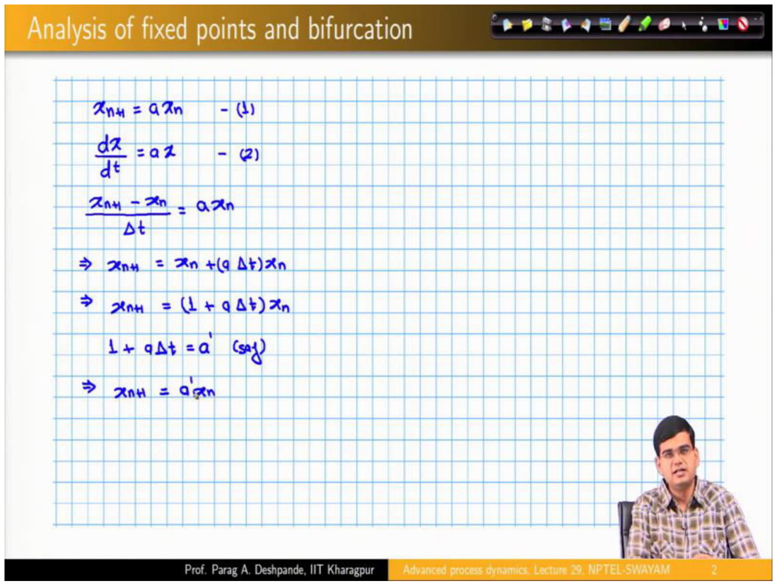
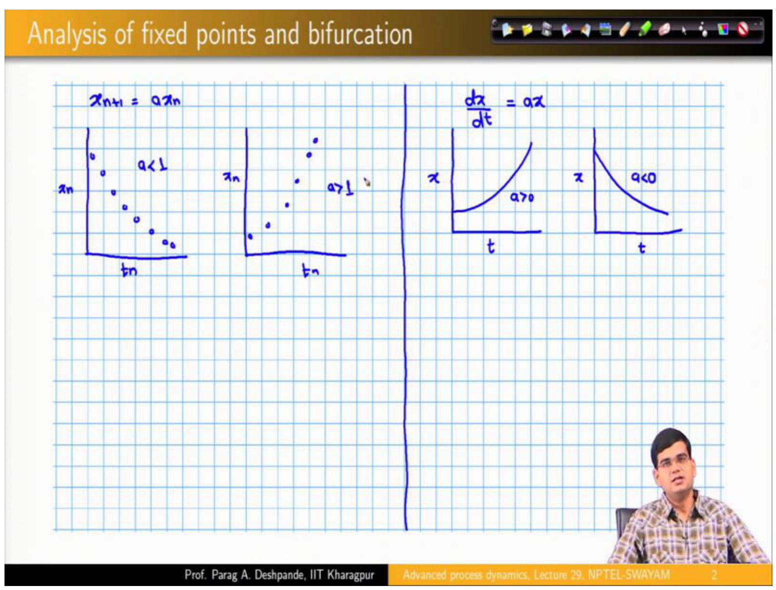
Advanced Process Dynamics

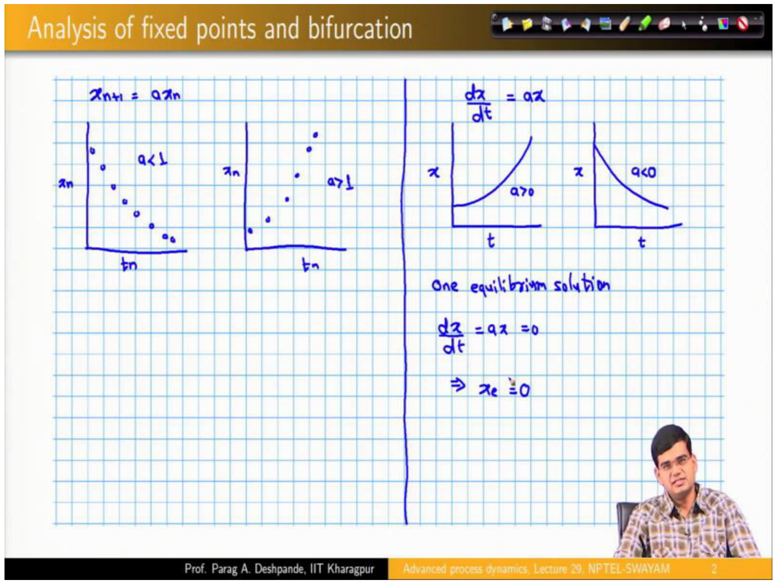
Professor Parag A. Deshpande
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture 29

