Advanced Process Dynamics

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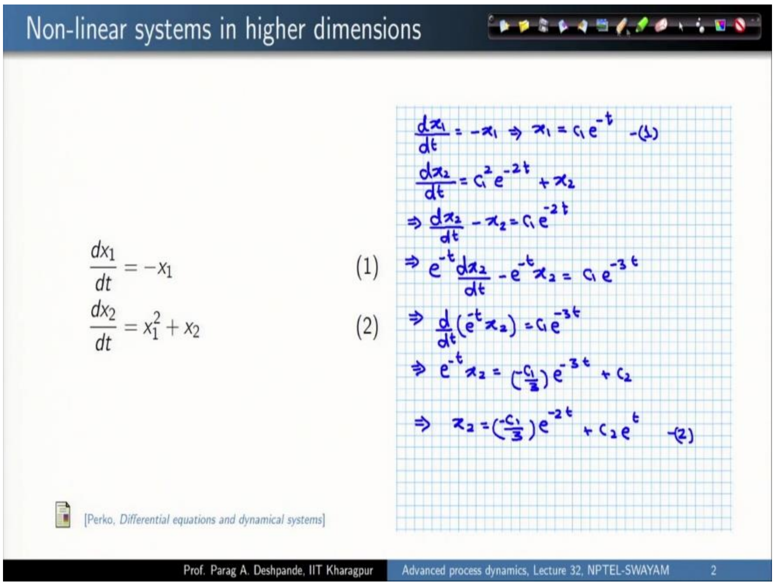
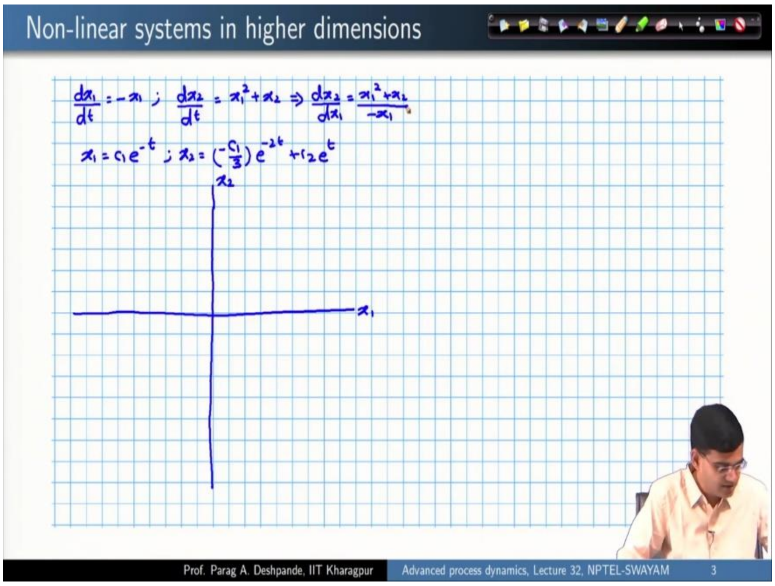
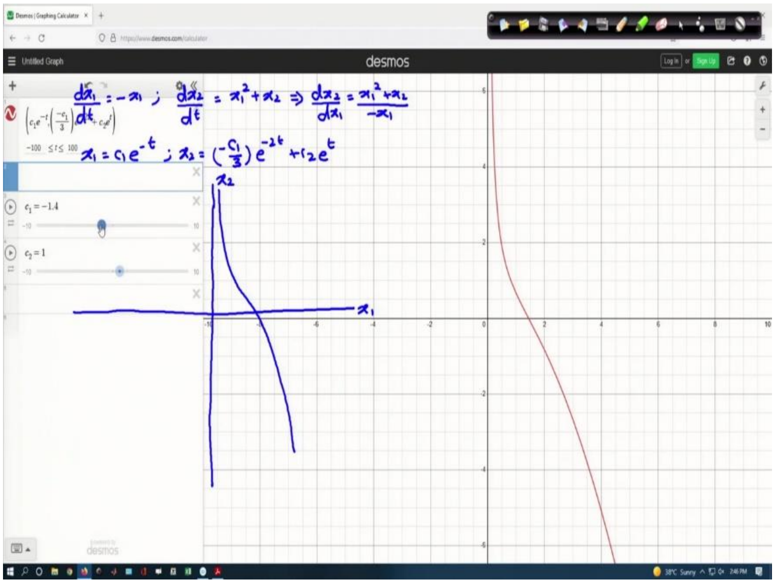
Indian Institute of Technology, Kharagpur

