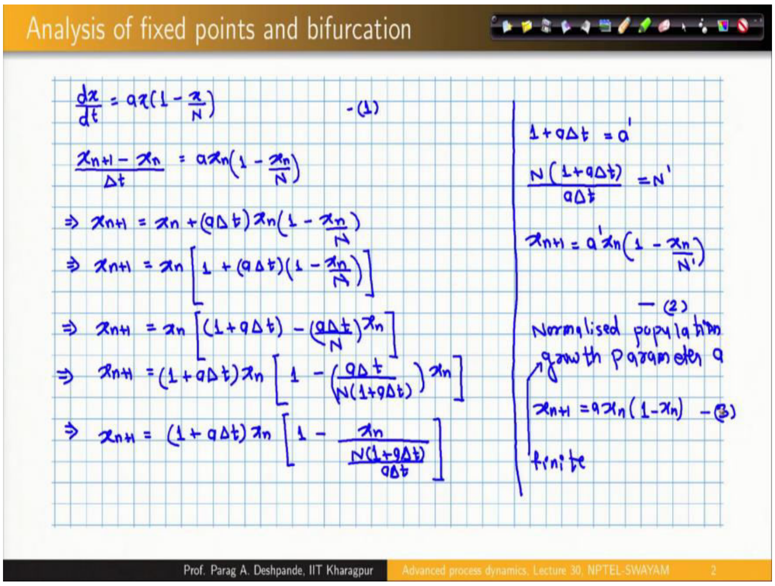
Advanced Process Dynamics
Professor Parag A. Deshpande

Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture 30
Analysis of fixed points and bifurcation in discrete domain continued...



