Advanced Process Dynamics

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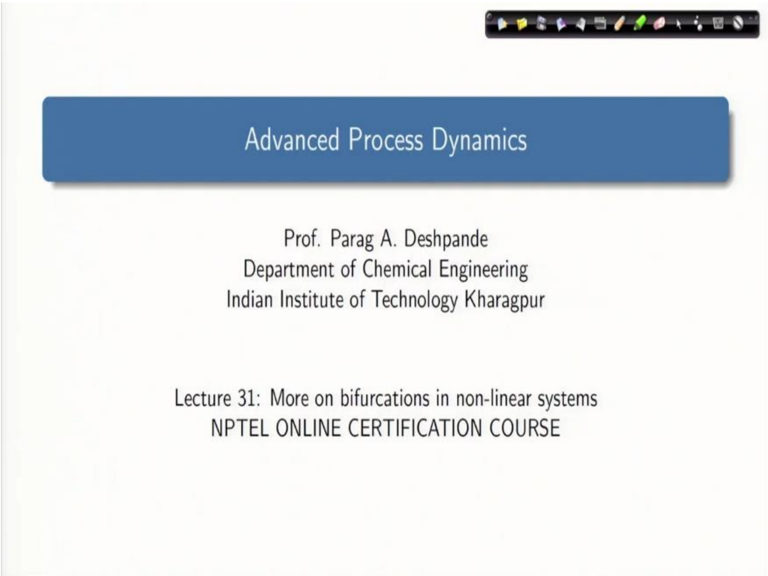
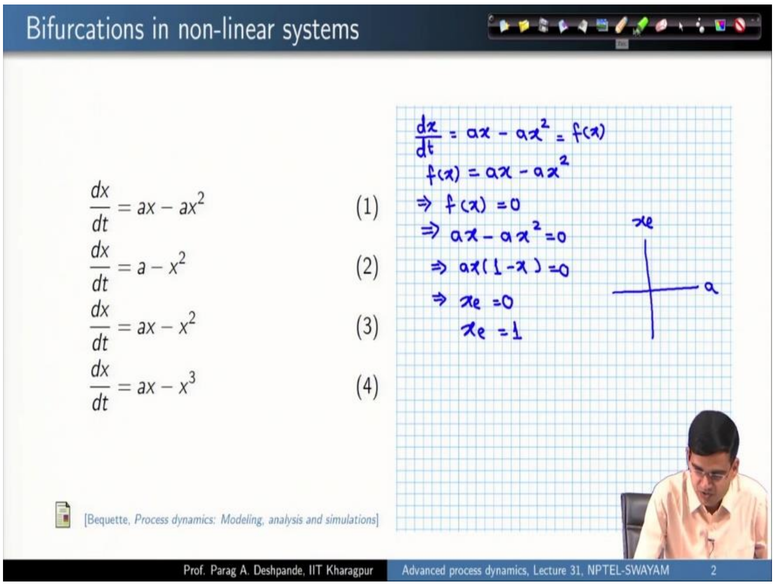
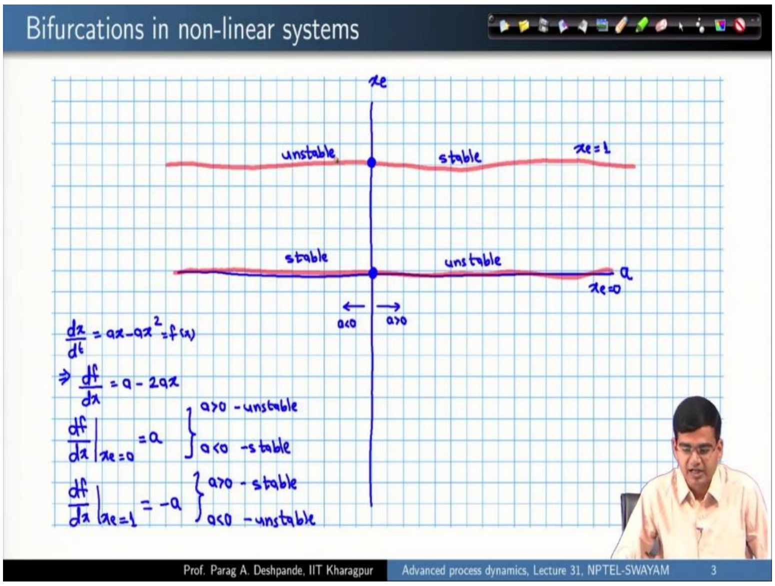
Indian Institute of Technology, Kharagpur

