Advanced Process Dynamics

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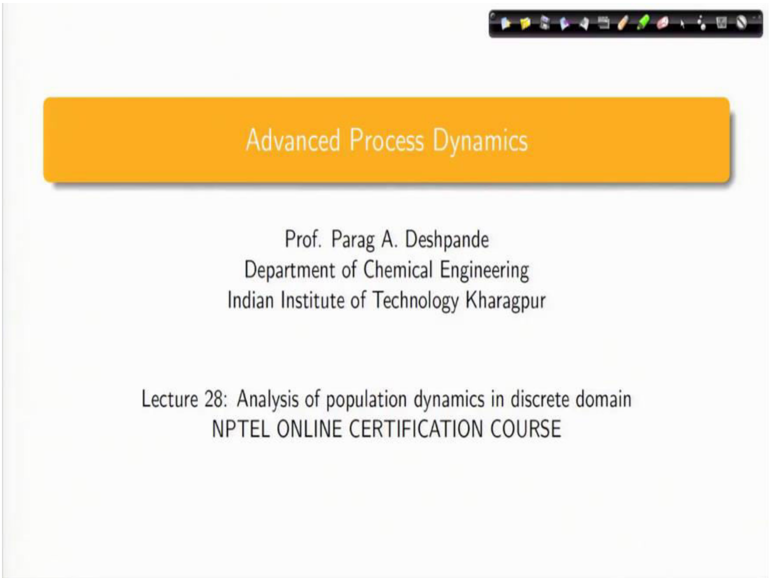
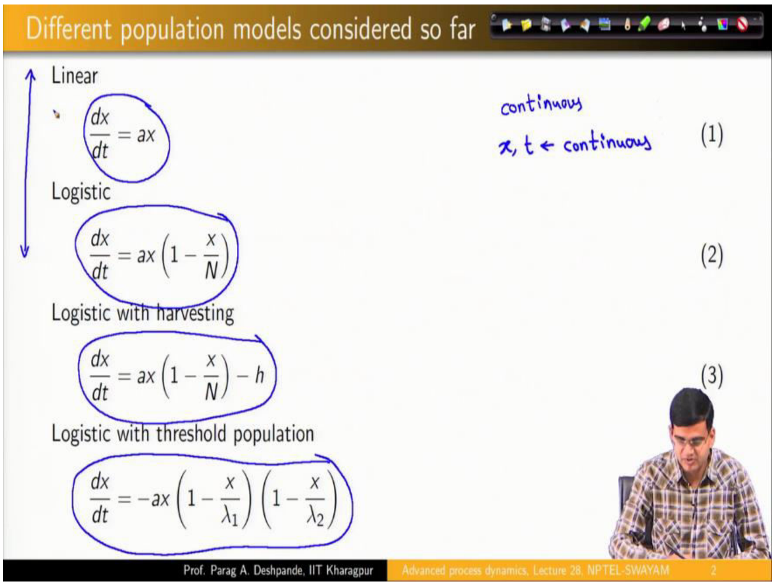
Lecture 28

Analysis of population dynamics in discrete domain





A fundamental problem with these models

Linear

$$x(t) = x(0)e^{at} \quad (5)$$

Logistic

$$x(t) = \frac{x(0)e^{at}}{1 - x(0) + x(0)e^{at}} \quad (6)$$

time	x(t) – linear	x(t) – logistic
0	10	10
1	73.89056099	45.08530604
2	545.9815003	85.84864498
3	4034.287935	97.81780512
4	29809.57987	99.69899243
5	220264.6579	99.95915675
6	1627547.914	99.99447051
7	12026042.84	99.99925163
8	88861105.21	99.99989872
9	656599691	99.9[illegible]29
10	4851651954	99.99[illegible]14

$x_0 = 10; a = 2$

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$x, t, a \in \mathbb{N}$

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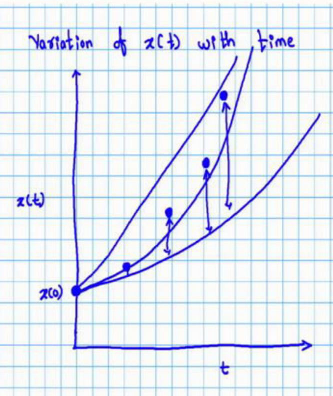
8	88861105.21	99.99989872
9	656599691	99.9[illegible]629
10	4851651954	99.9[illegible]314