$$x_{n+1} = a x_n (1 - x_n) \quad -(1)$$

$$f(x) = a x (1 - x)$$

$$f(x) = x$$

$$\Rightarrow a x (1 - x) = x$$

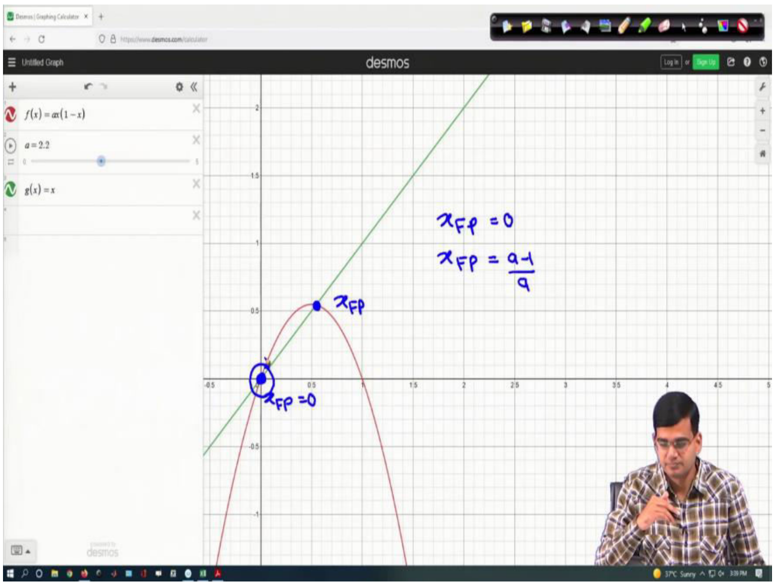
$$\Rightarrow x \{ a(1-x) - 1 \} = 0$$

$$\Rightarrow \boxed{x_{FP} = 0}$$

$$a(1 - x_{FP}) - 1 = 0$$

$$\Rightarrow 1 - x_{FP} = \frac{1}{a}$$

$$\Rightarrow \boxed{x_{FP} = \frac{a-1}{a}}$$



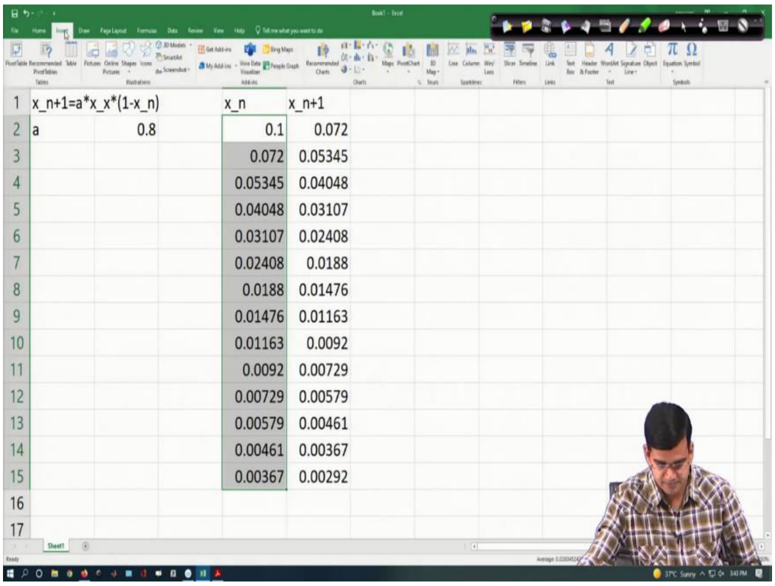
$$x_{n+1} = a x_n (1 - x_n)$$