Lecture 31

More on bifurcations in non-linear systems






Bifurcations in non-linear systems
xe
unstable
stable
xe=1
stable
unstable
a
xe=0
a<0
a>0
$\frac{dx}{dt} = ax - ax^2 = f(x)$
$\Rightarrow \frac{df}{dx} = a - 2ax$
$\left.\frac{df}{dx}\right|_{x_e=0} = a$ a>0 - unstable, a<0 - stable
$\left.\frac{df}{dx}\right|_{x_e=1} = -a$ a>0 - stable, a<0 - unstable
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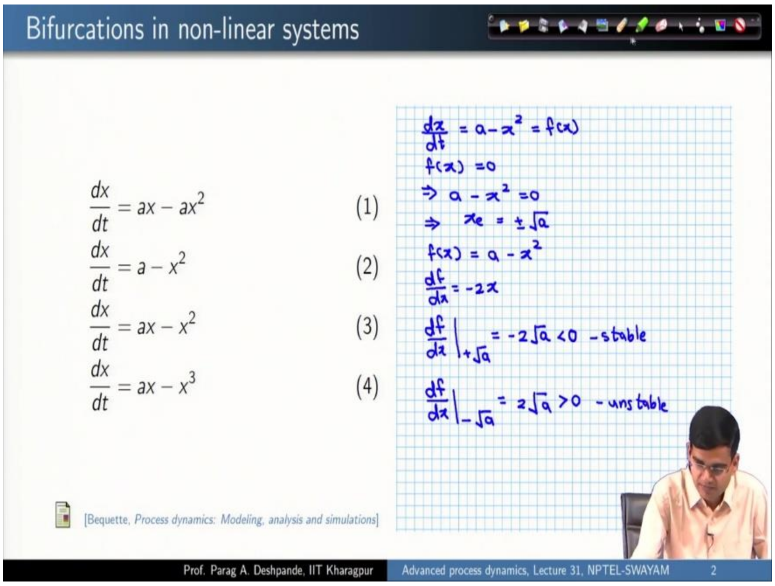
3


Bifurcations in non-linear systems
$\frac{dx}{dt} = ax - ax^2$ (1)
$\frac{dx}{dt} = a - x^2$ (2)
$\frac{dx}{dt} = ax - x^2$ (3)
$\frac{dx}{dt} = ax - x^3$ (4)
[Bequette, Process dynamics: Modeling, analysis and simulations]
$\frac{dx}{dt} = a - x^2 = f(x)$
$f(x) = 0$
$\Rightarrow a - x^2 = 0$
$\Rightarrow x_e = \pm\sqrt{a}$
$f(x) = a - x^2$
$\frac{df}{dx} = -2x$
$\left.\frac{df}{dx}\right|_{+\sqrt{a}} = -2\sqrt{a} < 0$ - stable
$\left.\frac{df}{dx}\right|_{-\sqrt{a}} = 2\sqrt{a} > 0$ - unstable
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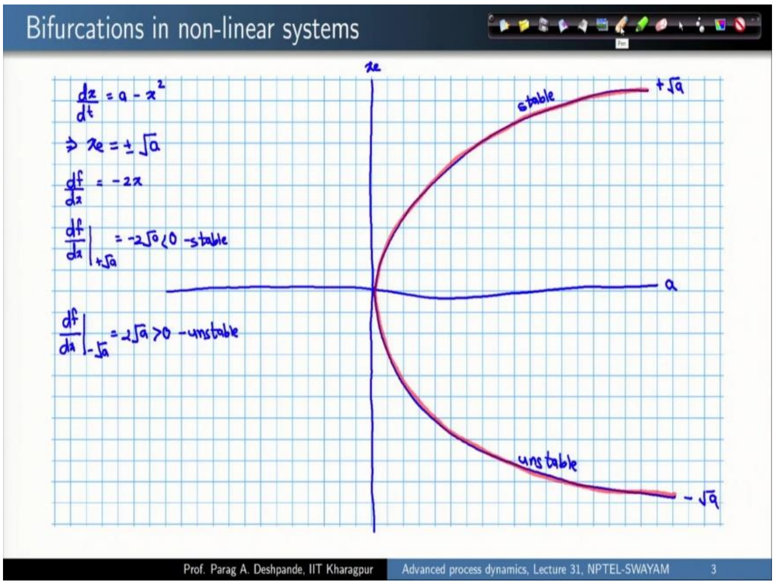
2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$\Rightarrow x_e = \pm\sqrt{a}$
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$\left.\frac{df}{dx}\right|_{-\sqrt{a}} = 2\sqrt{a} > 0$ - unstable
stable
$+\sqrt{a}$
a
unstable
$-\sqrt{a}$
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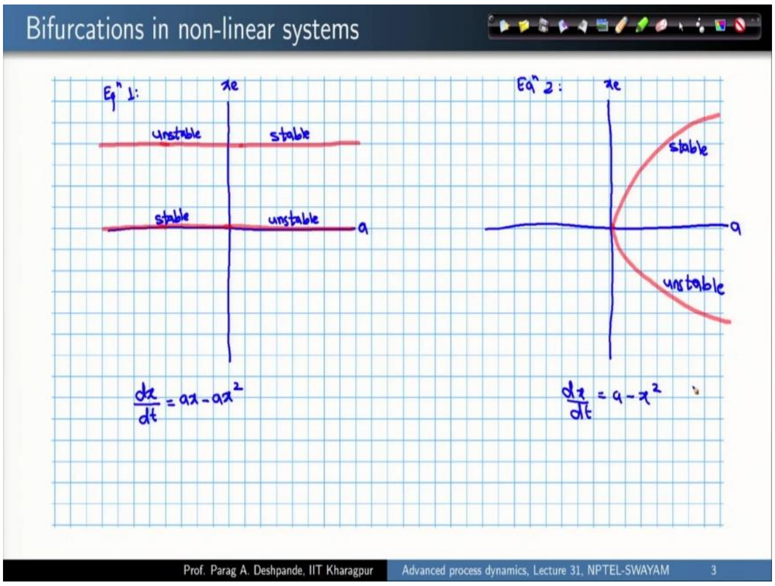
3