Lecture 32

Non-linear systems in higher dimensions






Desmos | Graphing Calculator
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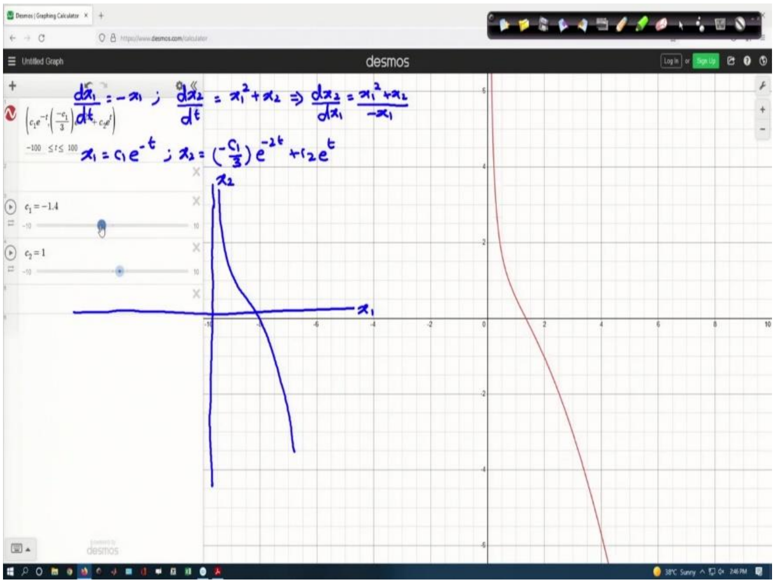
desmos
dx1/dt = -x1 ; dx2/dt = x1^2 + x2 ⇒ dx2/dx1 = (x1^2 + x2)/(-x1)
x1 = c1 e^-t ; x2 = (-c1/3) e^-2t + c2 e^t
-100 ≤ t ≤ 100
c1 = -1.4
c2 = 1
x2
x1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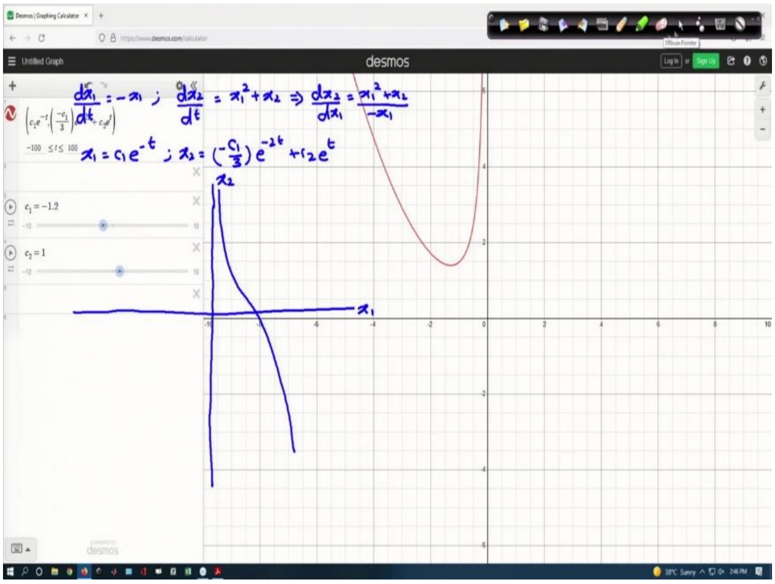
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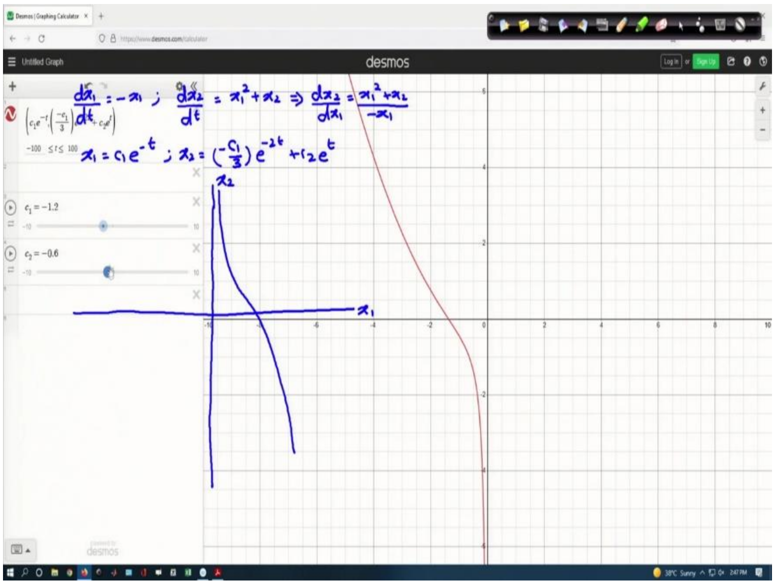
-100 ≤ t ≤ 100
c1 = -1.2
c2 = 1
x2
x1



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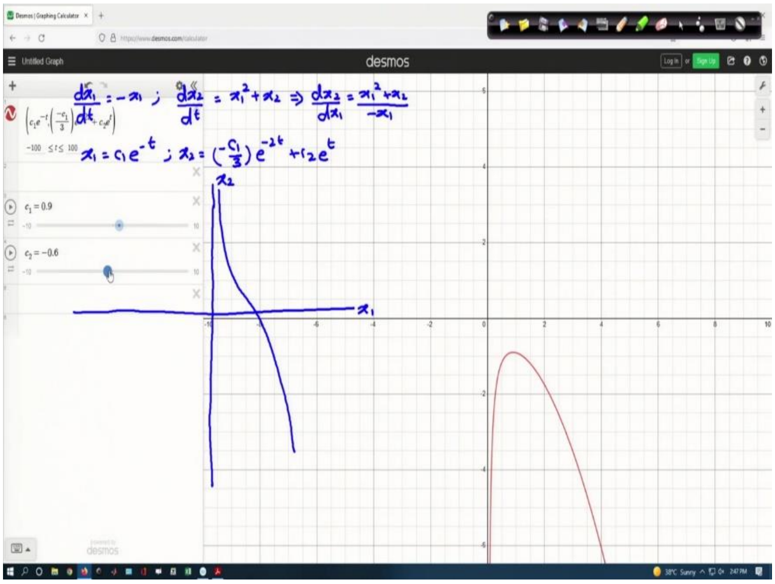
Untitled Graph
$\frac{dx_1}{dt} = -x_1$; $\frac{dx_2}{dt} = x_1^2 + x_2 \Rightarrow \frac{dx_2}{dx_1} = \frac{x_1^2 + x_2}{-x_1}$
$x_1 = c_1 e^{-t}$; $x_2 = \left(\frac{-c_1}{3}\right) e^{-2t} + c_2 e^{t}$
$-100 \le t \le 100$
$c_1 = -1.2$
$c_2 = -0.6$
x_2
x_1