Analysis of fixed points and bifurcation in discrete domain






Analysis of fixed points and bifurcation
$x_{n+1} = a x_n$
a<1
a>1
x_n
t_n
$\frac{dx}{dt} = ax$
a>0
a<0
x
t
One equilibrium solution
$\frac{dx}{dt} = ax = 0$
$\Rightarrow x_e = 0$
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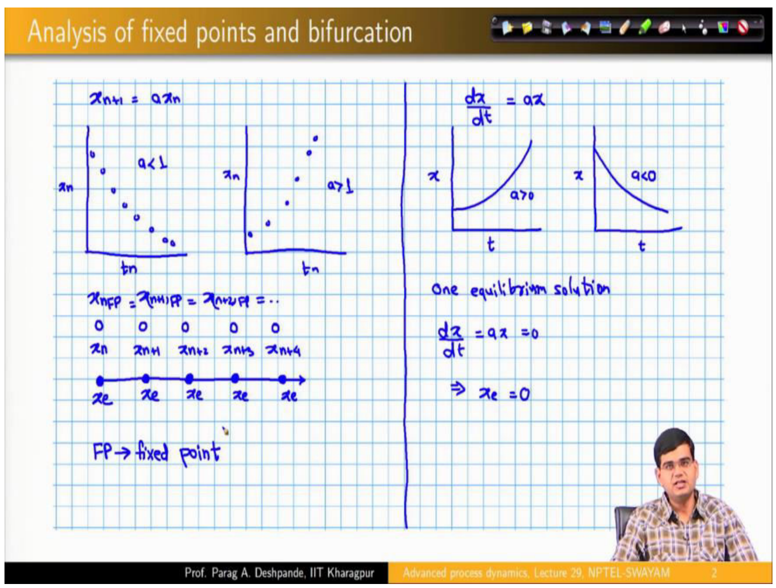
Analysis of fixed points and bifurcation
$x_{n+1} = a x_n$
a<1
a>1
x_n
t_n
$x_{n,FP} = x_{n+1,FP} = x_{n+2,FP} = ..$
0 0 0 0 0
x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4}
x_e x_e x_e x_e x_e
FP → fixed point
$\frac{dx}{dt} = ax$
a>0
a<0
x
t
One equilibrium solution
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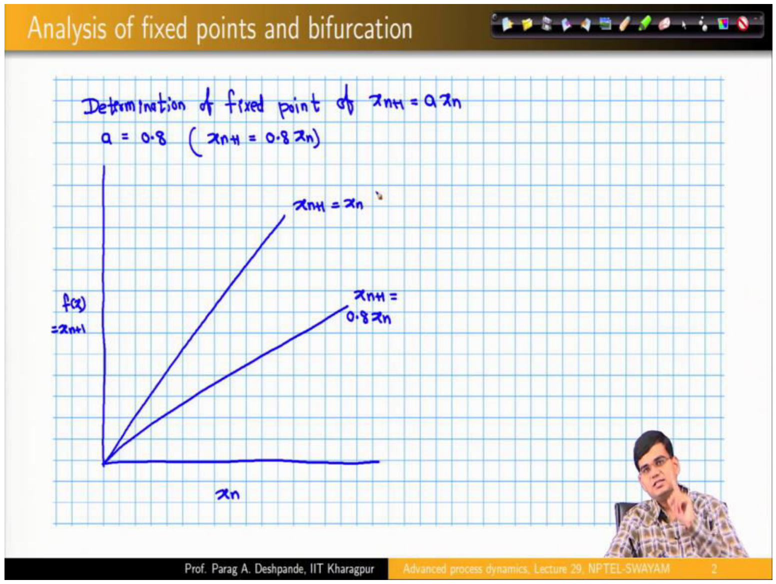