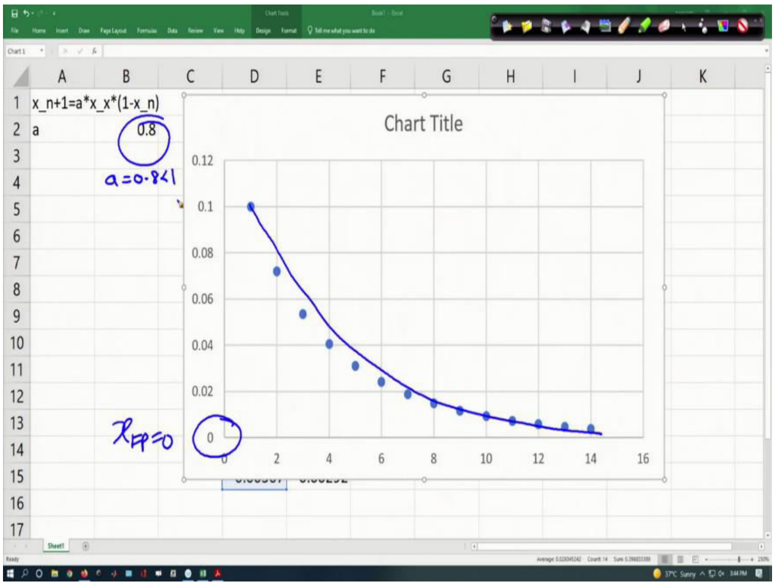
$$f(x) = a x (1 - x) = a x - a x^2$$

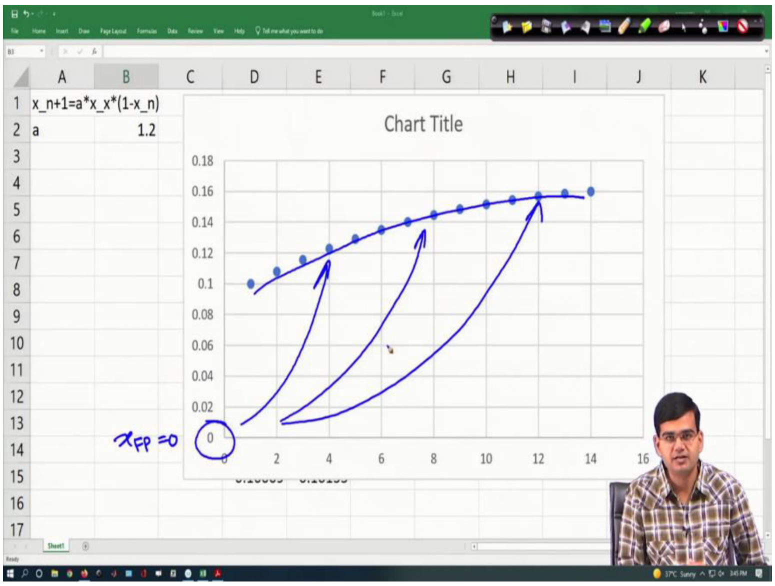
$$x_{FP} = 0 \; ; \; x_{FP} = \frac{a-1}{a}$$

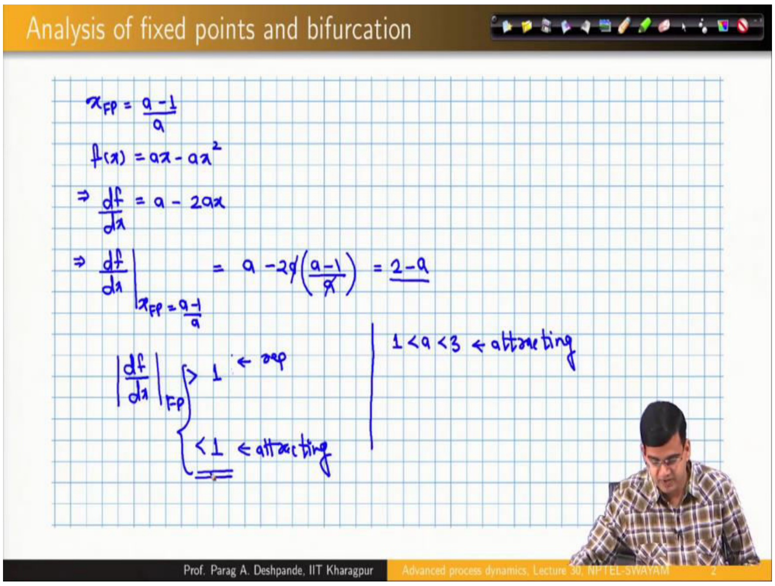
$$\frac{df}{dx} = a - 2 a x$$

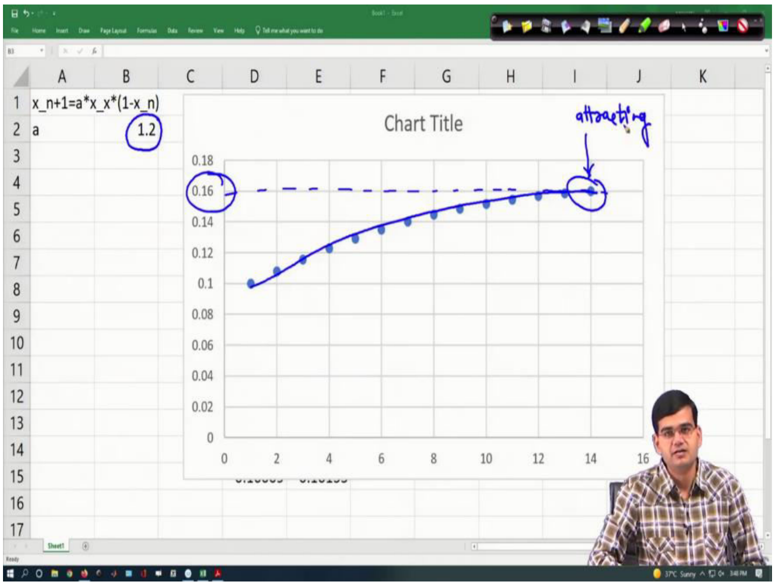
$$\left.\frac{df}{dx}\right|_{x_{FP}=0} = a \; \begin{cases} a < 1 & - \text{attracting} \\ a > 1 & - \text{repelling} \end{cases}$$