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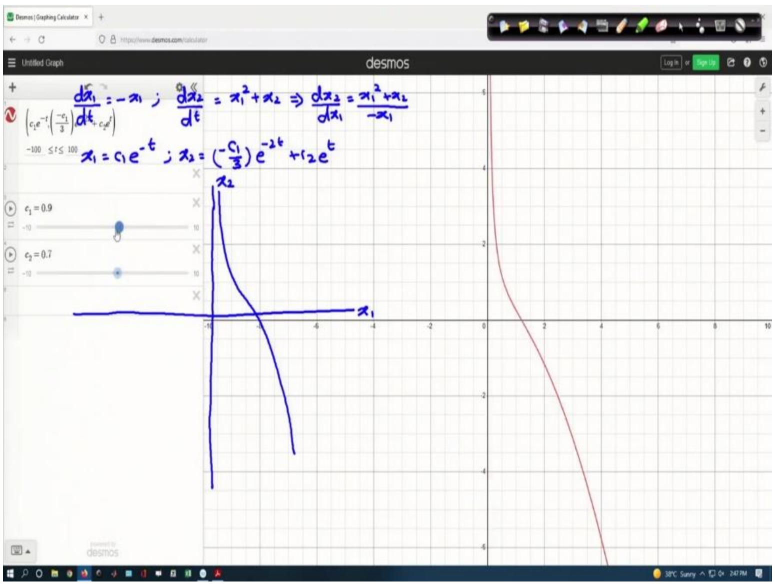
$x_1 = c_1 e^{-t}$; $x_2 = \left(\frac{-c_1}{3}\right) e^{-2t} + c_2 e^{t}$
$-100 \le t \le 100$
$c_1 = 0.9$
$c_2 = -0.6$
x_2
x_1



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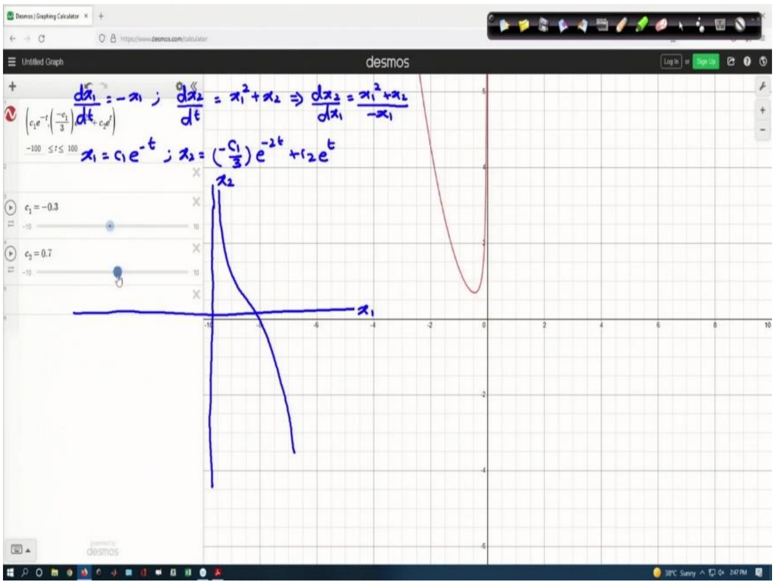
$x_1 = c_1 e^{-t}$; $x_2 = \left(\frac{-c_1}{3}\right) e^{-2t} + c_2 e^{t}$
$-100 \le t \le 100$
$c_1 = 0.9$
$c_2 = 0.7$
x_2
x_1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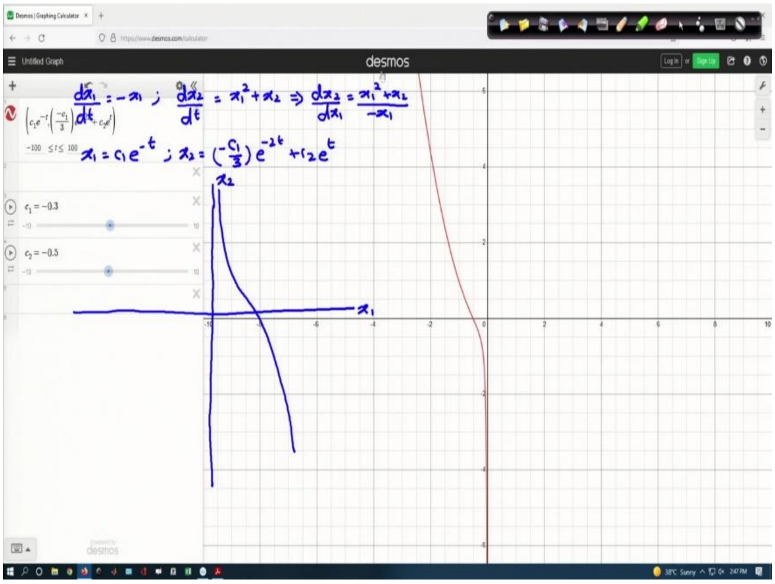
$x_1 = c_1 e^{-t}$; $x_2 = \left(-\frac{c_1}{3}\right) e^{-2t} + c_2 e^{t}$
$c_1 = -0.3$
$c_2 = 0.7$
x_1
x_2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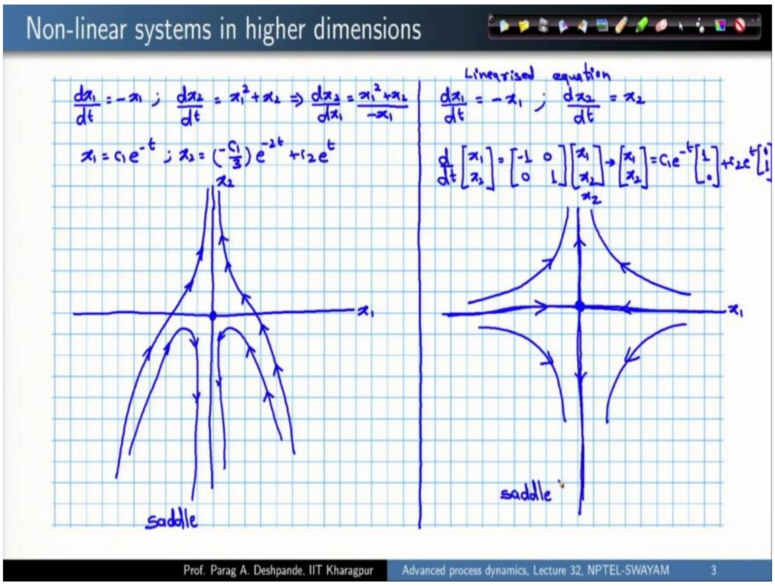
$x_1 = c_1 e^{-t}$; $x_2 = \left(-\frac{c_1}{3}\right) e^{-2t} + c_2 e^{t}$
$c_1 = -0.3$
$c_2 = -0.5$
x_1
x_2