Analysis of fixed points and bifurcation
Determination of fixed point of xn+1 = a xn
a = 0.8 (xn+1 = 0.8 xn)
xn+1 = xn
xn+1 = 0.8 xn
f(x) = xn+1
xn
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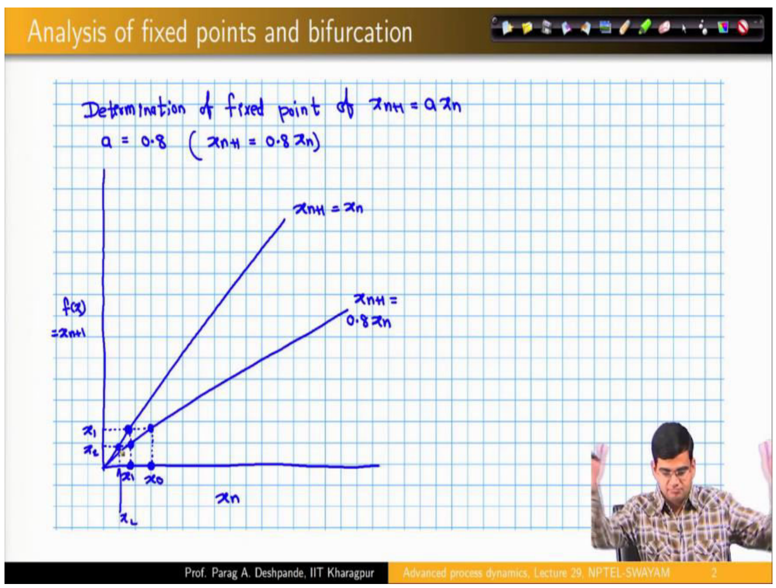
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x1
x2
x1
x0
x2
xn
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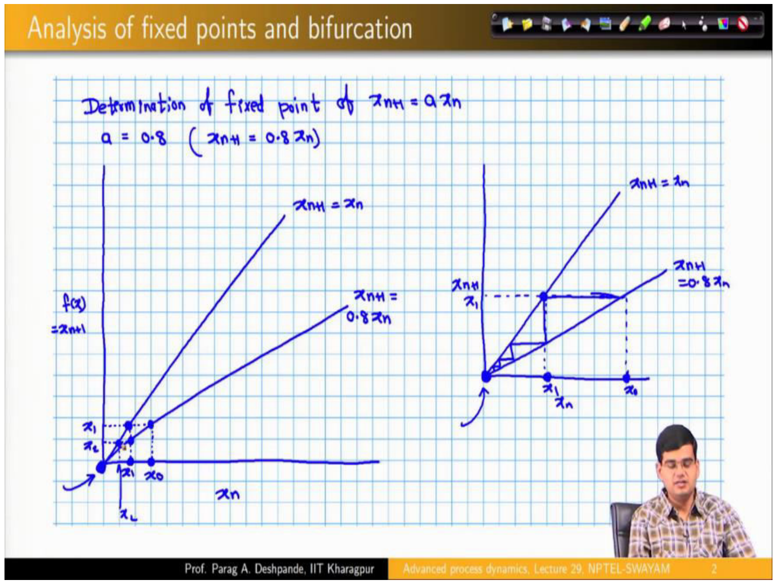
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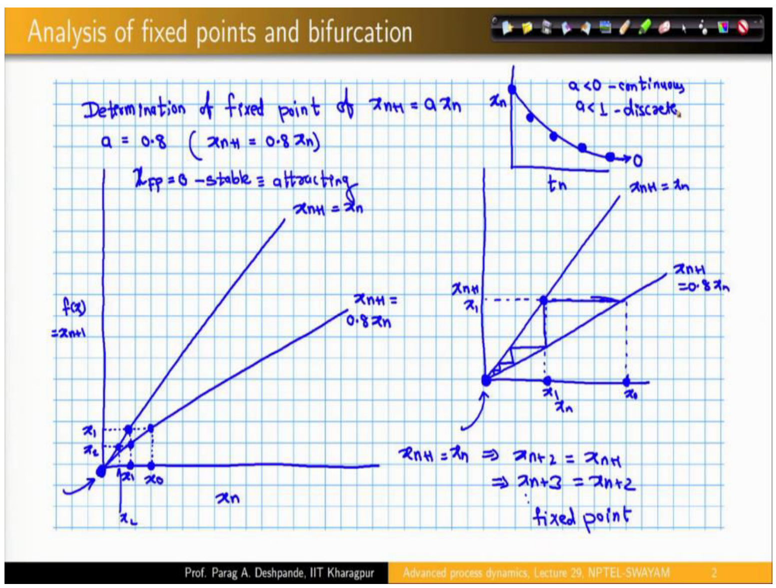
Analysis of fixed points and bifurcation
Determination of fixed point of xn+1 = a xn
a = 0.8 (xn+1 = 0.8 xn)
xFP = 0 - stable = attracting
a < 0 - continuous
a < 1 - discrete
xn
tn
xn+1 = xn
xn+1 = 0.8 xn
f(x) = xn+1
x1
x2
x0
xn