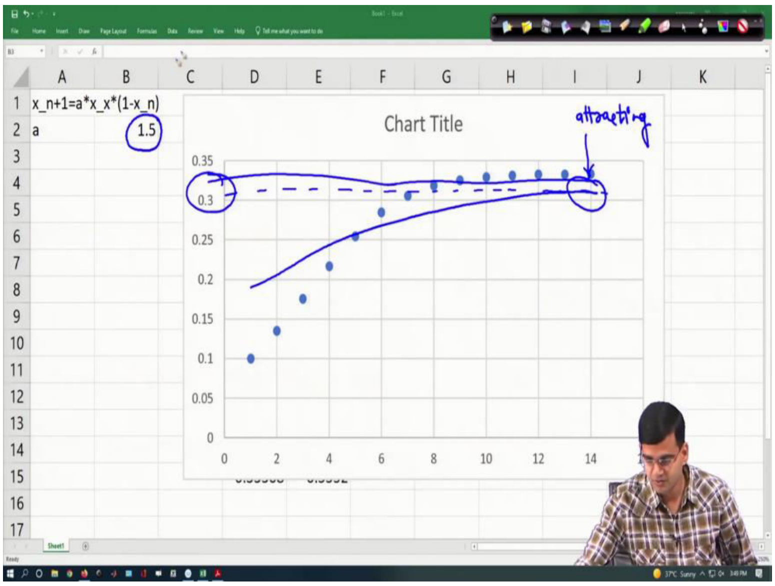

x_n+1=a*x_x*(1-x_n)
x_n
x_n+1
a
0.8
0.1 0.072
0.072 0.05345
0.05345 0.04048
0.04048 0.03107
0.03107 0.02408
0.02408 0.0188
0.0188 0.01476
0.01476 0.01163
0.01163 0.0092
0.0092 0.00729
0.00729 0.00579
0.00579 0.00461
0.00461 0.00367
0.00367 0.00292
Sheet1

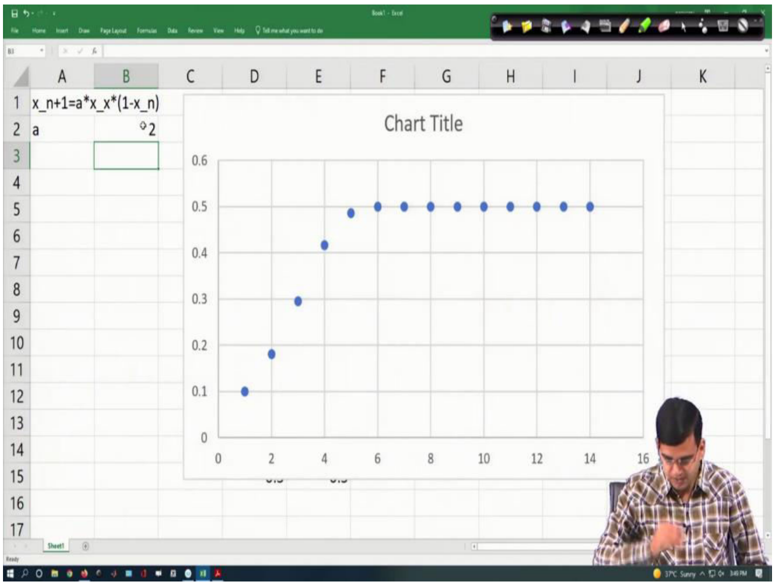

x_n+1=a*x_x*(1-x_n)
a
0.8
a=0.8<1
Chart Title
0.12
0.1
0.08
0.06
0.04
0.02
0
0 2 4 6 8 10 12 14 16
$x_{FP}=0$
Sheet1


x_n+1=a*x_x*(1-x_n)
a
1.2
Chart Title
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0 2 4 6 8 10 12 14 16
$x_{FP}=0$
Sheet1

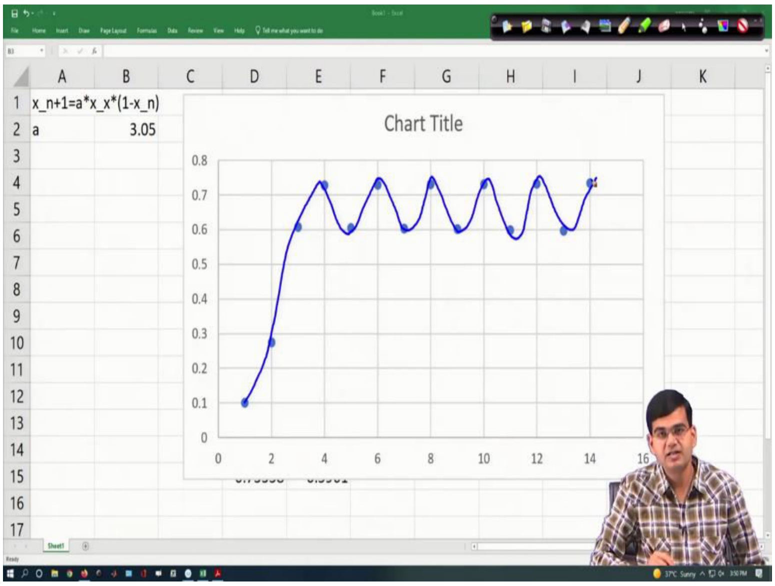

Analysis of fixed points and bifurcation
$x_{FP} = \frac{a-1}{a}$
$f(x) = ax - ax^2$
$\Rightarrow \frac{df}{dx} = a - 2ax$
$\Rightarrow \left.\frac{df}{dx}\right|_{x_{FP}=\frac{a-1}{a}} = a - 2\cancel{a}\left(\frac{a-1}{\cancel{a}}\right) = 2-a$
$\left|\frac{df}{dx}\right|_{FP} > 1 \leftarrow$ rep
$< 1 \leftarrow$ attracting
$1 < a < 3 \leftarrow$ attracting
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 30, NPTEL-SWAYAM