Non-linear systems in higher dimensions
$\frac{dx_1}{dt} = -x_1$; $\frac{dx_2}{dt} = x_1^2 + x_2 \Rightarrow \frac{dx_2}{dx_1} = \frac{x_1^2 + x_2}{-x_1}$
$x_1 = c_1 e^{-t}$; $x_2 = \left(-\frac{c_1}{3}\right) e^{-2t} + c_2 e^{t}$
Linearised equation
$\frac{dx_1}{dt} = -x_1$; $\frac{dx_2}{dt} = x_2$
$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
x_1
x_2
saddle
saddle
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3

Non-linear systems in higher dimensions

$$\frac{dx_1}{dt} = x_1^2 \quad (3)$$

$$\frac{dx_2}{dt} = -x_2 \quad (4)$$

$$\frac{dx_1}{dt} = x_1^2 \;;\; \frac{dx_2}{dt} = -x_2$$

$$\Rightarrow x_1 = \frac{1}{c_1 - t} \;;\; x_2 = c_2 e^{-t}$$

$$\frac{dx_1}{dt} = 0 \;;\; \frac{dx_2}{dt} = -x_2$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 0,\; \lambda_2 = -1$$

[Perko, *Differential equations and dynamical systems*]

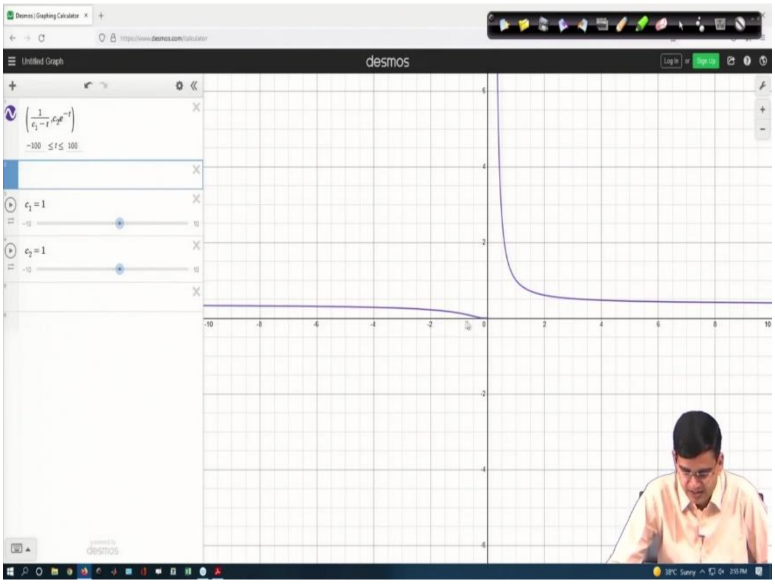
Non-linear systems in higher dimensions

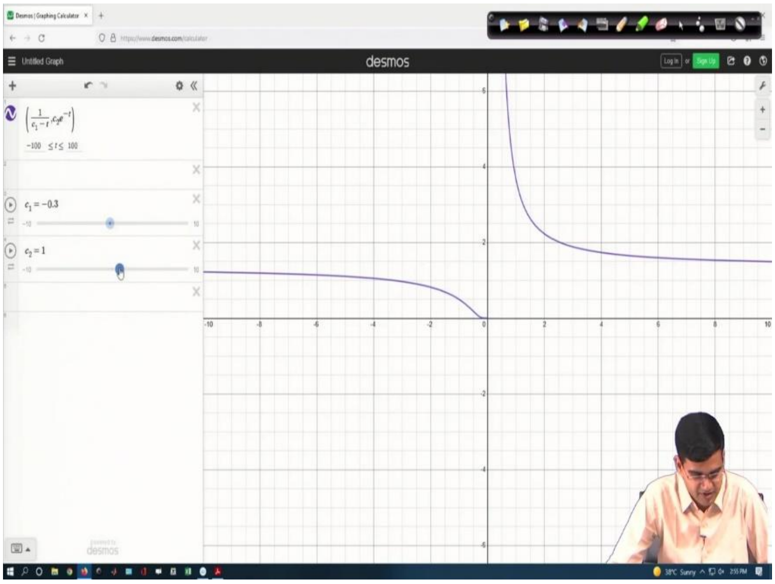
$$x_1 = \frac{1}{c_1 - t} \;;\; x_2 = c_2 e^{-t}$$

