles. And what I get is, I change this ratio, I change this ratio, I change this number of susceptible and this is what happens, I am going on changing here.

So, let us qualitatively see the nature. I am going on changing this, these are the phase lines, which I am getting. And similarly, I will change the initial population, these are the phase lands which I am getting. So, I got a general nature of variation of I with S, but is there any constraint in my system which I must take care off? Well, there is.

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So, if S_0 is the initial susceptible population, I_0 is the initial infective population then I know that $S_0 + I_0 = N$, the total population. But as the infections grow the susceptible number of susceptibles change and some of them become recovered, $S + I$ should become less than N. Why? Because some of them go to R. $S + I + R = N$. Therefore, $S + I$ should be less than N because R is again, because it is a population, a positive quantity.

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So, therefore, I must look at these solutions which are there I have changed c_1 you see here, I have changed c_2 you see here, I must look at only those solutions for which first I and S positive, negatives should not be considered because we are considering populations, but then I must consider the case where S plus I must be less than N.

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So, I must now here write $g(S)$ should be $N - S$. So, now, I have a triangle so, let me zoom it in. I have a triangle. So, this is the triangle. And any solution which lies within this triangle only is a feasible population. So, for the current case, for example, you see this curve. So, therefore, all of these points on the curve are the feasible populations at any given instant of time. So, now, if I need to look at this particular solution, then, let me do one thing, let me try to draw different phase lines conforming to this constraint.

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So, first of all what I will do is I will see I will fix a value for the parameter a / r. So, let me fix it at 1. So, for $c_2 = 0.73$ and $c_1 = 1$, which means $a / r = 1$, you do not have any solution. Do I get any solution is something which I need to see. So, let me change the population and I do see a population which is here.

So therefore, it is, it, for $c = a / r = 1$, which is just 1, ratio is 1 and S_0 is also 1. If you remember the reproduction ratio, this will result in reproduction ratio 1. So, we just have this point here. So now, what I will do is I will, this basically means that I have the entire population which is susceptible, but for this particular point here, the point is 1, 0 which means that there is no infectives present in my system.

Everyone is susceptible, but there is no infective. How would the infective grow if there is no initial number of infective. Therefore, this particular observation is consistent with our physical intuition that if you do not reduce introduce any infected members in the population, the infection is not going to rise. So, what do I need to do?

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Let me do one thing, let me make the *a / r* ratio 1.2, greater than 1, and let me reduce this number of infectives. Again, I do not see any solution. Physically, make sense. And therefore, now, if I conform to this situation here, c_1 is *a / r* ratio, c_2, very large number of infective in a, small number of infective, large number of susceptible population, I see some solutions appearing here.

What is the importance of several points here? c_2, which you can see here, $c_2 = S_0$, initial number of susceptible population. So, which means, I_0 will become $N – S_0$ which is 1 - 0.846. And if you go here, you will see that you have 0.86, if you see here, you have some number of initial population and then the population changes with time.

So, now, what you have to see is that how does the pandemic evolve with time. So, when I set c_1 as 0.44 and c_2 as 0.4, 0.846, you are evolving in time. So, let me draw this. You are evolving in time and you are coming here. So, what is happening to I? So, I have the axis I here, I have the axis S here. And remember, I am drawing I versus S. So, therefore, what happens is that I am going up, so my number of infectives go up.

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And let me now tweak around with these parameters. I go up, you see here, let me tweak around with this parameter further, and so on. And I can tweak around with this parameter like this. So, what is the general nature of the plot which you see? The general nature of the plot which you see here can be drawn here, let me draw that.

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So, the general nature is this. This is I, this is S. I need to confine myself within this line. This is equal to, this is the line which is $S + I = N$. This is the line which corresponds to $S + I = N$.

So, any solution to my system, which is within this triangle is the feasible population. And what are the phase lines that we drew?

We saw the phase lines which looked like this. So, suppose this is one phase line as an example. So, what is going on? I am starting with some initial population and therefore, my I as I go up, you will see here, I is increasing. I can have what is it dependent upon, its dependent upon S_0, it is dependent upon a upon r, which ultimately can be clubbed to the reproduction number R_0.

So, therefore, if S_0, *a* and *r* during the beginning of the pandemic itself, you are in a good position to tell whether you are going. So, if you look at this particular plot, can you say that the pandemic is going to happen? Yes, because you see that, you in fact, see an increase in the population of I.

So, during the back calculation of *a* and *r* and S_0 in the beginning, you can simply say that your pandemic is going to happen or epidemic is going to happen and the number of infections are going to increase. So, now, what happens further as you change these parameters, you see these kinds of plots and there would come a situation like this.

So, what happens in this situation? You are only going down. In the first situation, the infective increased, went till this point, and then they started decreasing. Do we see this these days for COVID-19 infection? Yes, every time we have a wave, number of infectives go on increasing, reach a peak, that is why we say that we have reached a peak, so that corresponds to this point.

So, once you reach the peak, then your infectives start coming down. But, will this always happen? In this last case, for example, you start with I and you always come down. So, therefore, for such case, the severity of pandemic would not, the pandemic would not occur at all, because the number of infectives will always come down.

What about the severity of pandemic? This is also one may comment upon. So, for that, what we can do is we can redraw this plot. I have I, I have S. So, therefore, you would go through maxima and this can be one of the solutions. To make it a little more elegant, this would be one of the solutions, for example.

So, when you start with this at any point, you see here, you compare this situation versus this situation. Here, you have reached a maxima, where you have larger number of infectives I_1

versus I_2, $I_2 < I_1$ and therefore, what you see is that although pandemic or epidemics is there in both the cases, the number of infectives increased with time in case of I_2, there did not occur much increase, or the peak value was smaller in case of I_1, the peak value was larger.

So, what we saw here today is that, you can in fact make certain predictions about the pandemic or epidemic right during the beginning of the spread itself. The parameters which would be associated for making these predictions would involve the infection rate, the removal rate as well as the initial susceptible population.

Depending upon the combination of these three parameters, which would result into the parameter called reproduction number, what you will see is that you will sample different phase lines on the I-S projection of your phase portrait. So, therefore, when you have this combination of a, r and S_0, such that your phase trajectory goes up along the I axis, you say that the pandemic has occurred.

But then by looking at this itself, that why peaks during wave are observed, because once you reach the maximum, the number of infectives again start coming down. In certain cases for combination of S_0, a and r, you can simply have the decline in the value of I right from the beginning, in which case you do not observe a pandemic or epidemic at all. We will further analyze the effect of these parameters in the next lecture. Till then, goodbye.