Bifurcations in non-linear systems
Eqn 1:
xe
unstable
stable
stable
unstable
a
$\frac{dx}{dt} = ax - ax^2$
Eqn 2:
xe
stable
a
unstable
$\frac{dx}{dt} = a - x^2$
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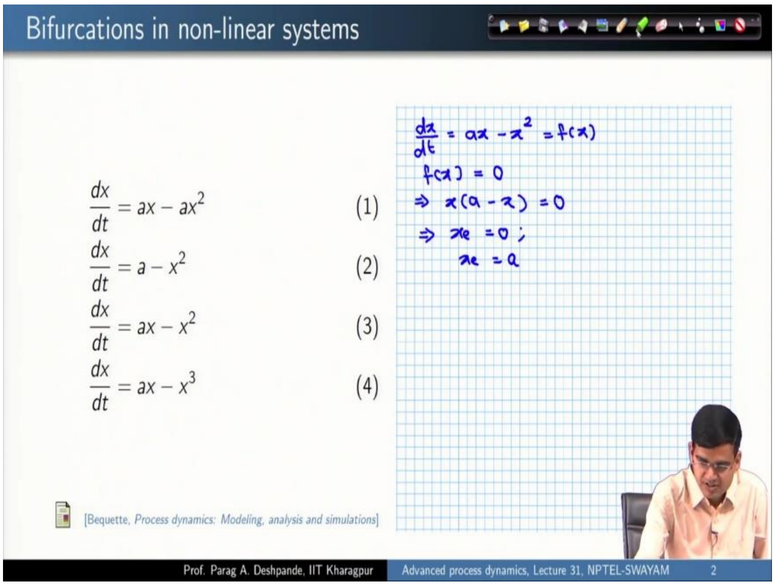
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[Bequette, Process dynamics: Modeling, analysis and simulations]
$\frac{dx}{dt} = ax - x^2 = f(x)$
$f(x) = 0$
$\Rightarrow x(a - x) = 0$
$\Rightarrow x_e = 0;$
$x_e = a$
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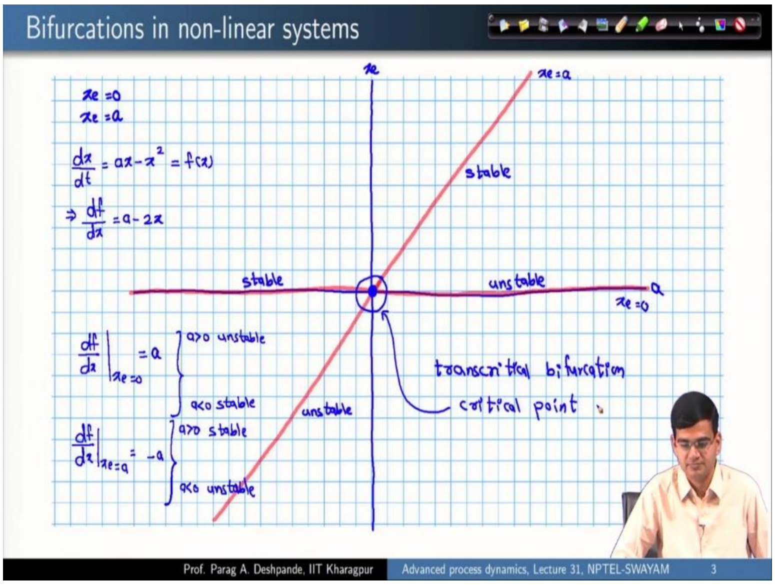
2