xn+1 = xn ⇒ xn+2 = xn+1
⇒ xn+3 = xn+2
fixed point
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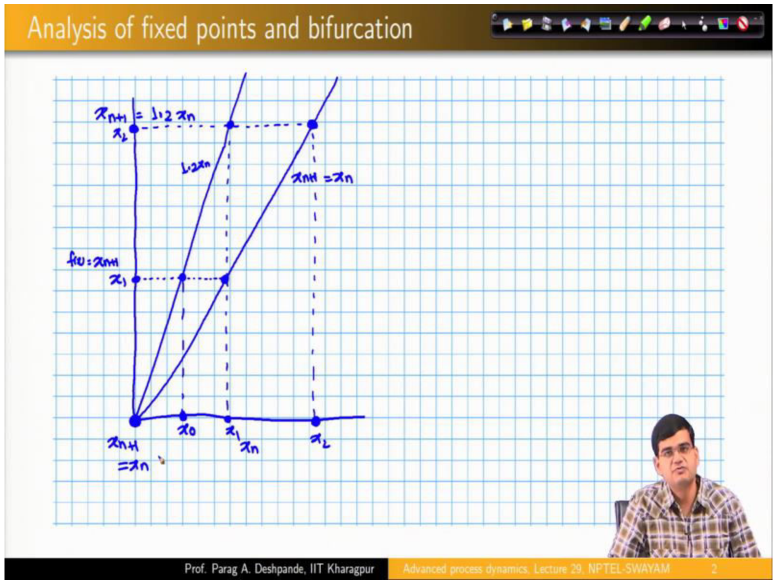
Analysis of fixed points and bifurcation
$x_{n+1} = 1.2 x_n$
x_2
$1.2x_n$
$x_{n+1} = x_n$
$f(x) = x_{n+1}$
x_1
x_0
x_1
x_n
x_2
$x_{n+1} = x_n$
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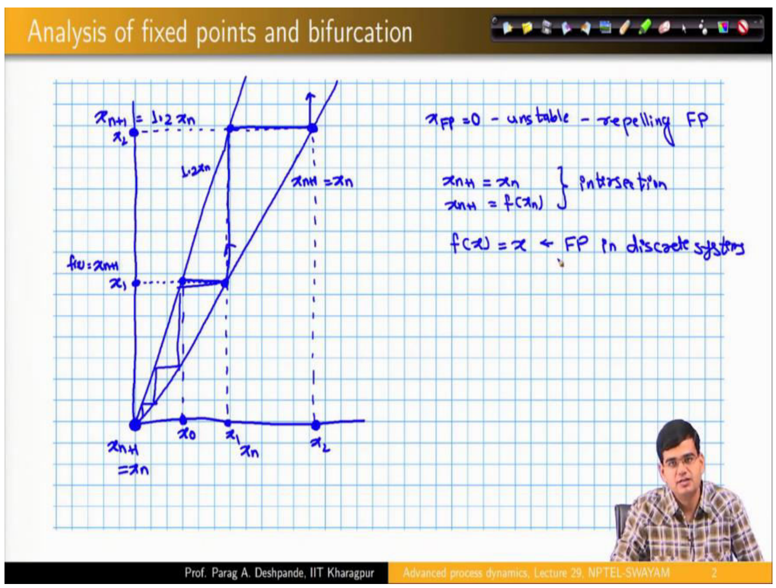
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x_2
$1.2x_n$
$x_{n+1} = x_n$
$f(x) = x_{n+1}$
x_1
x_0
x_1
x_n
x_2
$x_{n+1} = x_n$
$x_{FP} = 0$ - unstable - repelling FP
$x_{n+1} = x_n$
$x_{n+1} = f(x_n)$ } intersection
$f(x) = x$ ← FP in discrete systems
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Analysis of fixed points and bifurcation