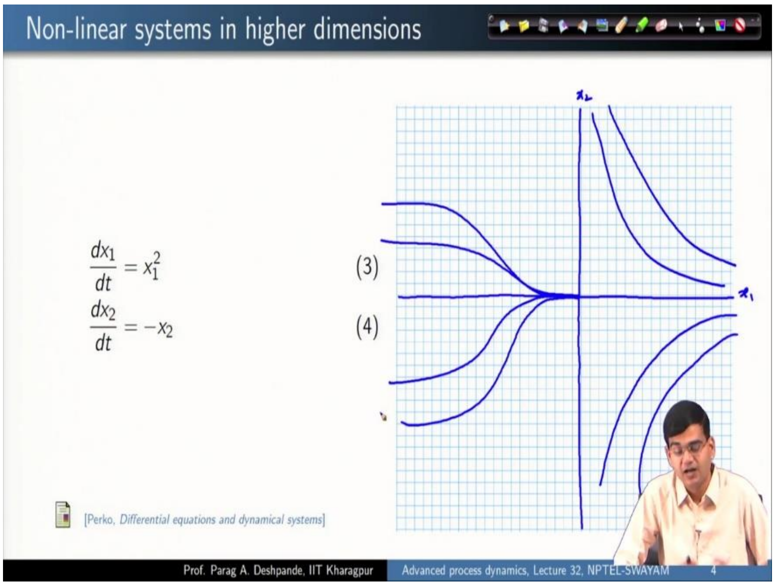
$$\frac{dx_1}{dt} = x_1^2 \quad (3)$$

$$\frac{dx_2}{dt} = -x_2 \quad (4)$$

[Perko, *Differential equations and dynamical systems*]




desmos
Untitled Graph
$c_1 = -0.3$
$c_2 = 1$


Non-linear systems in higher dimensions
$$\frac{dx_1}{dt} = x_1^2 \quad (3)$$
$$\frac{dx_2}{dt} = -x_2 \quad (4)$$
x_2
x_1
[Perko, *Differential equations and dynamical systems*]
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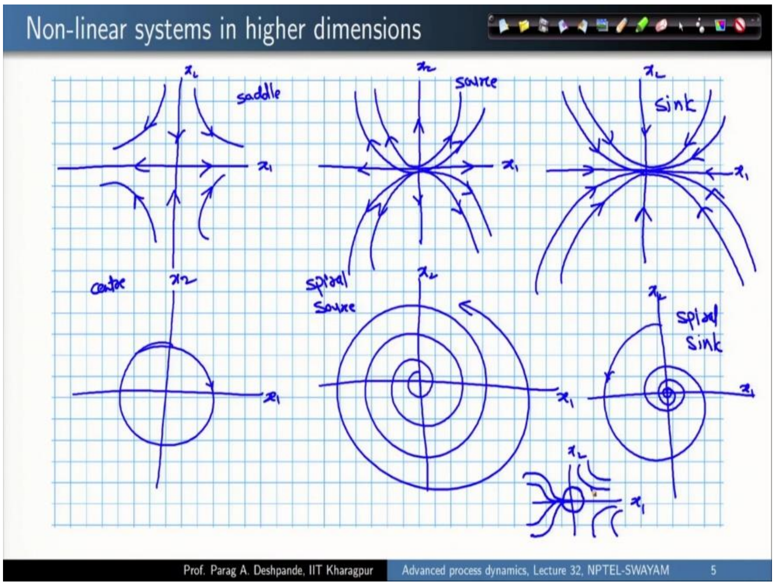
4


Non-linear systems in higher dimensions
saddle
source
sink
center
spiral source
spiral sink
x_1
x_2
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5

Hartman-Grobman theorem

The orbit structure of a dynamical system in the neighbourhood of a "hyperbolic" equilibrium point is topologically equivalent to the orbit structure of its linearised system

no eigenvalue will be "zero" or will have "zero" as the real part

[Perko, *Differential equations and dynamical systems*]

$$\frac{dx_1}{dt} = x_1^2 - x_2^2 - 1 \quad (5)$$

$$\frac{dx_2}{dt} = 2x_2 \quad (6)$$

$\frac{dx_1}{dt} = x_1^2 - x_2^2 - 1 = f_1$

$\frac{dx_2}{dt} = 2x_2 = f_2$

$f_1 = 0$ and $f_2 = 0$

$\Rightarrow x_1^2 - x_2^2 - 1 = 0$

and $2x_2 = 0$

$\Rightarrow x_2 = 0$

$x_1^2 - 1 = 0$

$\Rightarrow x_1 = \pm 1$

$\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

[Perko, *Differential equations and dynamical systems*]

$\frac{dx_1}{dt} = x_1^2 - x_2^2 - 1 = f_1$

$\frac{dx_2}{dt} = 2x_2 = f_2$

$\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{\partial f_1}{\partial x_1} = 2x_1 \quad ; \quad \frac{\partial f_1}{\partial x_2} = -2x_2$

$\frac{\partial f_2}{\partial x_1} = 0 \quad ; \quad \frac{\partial f_2}{\partial x_2} = 2$