Bifurcations in non-linear systems
$x_e = 0$
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$\frac{dx}{dt} = ax - x^2 = f(x)$
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$\left.\frac{df}{dx}\right|_{x_e=0} = a$
a>0 unstable
a<0 stable
$\left.\frac{df}{dx}\right|_{x_e=a} = -a$
a>0 stable
a<0 unstable
xe
xe = a
stable
stable
unstable
a
xe = 0
unstable
transcritical bifurcation
critical point
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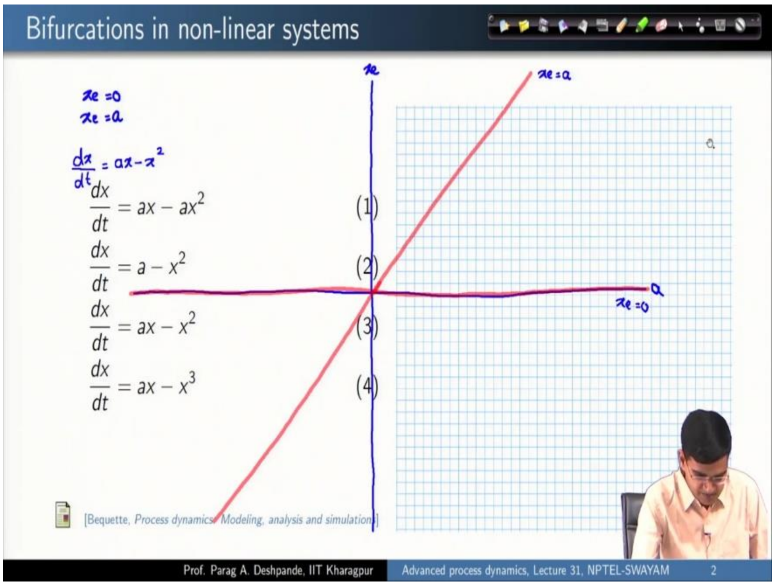
3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$\frac{dx}{dt} = a - x^2$ (2)
$\frac{dx}{dt} = ax - x^2$ (3)
$\frac{dx}{dt} = ax - x^3$ (4)
x_e
$x_e = a$
a
$x_e = 0$
[Bequette, Process dynamics: Modeling, analysis and simulation]
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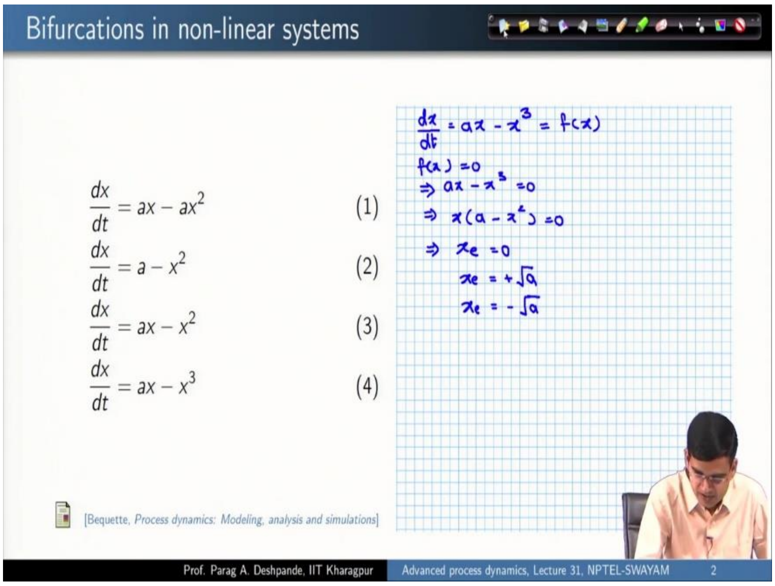