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)$$

$$\boxed{\frac{dx}{dt} = ax^2} \rightarrow x_e = 0$$

$$f(x) = ax^2$$

$$\frac{df}{dx} = 2ax$$

$$\left.\frac{df}{dx}\right|_0 = 0$$

Analysis of fixed points and bifurcation

$$x_{n+1} = a x_n^2$$

$$f(x) = ax^2$$

$$ax^2 = x$$

$$\Rightarrow ax^2 - x = 0$$

$$\Rightarrow x(ax - 1) = 0$$

$$\Rightarrow x_{fp} = 0 \ ; \ a x_{fp} - 1 = 0$$

$$\Rightarrow x_{fp} = \frac{1}{a}$$

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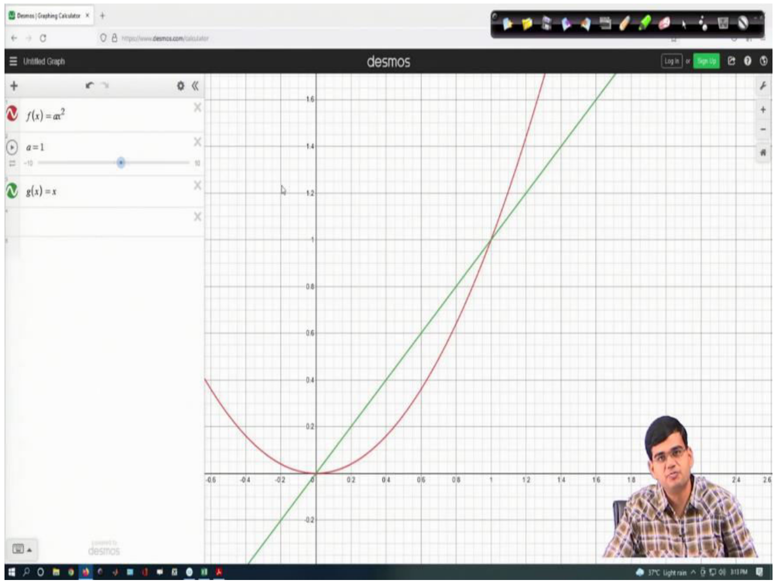
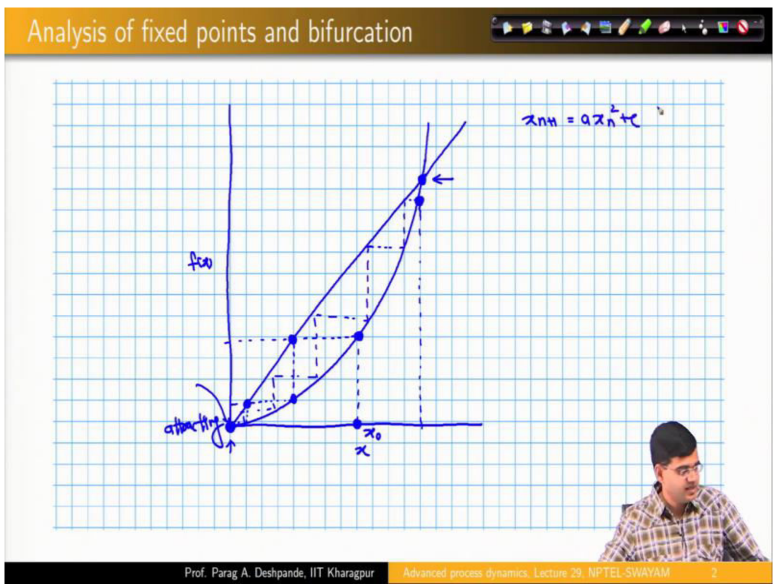

desmos
Untitled Graph
$f(x) = ax^2$
$a = 1$
$g(x) = x$


Analysis of fixed points and bifurcation
$x_{n+1} = a x_n^2 + c$
f(x)
attracting
x_0
x
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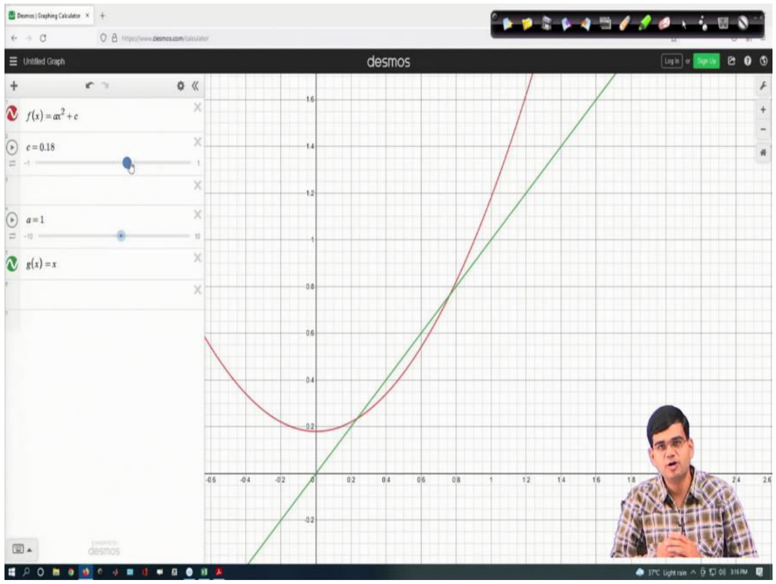