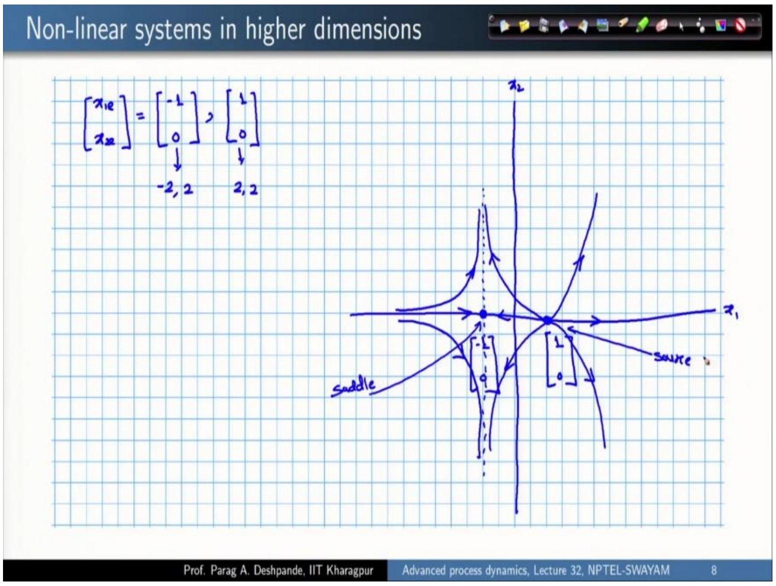
$\underline{\underline{J}} = \begin{bmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{bmatrix}$

$\underline{\underline{J}}\Big|_{\begin{bmatrix} -1 \\ 0 \end{bmatrix}} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

$\lambda = -2, 2$

$\underline{\underline{J}}\Big|_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\lambda = 2, 2$



Hello and welcome back. So, in the previous lecture, we said that we will start now the analysis of higher order nonlinear systems. One of the prominent examples, where you would find this in a physical example is the reactor dynamics, we will take this particular situation in the next three lectures, in fact, but before we go about analyzing a reactor system, which is an incredibly complex system, we first must equip ourselves with a basic understanding of higher order nonlinear systems, so that we can perhaps use those concepts in studying relatively complex problems.

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So, let us start our analysis, what we have here in front of us is second order system. So, I have a system of two equations,

$$\frac{dx_1}{dt} = -x_1 \dots \dots \dots (1)$$

$$\frac{dx_2}{dt} = x_1^2 + x_2 \dots \dots \dots (2)$$

We will try to analyze the system and see how what can we learn from the analysis of the system. So, not every system of equations, which is non-linear can be solved easily by sequentially solving individual equations, even if they are coupled. And therefore, linearization is something which we generally resort to for solving such systems.

But one question which arises is that can we always do linearization and analyze the system in linear domain and try to make conclusion about non linear domain will it always be possible, are there are some situations where we can get in fact, misleading results, this particular example and the next one will give us very good examples of when we can actually rely upon linearization and when we should not.

So, let us first try to solve this problem using our usual method, if at all it is possible to do so, so, the first equation has only x_1. So, I can write

$$\frac{dx_1}{dt} = -x_1$$

from where I can write

$$x_1 = c_1 e^{-t} \ldots\ldots\ldots (1)$$

So, solution of course, the question was not very difficult and now I will do this

$$\frac{dx_2}{dt} = c_1^2 e^{-2t} + x_2$$

and if this particular equation is not very difficult to be solved, what I can do is I can write here

$$\frac{dx_2}{dt} - x_2 = c_1^2 e^{-2t}$$

I can solve this equation using the method of integrating factor it has to be x_2. So, I can multiply both sides by e^{-t}.

So,

$$e^{-t}\frac{dx_2}{dt} - e^{-t}x_2 = c_1 e^{-3t}$$

from where I can write

$$\frac{d}{dt}(e^{-t}x_2) = c_1 e^{-3t}$$

and now, I will do an integration. So, I can write this is

$$e^{-t}x_2 = \left(-\frac{c_1}{3}\right)e^{-3t} + c_2$$

from where I can write

$$x_2 = \left(-\frac{c_1}{3}\right)e^{-2t} + c_2 e^t \ldots\ldots\ldots (2)$$

So, I have equation (1) and I have equation (2). So, now, for this particular example, it was possible to solve this equation easily and we explicitly got the answers for x_1 and x_2. So, let me write down the expressions. So, let me write down both the equation as well as the expressions.

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So, the equation is this

$$\frac{dx_1}{dt} = -x_1; \qquad \frac{dx_2}{dt} = x_1^2 + x_2$$

and my solutions were

$$x_1 = c_1 e^{-t}; \quad x_2 = \left(-\frac{c_1}{3}\right)e^{-2t} + c_2 e^t$$

These were the solutions. Now, can I develop the phase portrait of this system using these solutions should be fairly straightforward in the phase portrait what ......on one axis I should have x_1 .... on the other axis I should have x_2.

How can I know the curves on x_1 and x_2? I have the solution, so, I basically have two methods first method is pretty straightforward from here I can write

$$\frac{dx_2}{dx_1} = \frac{x_1^2 + x_2}{-x_1}$$

I will solve this equation again not very difficult equation to be solved and then I will get the solution for x_2 in terms of x_1 from that I can draw different curves. There is always a possibility there is another simpler solution I have x_1 in terms of t, I have x_2 in terms of t so, therefore, I can make a parametric plot to so as to get x_2 and x_1.

So, let me do let me adopt the second approach here I encourage you to solve $\frac{dx_2}{dx_1} = \frac{x_1^2+x_2}{-x_1}$ and plot the equation you should get the same answer. So, let us draw the parametric plot to get the equations to get the curves so, I have so, the way I would draw a parametric plot is this very simple $\left(c_1e^{-t},\ \left(-\frac{c_1}{3}\right)e^{-2t} + c_2e^{t}\right)$. So, now the plot is in front of you and what I can do is I can increase the parameter t from -100 to 100.

The plot is in front of you, one curve is in front of you, so, let me draw this curve and this curve would be this. Let me change the values of different parameters so as to get different curves, let me change c_1 and make it negative.

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Let us see what happens you get this curve, let me make another change and let us see what happens this is what happens. So, I get this curve, I get this curve this curve, I get this curve and so on.