x_n+1=a*x_x*(1-x_n)
a 1.2
Chart Title
attracting

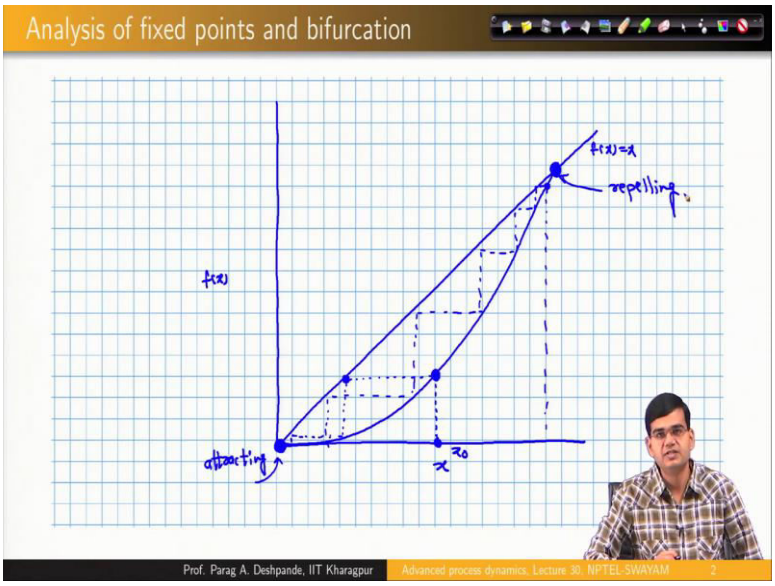

x_n+1=a*x_x*(1-x_n)
a 1.5
Chart Title
attracting



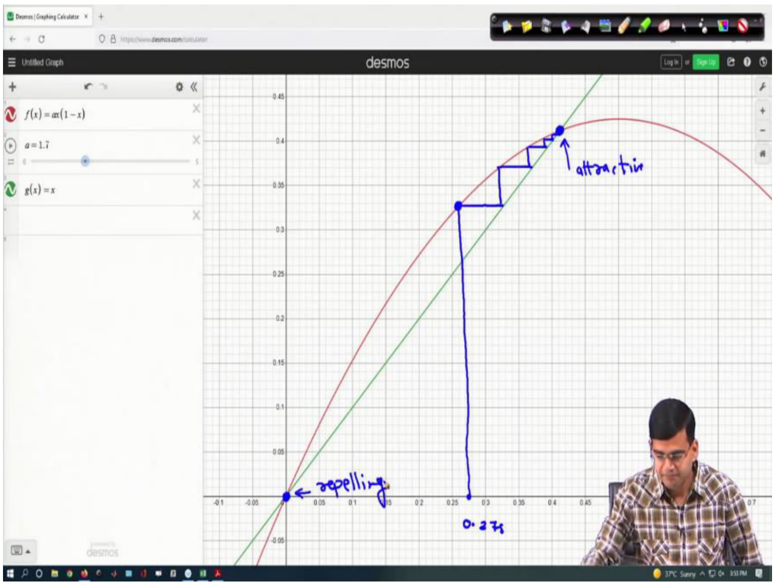
x_n+1=a*x_x*(1-x_n)
a
2
Chart Title

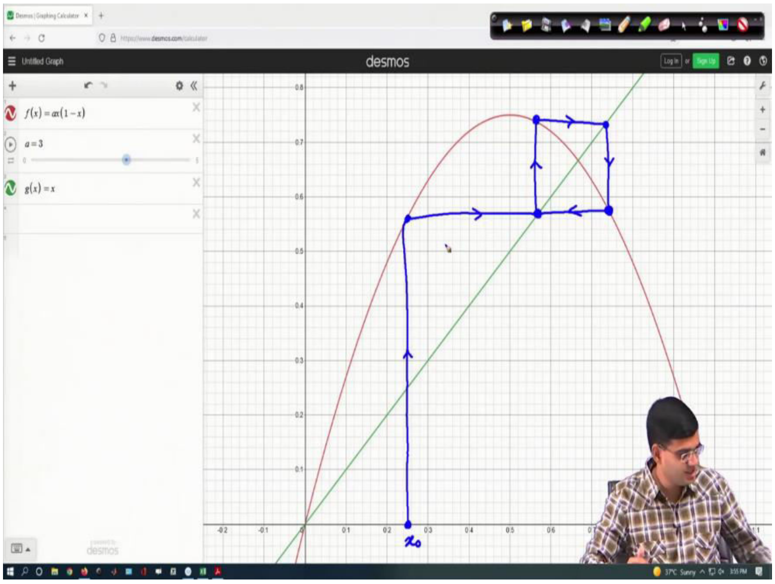


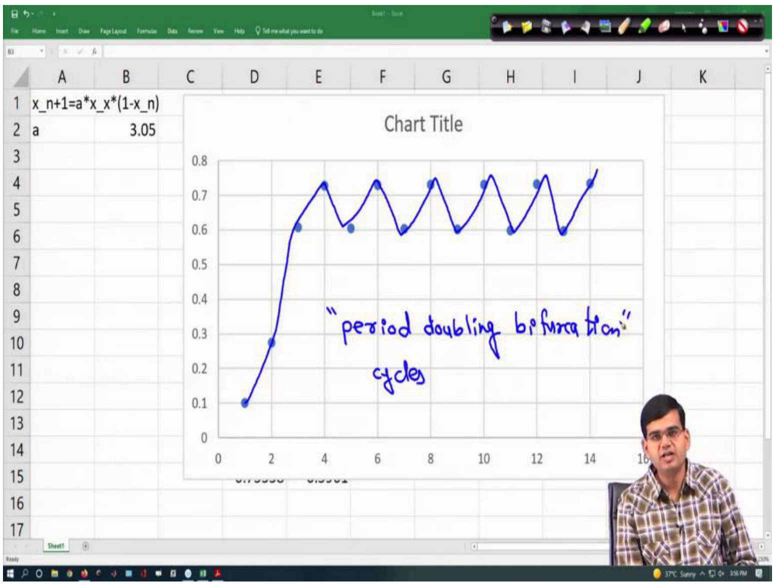
x_n+1=a*x_x*(1-x_n)
a
3.05
Chart Title

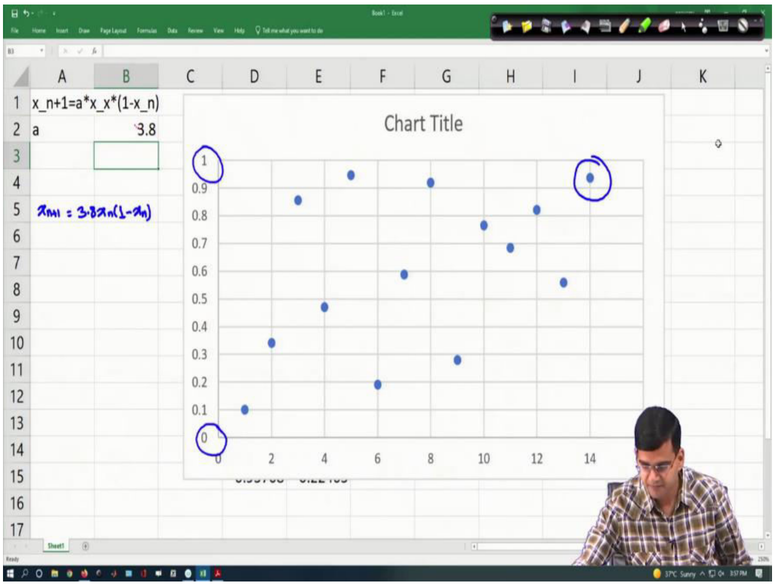


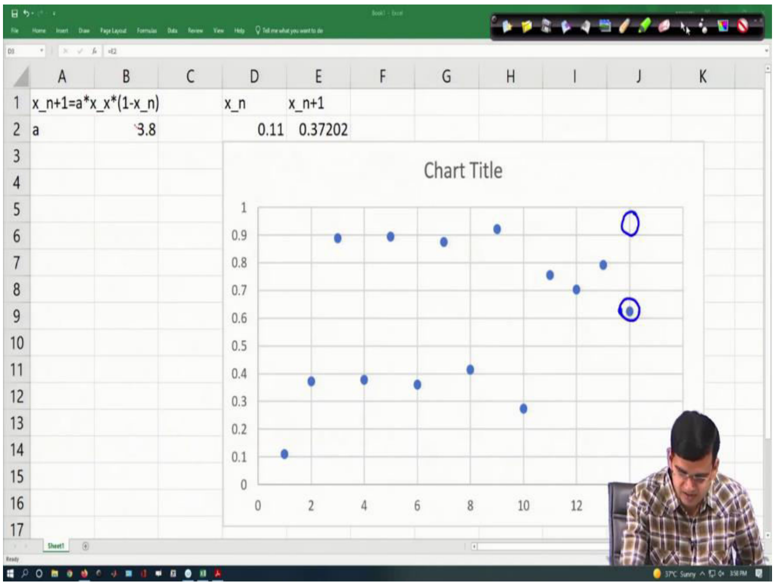
Analysis of fixed points and bifurcation
f(x)
f(x)=x
repelling
attracting
x
x0
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 30, NPTEL-SWAYAM