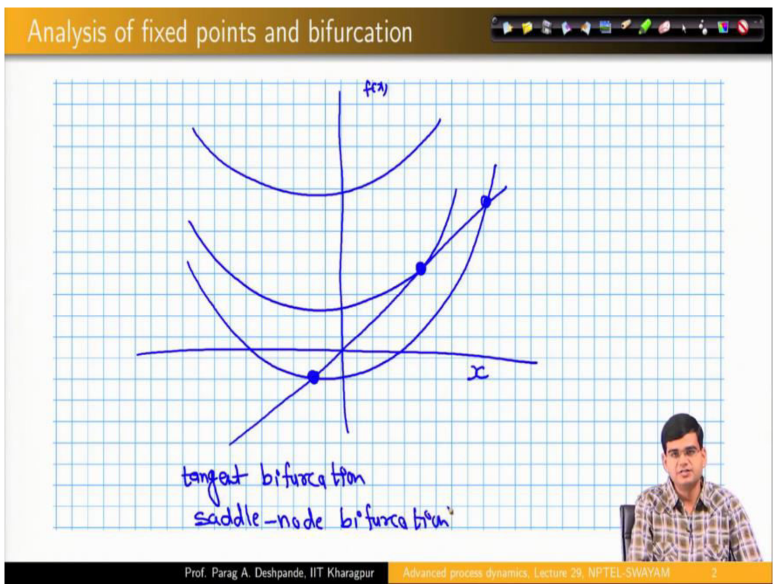

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So, let us continue our analysis of Dynamics in Discrete Domain. Today, we would come across the terms fixed points, we will try to understand the meaning and importance of the term fixed points in discrete domain dynamics. And what we actually did yesterday was took an example of the dynamics given as $x_{n+1} = a\, x_n$, this was our dynamical equation, and we saw that this equation qualitatively gave the same behavior as the equation $\frac{dx}{dt} = ax$.

So, does this mean that equation 1 is the discrete analog of equation 2? We can establish that so, to establish that let us consider equation number 2, $\frac{dx}{dt} = ax$ and if this be the case, we can discretize the system by using the definition of derivative as x_{n+1} - x_n and if this difference happens over a time period delta t then you can write this as a times xn from where I can write $x_{n+1} = x_n + a\ \Delta t\ x_n$ which further gives me $x_{n+1} = 1 + a\ \Delta t\ x_n$.

So, if I set $1 + a\ \Delta t\ x_n = a'$ then I can write let say this $x_{n+1} = a'\ x_n$. So, we in fact saw that the equation of the form $x_{n+1} = a\ x_n$ is the discrete analog of our first order linear autonomous system and when we say that this is an analog of first order linear autonomous system, then the features of such system should be offered by equation number 1 as well. So, we need to see whether that is really the case.

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So, I have on one hand $x_{n+1} = a\ x_n$ and on the other hand have $\frac{dx}{dt} = ax$ and at least qualitatively we know that this would be qualitative because a in the two cases are not different a on the right hand side is actually $1 + a\ \Delta t\ x_n$ of the a which is on the right hand side. So, therefore, the match would be only qualitative.

So, on the left hand side for the right hand side I got an increasing behavior x(t) for $a > 0$ and a decreasing behavior x(t) for $a < 0$ and we saw in the previous lecture that this in fact was the case for this also that you have x_n versus t_n and you saw a decreasing behavior for $a < 1$ and an increasing behavior for $a > 1$ greater than 1. So, the qualitatively as far as the solutions are concerned there is a one to one match.

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So, now in the continuous domain, we saw that there is one equilibrium solution there is one equilibrium solution, how did we determine the equilibrium solution? We determined it by setting $\frac{dx}{dt} = ax = 0$ thereby getting x equilibrium as 0. Now, how do I get that does the concept of equilibrium solution exist in the discrete domain as well and do I get an analogous solution here also, so, we need to check that. So, we need to find out first the meaning of equilibrium solution.

So, what happens at equilibrium solution in case of continuous domain modeling, your gradient becomes 0 and since your dynamical variable is along the y axis, what happens is that the dynamical variable ceases to change it remains constant with time. So, once you attain the equilibrium solution, if you keep on evolving your system in time you will keep on getting the same value of the variable, this was the meaning of equilibrium solution. Now, your dynamical equation for the discrete system is $x_{n+1} = a\, x_n$.

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So, I have discrete points in my system and I have say x_n, x_{n+1}, x_{n+2}, x_{n+3} and so on. If this were a continuous system then what would have happened I would have a line and if this is my x equilibrium since the gradient is 0 here also I would have x equilibrium here also I would have x equilibrium and so on. And I can move ahead in time indefinitely and I will still get the value as x equilibrium.

So, what should be the analogous definition in or the situation in discrete time the moment I get x_n. I will write here FP I will tell you what is the meaning of x_{nFP} if I attain x_{nFP} which is the analog of equilibrium solution then $x_{(n+1)FP}$ will also be the same $x_{(n+2)FP}$ will all be the same and so on and FP is called the fixed point FP is called fixed point of your system, which means that in case of discrete domain systems, if you reach a fixed point then all the subsequent observations would remain unchanged.

In case of continuous domain if you reach an equilibrium solution, the subsequent dynamical variable will remain the same in case of a discrete domain system if you attain a fixed point the subsequent values of the dynamical variable will remain same as that of the fixed point that is the change in terminology the physical meaning remains the same. Now, so, since the continuous domain model had one equilibrium solution $x_e = 0$, I should also get one fixed point for my system.

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So, let us see how can I determine the fixed point so determination of fixed point of $x_{n+1} = a\, x_n$, let us first consider the case where a = 0.8 meaning that $x_{n+1} = 0.8\, x_n$. So, for determining the fixed point what I will do is I will draw f(x) and what is f(x) is nothing but x_{n+1} as a function of

x_n, why, because, this plot will give me the evolution of my system how x_{n+1} is changing with with x_n and x_n is changing with time so, therefore, ultimately I am getting the evolution of my system.