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$x_e = +\sqrt{a}$
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[Bequette, Process dynamics: Modeling, analysis and simulation]
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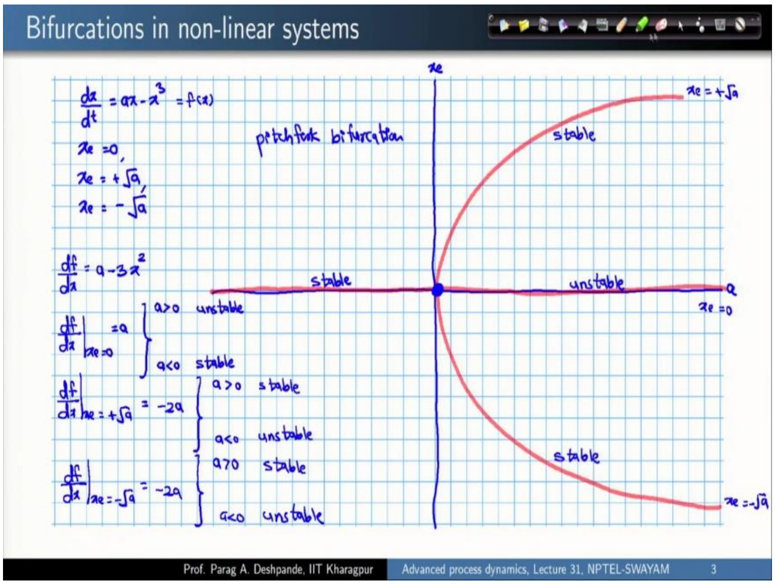
Bifurcations in non-linear systems
$\frac{dx}{dt} = ax - x^3 = f(x)$
$x_e = 0$,
$x_e = +\sqrt{a}$,
$x_e = -\sqrt{a}$
pitchfork bifurcation
$\frac{df}{dx} = a - 3x^2$
$\left.\frac{df}{dx}\right|_{x_e=0} = a$ $a>0$ unstable, $a<0$ stable
$\left.\frac{df}{dx}\right|_{x_e=+\sqrt{a}} = -2a$ $a>0$ stable, $a<0$ unstable
$\left.\frac{df}{dx}\right|_{x_e=-\sqrt{a}} = -2a$ $a>0$ stable, $a<0$ unstable
x_e
$x_e = +\sqrt{a}$
stable
stable
unstable
a
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stable
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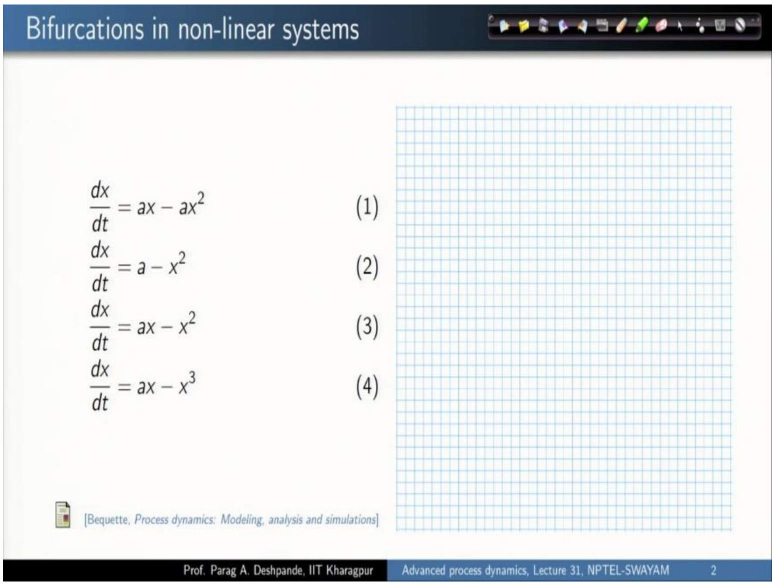