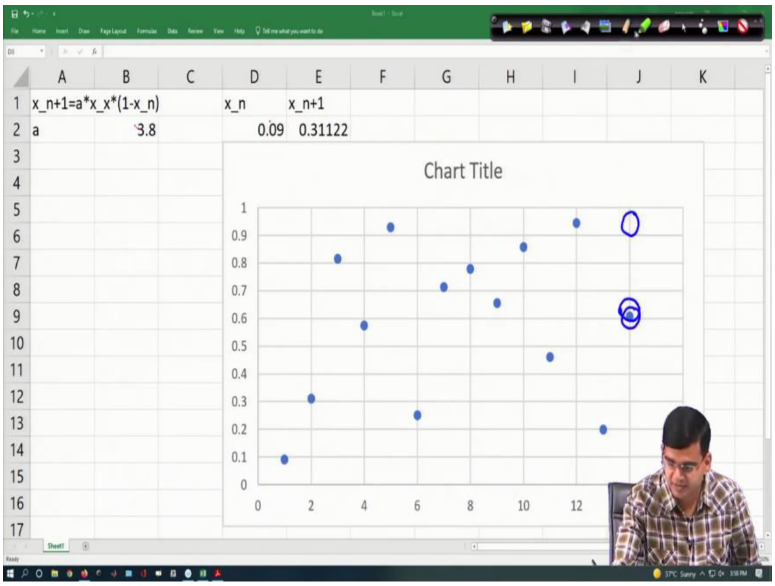

desmos
$f(x)=ax(1-x)$
$a=1.7$
$g(x)=x$
attraction
repelling
0.27

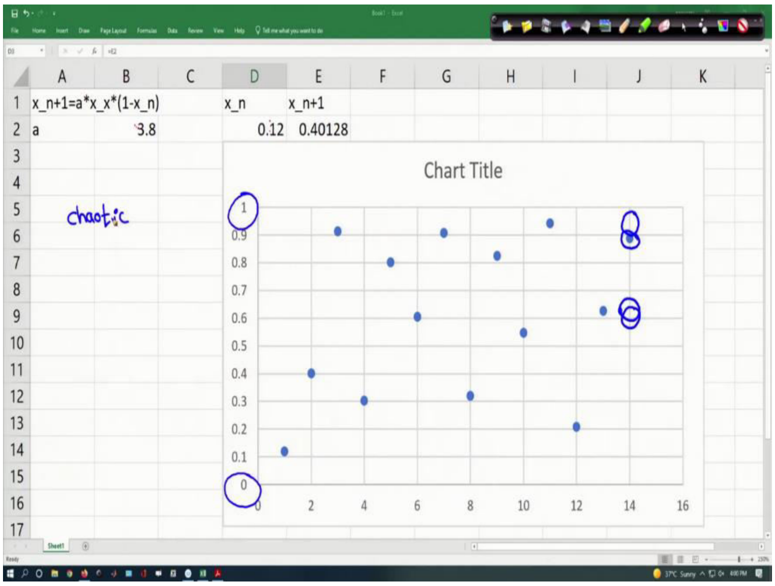

desmos
$f(x)=ax(1-x)$
$a=3$
$g(x)=x$
x_0


x_n+1=a*x_x*(1-x_n)
a 3.05
Chart Title
"period doubling bifurcation"
cycles


x_n+1=a*x_x*(1-x_n)
a
3.8
$x_{n+1} = 3.8x_n(1-x_n)$
Chart Title


x_n+1=a*x_x*(1-x_n)
x_n
x_n+1
a
3.8
0.11
0.37202
Chart Title


x_n+1=a*x_x*(1-x_n)
x_n
x_n+1
a
3.8
0.09
0.31122
Chart Title



(Refer Slide Time: 00:27)

So, let us continue our discussion on Fixed Points and Bifurcations in Discrete Domain. And today, let us take the case of logistic equation.

(Refer Slide Time: 00:32)

So, now we will analyze logistic equation in discrete domain. So, our logistic equation was $\frac{dx}{dt} = ax\ (1 - x / N)$. And we would like to know the discrete time counterpart of this equation. So, let us try to get that. Let me write the derivative as $\frac{x_{N+1}-x_N}{\Delta t} = a\ x_n\ (1\ -\ x_n / N)$ which can be written as $x_{N+1} = x_n + (a\ \Delta t)\ x_n\ (1 - x_N / N)$.

So, further simplify this as $x_{N+1} = x_N[1 + a\Delta t(1 - \frac{x_N}{N})]$. Now, let me do one thing let me simplify it a little further, $x_{N+1} = x_N[(1 + a\Delta t) - \frac{a\Delta t}{N} x_N]$, I will take 1+ (a Δt) out. So, I will get $x_{N+1} = (1 + a\Delta t)\left[1 - \frac{a\Delta t}{N(1+t)} x_N\right]$.

And I can write this finally in form which is known to us as

$$x_{N+1} = (1 + a\Delta t)\left[1 - \frac{x_N}{N(1+t)/\ a\Delta t}\right].$$

Now, why did I write this in this particular form, because if I denote $1 + a\Delta t$ by a new constant a' and N $(1 + a\Delta t)\ /a\Delta t$ by another new constant N' then I can write my dynamical equation is $x_{N+1} = a'\left[1 - \frac{x_N}{N'}\right]$.

And this form looks familiar to me because my dynamical equation in the continuous domain was $\frac{dx}{dt} = ax\ (1 - x / N)$ and in the discrete domain the form is very simple and very similar $x_{N+1} = a'\ x_N\ (1 - x_N / N')$. So, now I can analyze equation 2. So, let me call this equation 1, equation 1 is in continuous domain, equation 2 is in discrete domain.

And let me analyze this in discrete domain for a normalized population. So, let me say that I am considering a normalized population. And using the growth parameter as simply a, it is a matter of notation. So, my equation becomes $x_{N+1} = a'\ x_N\ (1 - x_N)$. So, I would like to analyze this equation.

In fact, mathematicians call this parameter as finite growth parameter, it is called finite growth parameter to emphasize that this particular equation is in discrete time domain. So, let me analyze equation 3 which is simply $x_{N+1} = a'\ x_N\ (1 - x_N)$.

(Refer Slide Time: 06:59)

So, the equation is $x_{N+1} = a'\ x_N\ (1 - x_N)$. And the first thing that I would like to do is to determine the fixed points of this population. The corresponding equilibrium solution concept would exist for the continuous domain. And we know that for the continuous domain equation, there are two equilibrium solutions. So, if we model the system in discrete time domain, I should get two fixed points, let us see if that happens.