So, f(x) which is x of x_{n+1} is 0.8 x_n so, therefore, this is going to be a straight line straight line with the slope point 8. So, this is 0.8 x_n, $x_{n+1} = 0.8\ x_n$ then what I will do is, I will draw x = y curve. So, this is $x_{n+1} = x_n$ remember I have drawn one curve which is my dynamical equation other curve which have drawn is x = y curve or $x_{n+1} = x_n$ curve or simply f(x) = x curve.

Now, what do I do in case of a continuous domain system I start with x_0 and then depending upon the slope which I get from my solution, I draw the solution now, I have here x_0. So, I have the observation at the first instant of time.

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So, let us say that this is x_0. So, corresponding to x_0, what will be the value of the function? This is going to be the value of the function. So, this is going to be x_1. See I am on the function I am on the I am on the line of the function so, therefore, this value along the y axis would be x_1, but what how do I evolve my system? My system evolves by now considering x_{n+1} as x_n remember the excel sheet that we used in the previous lecture, we said x_n same as x_{n+1}. So, this is an adaptive process the previous becomes the new one.

So, therefore x_1 is here. So, for next instant of time, this will become my x_1, I hope you are able to appreciate what is going on, I start with an initial value, go to the function find the corresponding value of my variable in the next time instant of time, that becomes the value of my function at the current instants of time and that is how, because that is how I am evolving the system. So now, I get the value of the function or my dynamical variable at new instants of time.

Now, for x_1 corresponding to x_1, this will be the value of x_2, this will be the value of x_2 and then I will repeat the procedure this will become this will become x_2 and so on.

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So, let me expand it. So, what is going to happen? What is going to happen is this that you have x_{n+1}, you have x_n and if I show you an expanded view, then what is going on is you have x_{n+1} =

0.8 x_n and this is $x_{n+1} = x_n$. So, I start with x_0 I go to on this function this gives me x_1 and therefore, I will come to this point this is $x_1 = x_{n+1}$. So, therefore, this becomes x_1 and therefore, I will keep on doing this I will keep on evolving my system in time and what will happen is, so, I am coming here then here, then here then here then here then here and so on.

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But, the moment I reached this point here or in this plot here what will happen? I was translating my system following these triangles start from here and do this like this, but, the moment I reached this point here the two curves intersect, which means that at this point $x_{n+1} = x_n$, at this point x_n has become equal to x_n which means that x_{n+2} will be equal to x_{n+1} which means x_{n+3} will be equal to x_{n+2} and so on which means that I have reached the fixed point. I have reach the fixed point and what is the fixed point my fixed point for this case x fixed point is 0, 0 is my fixed point.

So, if I start the evolution of my system, x_n in discrete time domain and I start with some initial population x_1 then the population will keep on evolving such that as t_n tends to infinity x_0 will tend to 0 very similar observation which we made for the case of discrete time domain for continuous time domain system as well.

Now, this means that this $x_{FP} = 0$ is a stable fixed point, it is a stable fixed point like your stable equilibrium solution you have a stable fixed point and there is a term which is used in stable fixed point the term is "attracting", this fixed point is an attracting fixed point and what is the continuous analog you see this decreasing behavior this was for a less than 0 for continuous and $a < 1$ for discrete there is a one to one correspondence.

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Let me draw this for a fixed point for this equation, $x_{n+1} = 1.2\ x_n$, what kind of behavior did we see for this equation in the previous case? The diverging behavior let us see if we can establish this graphically here. So, I have xn+1. So, $f(x) = x_{n+1}$ which is simply 1.2 x_n this is x_n. So, I have this equation this curve this is 1.2 x_n curve I will draw x = y line so, $x_{n+1} = x_n$, x = y line and then I start with some x_0 let us say this is x_0.

So, I will first go always to the equation dynamical equation and this becomes x_1 if this is x_1 on $x_{n+1} = x_n$, this will become my x_1. And now, it is not very difficult to see that when corresponding to x_1, the value of the function is this, which means this is x_2, this will become x_2, and so on. But if I am here at this point at this point the two curves intersect, which means $x_{n+1} = x_n$, which means that that point is a fixed point.

But what is the difference between the fixed point which you get here and the fixed point which you got in the previous case $x_{n+1} = 0.8\ x_n$.

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Now, this fixed point, so, my x fixed point is 0 in this case as well, it is simply the intersection of the two curves to x fixed point is 0, except that now this is unstable, why is it unstable? Because you are moving away from this point. So, let me try to draw the triangles, I will draw a triangle, I started from this point, I draw a triangle here and here and here and you see that you are just going away, you are never returning to that point.