So, this is fine this is the exact have we punched in let us cross check.

So, this is these are the various plots that I must get various curves that I must get.

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So, let me draw various curves from what I learned from the calculator I should get these curves and then as you would see from the solutions $x_1 = c_1e^{-t}$. So, therefore, it should have a conversion behavior along the x-axis. So, therefore, and then $x_2 = \left(-\frac{c_1}{3}\right)e^{-2t} + c_2e^{t}$. x_2 has e^{t} so, it should have a divergent behavior. So, in our previous terms it should have a saddle like behavior. So, I can draw the arrows like this it is not very difficult to see that this is how the phase portrait would look like. So, this is a saddle like behavior.

And now, this phase portrait was obtained without explicitly solving the nonlinear equation let me do one thing let me linearize this equation. So, I know that (0, 0) is a solution is an equilibrium solution to this system of equations. So, therefore, in close proximity of (0, 0), my x_1^2 will have a very small value. So, if $x_1 - x_e$ is a small quantity, $(x_1 - x_e)^2$ will be an even smaller quantity. So, therefore, one way of linearization is so, I have linearized model here and I can write the linearized model as this

$$\frac{dx_1}{dt} = -x_1$$

This equation was already linear. So, nothing problematic here the problem is that the other equation was nonlinear.

So, I make this other equation a linear by simply setting

$$x_1^2 = 0$$

and why am I doing this I can do this only because (0, 0) is my equilibrium solution and in close proximity of (0, 0), $(x_1 - x_e)^2$ will be a negligible quantity. So, if I write this as

$$\frac{dx_2}{dt} = x_2$$

and our linear model, so, I can write this as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and the solution is very straightforward. I know that the solution is going to be

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have seen this over and again.

So, for the linearized case, how would the phase portrait look like? Well, the phase portrait is very simple. This is x_1, this is x_2...... this would be the divergent axis, this would be the convergent axis and beyond this, you will have the saddle solution. Please refer to our previous lectures to see how we got this solution or this phase portrait and what kind of phase portrait is this, this is a saddle phase portrait.

So, therefore, about very close to (0,0), both the phase portraits give qualitatively similar behavior at least qualitatively the behavior is very similar, both of them show saddle solutions. Let us now take another example.

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Now, I have another case where you have

$$\frac{dx_1}{dt} = x_1^2; \;\; \frac{dx_2}{dt} = -x_2$$

The solution would be

$$x_1 = \frac{1}{c_1 - t} \; ; \; x_2 = c_2 e^{-t}$$

So, very simple equations the solutions are in front of you. So, now, can I develop and now, what would happen to the linearized form, the linearized form would be

$$\frac{dx_1}{dt} = 0; \;\; \frac{dx_2}{dt} = -x_2$$

So, my linearized equation would be

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Where,

$$\lambda_1 = 0 \; ; \; \lambda_2 = -1$$

Let us now compare the behaviors of these two cases.

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So, my first nonlinear dynamical solution is this

$$x_1 = \frac{1}{c_1 - t}\ ;\ x_2 = c_2 e^{-t}$$

Let me draw this. So, I will draw the parametric equation $\frac{1}{c_1 - t}$, $c_2 e^{-t}$. Let me increase this from -100 to 100. So, the plot what we get here is the decaying behavior asymptotically reaches just to 1 and let me change this, well qualitatively nothing changes now it gets inverted. So, let me draw this phase portrait here.

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The phase portrait would look like this, this is x_1, this is x_2 so, the phase portraits, the phase portrait is this. Now, this is strange because we have never come across any situation till now for the analysis of linear systems where the phase portrait would look like this is very different. How is this different?

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Let us see what are the different phase portraits which we came across? Well, we came across this. Saddle x_1........... x_2. we came across this, source x_1........x_2, depending upon the relative magnitudes of λ_1 and λ_2. The orientation will change so, this is saddle this is source. Then you have sink x_1........ x_2. All three for real eigenvalues, then we have we had centre x_1......x_2. We had spiral source x_1.......... x_2 and finally, spiral sink x_1......x_2 and compare it against this particular phase portrait which we caught in this study x_1........ x_2. There is no way that you can approximate this phase portrait to any one of the six phase portraits that you studied previously, which means that close to this (0,0), solution we are concerned about we anyway know that you can make approximation only close to (0,0), the equilibrium solution.

But even close to (0,0), the equilibrium solution the behavior is very different and why does this happen?

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This happens this happened quite simply because in this particular case, one of your eigenvalues was zero.

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So, we have Hartman-Grobman theorem, which will give us an idea about this. So, according to the Hartman-Grobman theorem, the orbit structure so, let us first try to understand what is the meaning of orbit structure. Orbit means evolution in a very general sense. So, in a discrete system for example, if you start with a point and the dynamical system is a system in which the variable changes with time, so, that particular point would keep on evolving in time. So, that would result in an orbit, an orbit structure means the arrangement of different phase lines in a particular region.

So, the orbit structure of a dynamical system in the neighborhood of a hyperbolic equilibrium point, so, we know the meaning of equilibrium point, but now, we have a new term hyperbolic equilibrium point we will come to this definition a little later. So, in the neighborhood of a special type of equilibrium point known as hyperbolic equilibrium point is topologically equivalent. Topologically equivalent means the geometrical features of the arrangement of the phase lines. So, it would be topologically equivalent to the orbit structure of the linearized model linearized system.

So, what is the meaning of this? So, what it means is that, if you take a non-linear system and imagine that you have some way to determine its complete phase portrait then you would be drawing the phase portrait. So, that phase portrait will be a collection of phase lines and since it is a collection of phase lines, there would be arrangement of these phase lines around the equilibrium point. And then what you would do, you would linearize the system and again determine the topological features which means you will determine the orbit structure which means you will determine the arrangement of phase lines close to the equilibrium point near the for a linearized system.