So, I would denote f(x) as $ax\ (1 - x)$ and the fixed points can be determined by setting f(x) = x. So, I get $ax(1 - x)$ = x or x a (1 – x) - 1 = 0. So, therefore, my first fixed point x fixed point is 0 and a 1 minus x fixed point minus 1 is equal to 0 implies that 1 minus x fixed point is equal to 1 upon a from where I get x fixed point to be equal to a minus 1 upon a. So, this is my first fixed point. And this is my second fixed point. Will always get two fixed points, is there any condition under which I may not get two fixed points? We can quickly check this by solving this problem using graphical method.

(Refer Slide Time: 09:29)

So, following the graphical method I have $f(x) = ax(1 - x)$ and $g(x) = x$. So, let me analyze this. We are considering populations, so, x would be a positive number. So, I would consider only the first quadrant. So, in such a case, what I see is that when a is a group parameter, so, it must be a positive quantity in fact, finite growth parameters, so, it must be a positive quantity.

So, when a is equal to 1, you see that there is only one fixed point. So, for a is equal to exactly one there will be a fixed point there will be only one fixed point. When I make a greater than one what you will see is that there are in fact two fixed points. So, you can see here that you have this as one fixed point, x fixed point which is equal to 0 and then you have another fixed-point x fixed point and this happens when the value of a is not equal to 1.

And now, anyway I previously wrote that x fixed point was 0 or x fixed point was a minus 1 upon a which means the other fixed point the second point would depend upon the value of n. It is the case because as I keep on changing the value of a this fixed point can be seen to change, but this fixed point the lower one at x equals 0 remains the same.

So, therefore, I will have two fixed points which means, the way I used to see equilibrium solutions, two equilibrium solutions for continuous logistic equation, I actually see two fixed points here for the discrete domain equation as well.

(Refer Slide Time: 12:03)

Now, the question is that if I have two fixed points, what are the natures of the fixed points. So, I have the equation as $x_{N+1} = ax_N$(1- x_N), *f(x) = ax(1 – x)*, x fixed points first one is 0 and x fixed point second one is a minus 1 / a. So, let us check the natures of the fixed points.

So, I will write df /dx, I will determine df /dx and f is ax - a x^2. So, df /dx would be a - 2ax, this is my df/dx. Now, I need to determine df/dx at the fixed points. So, df /dx at x fixed point is equal to 0 is a. So, what would be the nature of the fixed point? It would depend upon the parameter a.

So, for a < 0, so, a < 1 it is very important that we do not, by mistake use 0 here for a less than 1 you would have attracting fixed point which means a stable solution tending to that fixed point and for a greater than 1 you will have repelling fixed point. Let us check if this is the case.

(Refer Slide Time: 14:03)

So, let us try to draw this dynamical system. So, my equation is x_{N+1} = a x_N (1 - x_N), this is my equation. I would set the value of a as some value and let me first start with 0.8 which is less than 1, 0.8 my variables are x_N and as the time progresses I would discreetly move ahead. So, let me start with some initial seed for my series. So, I will have let us say that I use 0.1.

So, 10 percent of the saturation population this is the meaning of initial population as 0.1 because we are using normalized populations. So, what would happen here this would be a means this multiplied by x_N (1 - x_N)with a small change that I will make a constant. What was our working algorithm? For the next step x_{N+1} becomes x_N.

So, therefore, I would write this is equal to this and then I would propagate my system let me propagate my system like this. And now, I have various values of x_N. Let me see how the system evolves. So, I will draw this and it is pretty clear from this plot that you in fact have a convergent system. So, the system is tending to this particular.

So, you are coming down here like this. So, this x fixed point is equal to 0 for a is equal to 0.8 which means *a* is equal to 0.8 which is less than 1 is an attracting fixed-point convergent population, converging to 0 in fact. Now, I will go to the second situation and what I will do is, I will set up this as 1.2. So, let me make this 1.2. What do I see?

What now what I see is that the population is going away from this fixed-point, x fixed point is equal to 0, which means for a greater than 1, the fixed point 0 is a repelling fixed point. So, you are getting repelled from x fixed point is equal to 0. So, what our analysis we did sounds to be correct. Now, let us look into the second fixed point.

(Refer Slide Time: 17:47)

So, our second fixed point x fixed point was a minus 1 upon a and f(x) = $ax - ax^2$ which means df / dx was $a - 2ax$ which means that df/dx at x fixed point is equal to a – 1/ a would be what $a - 2a(a - 1/ a)$. And what this would be equal to? This should be equal to 2 - a because these a's go away and I have 2 - a.

And my condition for stability now, for attracting or repelling fixed points were that this df / dx at fixed point should be greater than 1 for repelling and less than 1. So, greater than 1 is repelling and less than 1 attracting. So, for a in the range using this inequality you can see that for a in the range $1 < a < 3$ you will have attracting fixed point, this inequality can be very easily established from here and from this condition.

So, if I have the value of a between 1 and 3, my upper fixed point would be an attractive fixed point. Let us see if that is the case which we see in the graph.

(Refer Slide Time: 20:23)

You see in fact when I set the value of a as 1.2 which is greater than 1. So, in the range of 1 and 3, the upper one would be the attracting fixed point. So, when a is 1.2. You see that the curve is saturating. That means, you can expect the fixed point to be some basis, you can expect this 0.16 to be the fixed point. And the system is asymptotically tending towards these fixed points that means this is attracting.

Let me see if this is the case for other values of a in this range as well. So, I will change this to 1.5, you see your system is saturating again to this value, your system is saturating to this value. Let me do it further. So, I will make it 2. Again, you see saturation. So, as I go on increasing the value of a, my system now has upper fixed point which is in attracting fixed point.