Welcome back. We are currently studying nonlinear dynamics in this course on advanced dynamical systems.

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In the past few lectures, we came across the concept of bifurcation diagrams. What bifurcation diagrams tell us is how the bifurcation in the system works, what are the different values of the parameters under which the system can show stability or lack of stability. We will take up this concept in a little more detail today. And we will see how are seemingly very similar looking dynamical equations can behave very differently and how we can analyze the same using bifurcation diagrams. So, let us look into some equations which we have in front of us.

(Refer Slide Time: 01:14)

The four equations which you see here are four nonlinear dynamical equations where the right-hand side of these equations are different but they are very similar.

$$\frac{dx}{dt} = ax - ax^2 \dots\dots\dots (1)$$

$$\frac{dx}{dt} = a - x^2 \dots\dots\dots (2)$$

$$\frac{dx}{dt} = ax - x^2 \dots\dots\dots (3)$$

$$\frac{dx}{dt} = ax - x^3 \dots\dots\dots (4)$$

So, they look very similar, but we will see very soon how they are very different in terms of their dynamical behavior. So, let us look into the first equation. So, the equation is

$$\frac{dx}{dt} = ax - ax^2$$

Have we come across this equation before?

Well, this equation is our logistic equation for population growth, except that the carrying capacity parameter N has been set to one here. So, we know the behavior of this equation very well. So, in order to set up the procedure for following equations, equations (2), (3) and (4), let us first analyze this equation, so that we know the steps very well. So, the first step that we will do is we would set this as f(x). So,

$$f(x) = ax - ax^2$$

The bifurcation diagram is the diagram on which you are on the x-axis you have the bifurcation parameter and, on the y-axis, you have the equilibrium solution.

So, we need to determine the equilibrium solutions of this and this can be obtained by setting

$$f(x) = 0$$

which means

$$ax - ax^2 = 0$$

which means

$$ax(1 - x) = 0$$

which means I have two equilibrium solutions for my case,

$$x_e = 0 \, ; x_e = 1$$

So, what would be the axis for my bifurcation diagram, the axis would be simply a versus x_e. The information that would go in this bifurcation diagram would be that $x_e = 0$ and $x_e = 1$. So, let us draw in detail the bifurcation diagram.

(Refer Slide Time: 04:21)

So, I have axis as the bifurcation parameter a, equilibrium parameter, the equilibrium solution x_e, so I have two equilibrium solutions. $x_e = 0$, this is the first equilibrium solution, then I have $x_e = 1$, this is the second equilibrium solution. So, let me write here $x_e = 0$ and $x_e = 1$. Now, the question which is in front of me is that on what parts of this diagram is the state system stable and where is the system unstable. To do that, I will need to look into the derivative conditions.

$$\frac{dx}{dt} = ax - ax^2 = f(x)$$

from where I can write

$$\frac{df}{dx} = a - 2ax$$

and this $\left.\frac{df}{dx}\right|_{x_e}$ has to be determined.

So, therefore,

$$\left.\frac{df}{dx}\right|_{x_e=0} = a, \qquad a > 0, unstable \; ; \; a < 0, stable$$

$$\left.\frac{df}{dx}\right|_{x_e=1} = -a, \qquad a > 0, stable \; ; \; a < 0, unstable$$

So, let me identify these focuses on this diagram, I have here this point and towards the right of this point I have a > 0 and left of this point a < 0. So, for $x_e = 0$ and a > 0, I have this part is unstable and this part is stable. Similarly, this point now x = 1 for a > 0, I have stable you can see here, so, therefore, this is stable and this is unstable. So, this is the bifurcation diagram, which I get for $\frac{dx}{dt} = ax - ax^2$ which is the logistic population growth model which we have studied previously. Now, in a similar manner, can we develop the bifurcation diagram for other systems?

(Refer Slide Time: 8:33)

Let us look into that. So, now my equation is

$$\frac{dx}{dt} = a - x^2 = f(x)$$

So, how would I determine the equilibrium solutions? I will set

$$f(x) = 0$$

which means

$$a - x^2 = 0$$

from where I get

$$x_e = \pm\sqrt{a}$$

So, my bifurcation plot will have the curves $\pm\sqrt{a}$. Now, I need to determine the stabilities.