Now, if the equilibrium point is hyperbolic then according to Hartman-Grobman theorem, the topological features would be equivalent, which means, the nature the dynamical nature would be similar. Now, the only thing which is left is to understand what is the meaning of hyperbolic here. Hyperbolic means no eigenvalue will be zero this is important. If there is even one eigenvalue, which is zero or no eigenvalue will be zero or will have zero as the real part the eigenvalues can be either complex or real. So, in cases where the eigenvalues are all real, no eigenvalue should be zero and if the eigenvalues are complex then the real part of none of the eigenvalues can be zero.

And if that is the case then the equilibrium point is called a hyperbolic equilibrium point. So, how would you then assess whether it would be possible for you to linearize the system and have a guarantee that the linearized system will indeed have the same features as the nonlinear system while you will determine the equilibrium point you will determine the eigenvalues and with the help of the eigenvalues if the none of the eigenvalues are zero or zero is the real part, then you will declare that there was equilibrium point as hyperbolic.

And if the system has hyperbolic equilibrium points, then you can say, then you can assure yourself that you in fact have a system where linearization would work. Now, in the previous examples, you could trivially do linearization by setting the non-linear part on the right-hand side, zero and that was possible because you had the equilibrium points as (0,0) and so on. But what will happen when you do not have the equilibrium point (0,0) or equilibrium solution at origin, then you will have to do a proper Taylor series expansion for linearization. Let us see that.

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I have

$$\frac{dx_1}{dt} = x_1^2 - x_2^2 - 1 = f_1$$

and

$$\frac{dx_2}{dt} = 2x_2 = f_2$$

What would be the equilibrium solution? Equilibrium solution would be obtained by setting up

$$f_1 = 0 \quad \& \quad f_2 = 0$$

So, I can determine the equilibrium point as

$$x_1^2 - x_2^2 - 1 = 0 \quad \& \quad 2x_2 = 0$$

from where I get

$$x_2 = 0 \quad \& \quad x_1 = \pm 1$$

So, what would be x_{1e} and x_{2e} ?

$$\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Both of them are equilibrium solutions and they are known not $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

In this case, you cannot simply do this linearization trivially by setting up the square terms to zero. And therefore, we would determine the linearized model by following the proper Taylor series expansion.

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So, let me see if we can do that. So,

$$\frac{dx_1}{dt} = x_1^2 - x_2^2 - 1 = f_1$$

$$\frac{dx_2}{dt} = 2x_2 = f_2$$

and

$$\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

I will determine the Jacobian. So,

$$\frac{\partial f_1}{\partial x_1} = 2x_1 \; ; \; \frac{\partial f_1}{\partial x_2} = -2x_2$$

$$\frac{\partial f_2}{\partial x_1} = 0 \; ; \; \frac{\partial f_2}{\partial x_2} = 2$$

So, therefore, my Jacobian matrix is the general Jacobian is

$$\underline{\underline{J}} = \begin{bmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{bmatrix}$$

So, in order to get the idea about the solutions what I will do is I will determine this Jacobian at $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and this is going to be

$$\underline{\underline{J}} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \quad ; at \; \begin{bmatrix} -1 \\ 0 \end{bmatrix} \; \& \; \lambda = -2, 2$$

which have the eigenvalues

For the second Jacobian, Jacobian at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$$\underline{\underline{J}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad ; at \; \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; \& \; \lambda = 2, 2$$

So, from these eigenvalues, I can have an idea about the nature of the solutions, the nature of the linearized solutions now, would linearization work in this case, yes linearization would work because none of the eigenvalues are zero. So, I can expect the linearization to work and finally, can I get an idea about the phase portrait?

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Well, I can, so, to develop the phase portrait what I would do is I know that my equilibrium solutions and corresponding eigenvalues are

$$\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}; \ \lambda = -2, 2 \quad \& \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \ \lambda = 2, 2$$

So, let me draw x_2 here........ x_1 here. Now, my first equilibrium solution is $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. So, this is $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, my second equilibrium solution is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, this is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Now, $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is associated with two eigenvalues one of which is positive the other one is negative. So, it should have a saddle behavior about that point and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is associated with both positive eigenvalues there should for it should have a source behavior about that point.

So, how can I draw phase lines which would both confirm to this particular situation? Let me see, can I draw this. So, let me do this what if I drew curves like this. Now what is left? What is left is to draw the arrows, so, since this is a source, this would go up, this would go up and from here since it is a saddle, it should go like this. So, since this $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a source, you should have corresponding arrow here, corresponding arrow here and following the arrows of the leftmost portion, you should have the arrow here. And again, you will see here that this is a saddle, all the arrows confirmed to a saddle behavior and this is a source all the arrows conform to the source behavior.

So, this is how we will handle higher order nonlinear dynamics, what you do is the first thing which you establish is the equilibrium solutions, the equilibrium solutions are arrived by setting up the individual equations to zero. Now, in most cases since, it is not possible to use the nonlinear model directly, you would like to linearize the system. How can you be confident that your linearized behavior is similar to the behavior of nonlinear system? Well, what you do is you would establish whether the equilibrium solutions are hyperbolic or not.

So, following the Hartman-Grobman theorem, if the sorry if the point equilibrium points are hyperbolic, which means that none of your eigenvalues are zero, then you can in fact use linearized phase portraits and the orbits would be similar to those of a nonlinear system. The method which we develop today will be used from tomorrow onwards to analyze the system of reactors in case of transient operation of reactors. Till then goodbye.