Now, the question is what happens at the condition when you go beyond 3. So, let us punch in some value which is more than three, let us say 3.05. What do I see? Well, I see an interesting behavior that there is an initial dynamics. So, there is an initial dynamics and once that initial dynamics is finished, you start seeing oscillations, start seeing oscillations and this happens at the value of parameter which is greater than 3. Let us see why does this happen.

(Refer Slide Time: 22:40)

So, we came up with a working algorithm to determine how we can determine the fact whether the particular fixed point is attracting or repelling. What you will do is you will do simply this, you will plot f(x) versus x and let us say that this is the curve. Then what you will do you will plot $f(x) = x$.

Now, I have two fixed points, this is one fixed point this is another fixed point. I need to know the natures of the fixed point how would I do I would start with an x_0. So, this is x_0, I will go to the corresponding point on f(x) curve I will get the value of x_1, I will go here on $y = x$ line I will again from here keep on making triangles and what I see is that I am slowly reaching towards this point.

So, this is an extracting fixed point and even if I start from here which is very close to the other fixed point what I see is if I follow the same method of drawing triangles, what I see is that I go like this, I ultimately reach this point. So, therefore, this is the repelling fixed point. This was our working algorithm. Let us see if this algorithm works for the logistic equation as well.

(Refer Slide Time: 24:47)

So, I will go to the logistic equation what I have is let me expand it and let me set the value of this to say 1.7, 1.7 is good enough. So, I have two fixed points here. So, my first fixed point is this, my second fixed point is this. Now, let me start with some x0. So, let me say that I will start with this x0 is equal to 0.275. When I do this, what I will do is I will go here on the curve on f of x, then I will move like this.

And what we saw just now, in fact holds true here that this is an attractive fixed point and this therefore is repelling fixed point. Now, what happens if I change the value of the parameter a? Let me do one thing. Let me make the parameter a say 3, let me make it 3. So, this is what happens when I make it 3. So, I had previously started with 0.275 just one example.

So, this is 0.275, x0. So, I will go up all the way here 0.275, then I will go here, then I will go here, I will go here and I will come here and then an interesting thing happens. What is the next step? I will come here. Once I reach here, I keep on repeating my values inside this box. So, I go here and then go here, here, here, but till I have not reached that value. There is an initial dynamics and this is precisely what you see here.

(Refer Slide Time: 27:27)

There is an initial dynamics and then you keep on repeating the exact same values and this phenomenon is called period doubling bifurcation, when you start seeing cycles in your system, so, these are cycles. Why these are cycles? When you keep on moving in a cycle the values repeat. And therefore, period doubling bifurcation happens in your system. Now, what if I increase the value of a further, let us see what happens.

So, let me make this say 3.8. Well, do you see a convergent trend? The answer is no. Do you see a divergent trend? No. Do you a convergent trend? No. Do you see oscillations? No. The points you will see are all over the place perfectly randomly distributed. So, therefore, you say that $x_{N+1} = 3.8x_N (1 - x_N)$ will give me the points which are uniformly distributed in the entire range, they go all the way from 0 to 1 in the range 0 to 1 and they are perfectly random, you cannot figure out any trend.

They are not converging to a higher value, they are not converging to a lower value, they are not diverging, they are not oscillating with any specific cycle. So, therefore, you have perfectly random points. And what is interesting is that you will be interested in knowing the ultimate fate of the system and ultimate fate of the system would be given by the last point, this is given by the last point.

So, let me see, let me change the initial value by a small number. So, I will change the initial value. So, my, say ultimate value for initial value 0.1 was this, 0.95 or something, and then so I was here. And then what I would do is I will make this as 0.11. And let us see where I am now, I am now here. I change the value by a very small number. I will do it further.

Instead of 0.11, I will make it 0.09. I increase it and again you see that your value changed. Let me do it further. I will do it, I will make it 0.12. again, come here. And then if you try to find out the trend, which this entire set of points is trying to follow then you will see that there is no trend which is followed they are perfectly randomly distributed in this entire range of 0 to 1.

And why does this happen? This happens because the equation is chaotic. So, the equation is set to display chaos. And what is the meaning of chaos? The meaning of chaos is that your fate of your system is highly dependent upon the initial condition. Well, that is always the case, that if you change the initial condition the dynamics will change, but to what extent, when you see a convergent trend then or whatever initial value of the variable that you take your system will converge to the same ultimate value as time t tends to infinity.

For a divergent trend does not matter what initial value you take, your system will diverge to infinity. So, if you change the initial value, the fate of the system is not going to change, and therefore, the fate of the system is very predictable. But on the other hand, if you have a chaotic system, then even a small change in the initial condition of your system will result in a very, very large difference in the ultimate fate of your system.

And this happens in case of the logistic equation, in case of discrete time logistic equation when you have the value of a close to 4. So, therefore, we saw that we can have a very simple equation discrete equation of the form $x_{N+1} = a x_N (1 - x_N)$, and depending upon the value of a you can get very, very different behavior. The first behavior was that you see a convergence to the lower fixed-point $a < 1$.

You see convergence to the upper fixed point in the range is was from 1 to 3, 3 above, in fact at 3 that happens a period doubling bifurcation where you will see an initial dynamics and then you see cycles in your system. And why does this happen? We saw that in our plot the values of x_{N+1} and x_N keep on repeating. And when you increase the value further close to 4, you see complete chaos in the system.

So, these were some very interesting features which we observed in case of discrete dynamical system. We will continue our discussion on nonlinear dynamics and we will see some more interesting features of dynamical systems especially with reference to bifurcations in nonlinear systems in the next week. Till then, goodbye.