So, let me do one thing, let me write here

$$f(x) = a - x^2$$

where I get

$$\frac{df}{dx} = -2x$$

so, from here I can make this as

$$\left.\frac{df}{dx}\right|_{x_e=+\sqrt{a}} = -2\sqrt{a} < 0, stable$$

and

$$\left.\frac{df}{dx}\right|_{x_e=-\sqrt{a}} = 2\sqrt{a} > 0, unstable$$

So, how would I use this information to develop the bifurcation plot.

(Refer Slide Time: 11:21)

So, my equation is $\frac{dx}{dt} = a - x^2$, from where I get $x_e = \pm\sqrt{a}$. So, on this curve on this plane will draw the axis as a and x_e, $x_e = +\sqrt{a}$ would look like this. So, this is $x_e = +\sqrt{a}$ and $x_e = -\sqrt{a}$ would look like this, this is $-\sqrt{a}$. Now, I need to determine whether these two parts are stable or unstable. So, what did I get from the previous analysis. I got a $\frac{df}{dx} = -2x$. So, therefore, $\left.\frac{df}{dx}\right|_{x_e=+\sqrt{a}} = -2\sqrt{a < 0}$ which means stable and I can write this here stable.

And then $\left.\frac{df}{dx}\right|_{x_e=+\sqrt{a}} = -2\sqrt{a}$ which means unstable and this means this is unstable. Now, to differentiate the curves on this plane from the axis themselves let me draw these lines using these red highlighters. This will simply show that your axis themselves are not the equilibrium solutions because in certain cases your axis themselves maybe equilibrium solutions, so, to tell that apart, we have used a red curve for representing the equilibrium solutions. So, now, in the previous case what you saw was that the bifurcation diagram looked like this.

(Refer Slide Time: 11:21)

In the previous case equation (1), the bifurcation diagram a versus x_e and equation (2), the bifurcation diagram a versus x_e, they looked like this. In the previous case, you had these curves and now you have these curves. We can quickly write stability and like thereof here we have unstable, here stable, here stable, here unstable four parts and here stable and unstable and how did the equations look like here you have $\frac{dx}{dt} = ax - ax^2$ and here you have $\frac{dx}{dt} = a - x^2$. We made a small change in the equation and what you found was that the bifurcation behavior completely changed.

(Refer Slide Time: 16:24)

We can further appreciate this by looking at other equations. So, let us look at equation number (3). Now, the equation is

$$\frac{dx}{dt} = ax - x^2$$

So,

$$\frac{dx}{dt} = ax - x^2 = f(x)$$

I can determine the equilibrium solution by setting

$$f(x) = 0$$

which means

$$ax - x^2 = 0$$

So, from here I get

$$x_e = 0\ ;\ x_e = a$$

Let me draw these curves.

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So, I have $x_e = 0$ and $x_e = a$, these are the two equilibrium solutions, let me draw the axis for the plane I have a.... I have x_e........so, what would be the first curve that would correspond to $x_e = 0$ and this is what I was referring to that I have the curve which is the same as your axis. So, this is my first solution. So, this is $x_e = 0$ and then the second solution second equation is

$x_e = a$. Now, x_e is your y-axis and a is your x-axis. So, this is basically the equation y = x, so, $x_e = a$ is the equation of the straight-line y = x. so, this is going to be your curve and this is $x_e = a$.

Now, let me analyze different parts of these two lines and analyze whether the which part is stable and which part is unstable. So, my equation was

$$\frac{dx}{dt} = ax - x^2$$

So, then what I will do is I will assign this as f(x)

$$\frac{dx}{dt} = ax - x^2 = f(x)$$

from where I get

$$\frac{df}{dx} = a - 2x$$

and this has to be determined at x = 0 and x = a. So, therefore,

$$\left.\frac{df}{dx}\right|_{x_e=0} = a, \qquad a < 0, stable \;\&\; a > 0, unstable$$

Then I will determine

$$\left.\frac{df}{dx}\right|_{x_e=a} = -a, \qquad a > 0, stable \;\&\; a < 0, unstable$$

so, now, I can determine various parts of my bifurcation diagram and determine whether they are stable or unstable. So, let me see here I have $\left.\frac{df}{dx}\right|_{x_e=0} = a$, which means this horizontal line and for a > 0, this is unstable. So, this means this is going to be unstable and for a < 0, this is going to be stable.