But you could have started from somewhere close to this point and you could have drawn the triangles in following the method which I did and so on, and you would only have diverged away from the fixed point is equal to 0. So, therefore, this is an unstable equilibrium solution analog. In other words, this is called repelling fixed point this is the repelling fixed point.

So, what do I understand? How do I look into these two cases and conclude whether the system is the fixed whether the system has a fixed point or not, very simple, I will draw $x_{n+1} = x_n$ and I will draw $x_{n+1} = f(x_n)$ and the intersection would be the fixed point you can do this graphically, you can do this analytically as well.

How would you do how would you solve for this analytically? Simply f of x = x this is fixed point in discrete systems. If you want to solve it analytically, solve for f of x = x. If you want to solve it graphically, draw the two curves intersection would be the fixed point.

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If this is the case, then can we have another model which can be analyzed in the same manner so, another model was $\frac{dx}{dt} = ax(1 - x / n)$, so, before we analyze this model, this this model has this x squared term, before we analyze this model, let us take a simpler model which is $\frac{dx}{dt} = ax^2$.

So, I need to solve I need to analyze this model and how do I analyze this continuous model, not very difficult to analyze, I had x equilibrium as what 0, one equilibrium solution. What will you do to determine whether the equilibrium solutions are stable or unstable, you would assign $f(x) = ax^2$ you will do $\frac{df}{dx} = 2ax$. So, therefore, you will do $\frac{df}{dx}$ at 0 is equal to in fact 0.

Now this basically means that you have a bifurcation in the system about $x = 0$ and $a = 0$. So, now can we do a similar analysis if this point if this point if this equation is treated in discrete domain.

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So, let us say that discrete domain is equation is $x_{n+1} = ax_n^2$. I want to determine the behavior dynamical behavior of this particular equation. So, how do I do this? I will assign f(x) as ax^2, then you can do one thing you can solve for $ax^2 = x = 0$, in other words $x(ax - 1) = 0$ which means x fixed point = 0 or ax fixed point - 1 = 0 in other words x fixed point = $1 / a$. This means that you have two fixed points in this in this particular system.

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Can we quickly see if this is the case graphically, so, I will draw $f(x) = ax^2$ and I will draw $g(x)$ simply as x. And then you see here that in fact there are two intersection points one is this 0, 0 that is what we obtained other one is non-zero it is a function of a because if I change this you see as I change this location of intersection changes. So, therefore, you in fact have two equilibrium 2 fixed points, if this were a corresponding system in continuous domain you would have two equilibrium solutions, but here you have two equilibrium points which can be very easily seen from this particular plot.

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Now, if this is the case our plot looked something like this. *f(x)*, this is x, these are the two fixed points. So, how do I determine whether the lower fixed points fixed point is stable or unstable the upper fixed point is stable or unstable? The same method which we followed previously I start with an x_0 x_0 I will go up on the curve, I will move horizontally then I will move vertically towards the curve thereby making triangles and then you see here that I am moving towards this lower fixed point which means this is the attracting fixed point.

And even if you start very close to the upper fix point if you start very close to the upper fix point, what is going to happen, you are going to make triangles in this manner, which means that you are again moving towards this fixed point, you are always moving towards this fixed point. But now, in this case you concluded that you had two fixed points and one of them is attracting the other one is repelling let me change my model equation very slightly as $x_{n+1} = ax_n + c$.

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And let me plot this now. So, now I have $ax^2 + c$. And then you see for this particular case, there is no intersection of the two curves at all which means there is no fixed point. In other words for a continuous domain an analogous system there would have been no equilibrium solution.

If you remember the case of population dynamics with harvesting, you did not have equilibrium solutions for certain cases this also is the case where the two curves do not intersect and therefore, you do not have any fixed point but if I change see here now you see for certain value of c you can have just exactly 1 for if I set c as 0, so, let me move it between -1 to 1.

So, if I set this as a value here at 0.25, 0.25 there is only one solution that means, there is one fixed point greater than 0.25 there is no fixed point and lesser than that there are two fixed points. So, therefore, now, I see that as I change the value of the parameter c my number of fixed points go from no fixed point to one fixed point to two fixed points.

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So, very quickly we can draw this situation like this, you have the parameter c, sorry this is x this is *f(x)* so, I have this case where there is no fixed point then I can have this case where there is one fixed point and then I can have this case where I have two fixed points. So, one fixed, point 2 fixed points and no fixed point.

So, as the value of parameter c changes, my fixed points vanish or appear and such kind of bifurcations are called tangent bifurcation. In fact, there is another term for it, it is called saddle node bifurcation. So, as the value of my parameters c changes, one or the one or more equilibrium solutions of fixed points can appear and such bifurcations are called saddle node bifurcations. We will continue this discussion in the next lecture till then, goodbye.