So, about this point I determine the stability, what about the x = y line or x_e = a line, for a > 0, I have stable which means this part the upper part is stable and a < 0 is unstable. So, this part is the unstable part and what I see is that as I change the value of a in this particular bifurcation diagram I go if I start from stability, I pass through a point beyond which my system becomes unstable or if I start with unstable point, I reach a particular point beyond which my system becomes stable. So, you go from stable to unstable, stable to unstable or unstable to stable, unstable to stable passing through one specific point and therefore, the bifurcation diagrams of this nature are called transcritical and the bifurcation is called transcritical bifurcation.

So, the bifurcation diagram shown here is called transcritical bifurcation and what would you call that particular point where this point where these two curves meet where you go from you go from stable to unstable, unstable to stable and so on. You would call this point as critical point. And again, you would see that you changed the nonlinear equation a little bit and what happened was that you changed your pacification diagram altogether, the features the curves on the bifurcation diagram changed. Let us look into the final equation.

(Refer Slide Time: 24:45)

So, the equation now is

$$\frac{dx}{dt} = ax - x^3$$

this is a cubic equation so, therefore, you may expect three equilibrium solutions. Let us see, so let us assign this as

$$\frac{dx}{dt} = ax - x^3 = f(x)$$

Then,

$$f(x) = 0$$

Means

$$ax - x^3 = 0$$

which means

$$x(a - x^2) = 0$$

So, now, if this is the case, then I get

$$x_e = 0\,, x_e = +\sqrt{a}\,, x_e = -\sqrt{a}$$

So, before I decided anything about the stability, let us first draw the various portions of the bifurcation plot itself.

(Refer Slide Time: 26:14)

So,

$$\frac{dx}{dt} = ax - x^3 \; ; \; x_e = 0 \, , x_e = +\sqrt{a} \, , x_e = -\sqrt{a}$$

So, I will draw the curves, I will draw the draw the first a plane the plane is this I have a.... I have x_e. So, my first solution $x_e = 0$, my second solution is $x_e = +\sqrt{a}$ and my third solution is $x_e = -\sqrt{a}$.

Now, time to identify, the stability and instability of various parts of this plane. So, if I assign

$$\frac{dx}{dt} = ax - x^3 = f(x)$$

then I can write

$$\frac{df}{dx} = a - 3x^2$$

So, therefore,

$$\left.\frac{df}{dx}\right|_{x_e=0} = a, \qquad a < 0, stable \;\&\; a > 0, unstable$$

Now,

$$\left.\frac{df}{dx}\right|_{x_e=+\sqrt{a}} = -2a, \quad a > 0, stable \;\&\; a < 0, unstable$$

Finally,

$$\left.\frac{df}{dx}\right|_{x_e=-\sqrt{a}} = -2a, \qquad a > 0, stable \;\&\; a < 0, unstable$$

So, time to mark the plane here. First the equilibrium solution $x_e = 0$, for a > 0, I have unstable part. So, this is unstable. a < 0, I have stable so, this is stable. Now for the upper curve $\left.\frac{df}{dx}\right|_{x_e=+\sqrt{a}}$ the curve exists only in the first quadrant there is nothing in the -a. So, therefore, the only curve which is present here is the stable one, this is stable. You can see from $\left.\frac{df}{dx}\right|_{x_e=-\sqrt{a}} = -2a$, and a > 0 and this and same is the case, for the lower curve this is also stable and all these three points meet here.

And to mathematicians when they looked at this diagram it reminded them of one specific tool which is used in farms and let us call a pitchfork these lines which are highlighted in red, they look like a pitchfork and therefore, this bifurcation is called pitchfork bifurcation. And this again has very different characteristics as we saw in this particular case, we service all the other cases.

(Refer Slide Time: 26:14)

So, therefore, when you compare equations (1), (2), (3) and (4), what you see is that they look similar, they are not same obviously and if they are not same, the dynamical behavior may not be expected to be the same obviously, but they are very similar and small changes in the parameters or the variables or their powers result in drastic changes in the bifurcation behavior of the system.

So, we will stop here today, appreciating that bifurcations in nonlinear systems can show very, very different behaviors depending upon the functional form of the function which you have in the equation. And we have taken a lot of examples of different first order nonlinear systems. Now, I believe it is a good time to go to higher order systems. So, in the next lecture, we will start with higher order nonlinear dynamical systems. Thank you.