$x_0 = 10; a = 2$

Population growth model in discrete domain



$$\frac{dx}{dt} = ax$$

$$\Rightarrow x(t) = x(0)e^{at}$$

determine the value of "a" by curve fitting

RMSE < $\boxed{\alpha}$

$a = 2 \pm \beta$

uncertainty

A fundamental problem with these models

Linear

$$x(t) = x(0)e^{at} \quad (5)$$

Logistic

$$x(t) = \frac{x(0)e^{at}}{1 - x(0) + x(0)e^{at}} \quad (6)$$

model predicted population

time	x(t) – linear	x(t) – logistic
0	10	10
1	73.89056099	45.08530604
2	545.9815003	85.84864498
3	4034.287935	97.81780512
4	29809.57987	99.69899243
5	220264.6579	99.95915675
6	1627547.914	99.99447051
7	12026042.84	99.99925163
8	88861105.21	99.99989872
9	656599691	99.9[illegible]8629
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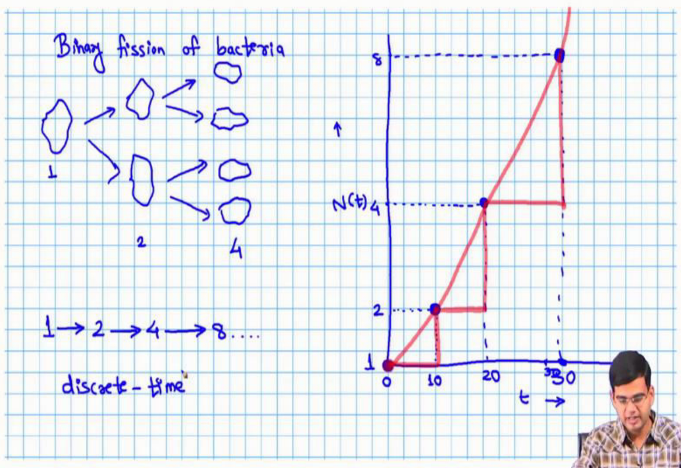
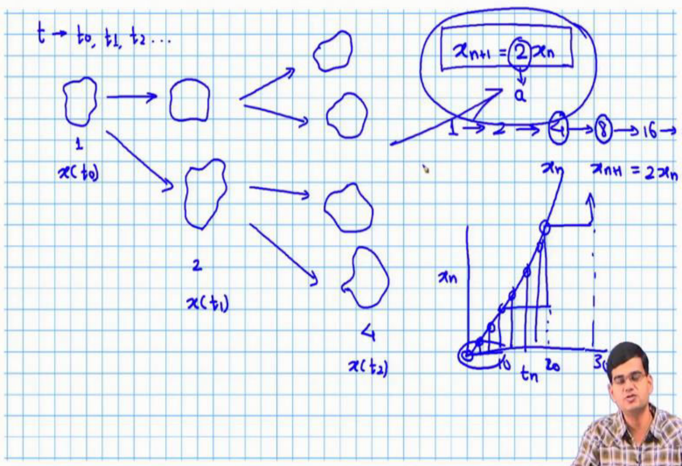
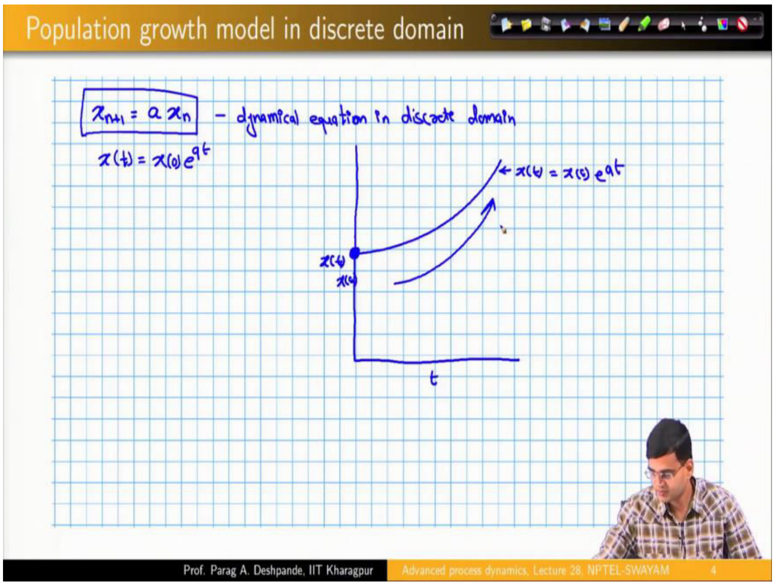
$x_0 = 10; a = 2$

Population growth model in discrete domain



Population growth model in discrete domain




Population growth model in discrete domain
xn+1 = a xn – dynamical equation in discrete domain
x(t) = x(0)eat
x(t) = x(0) eat
x(0)
t
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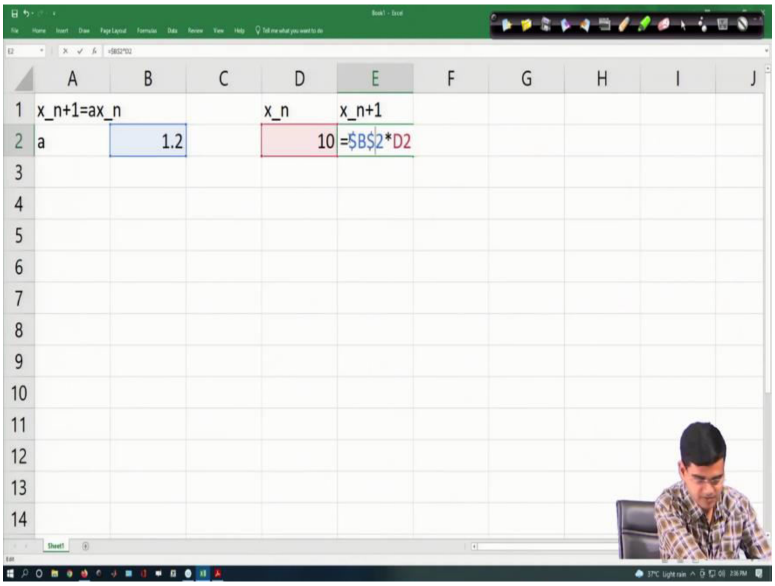
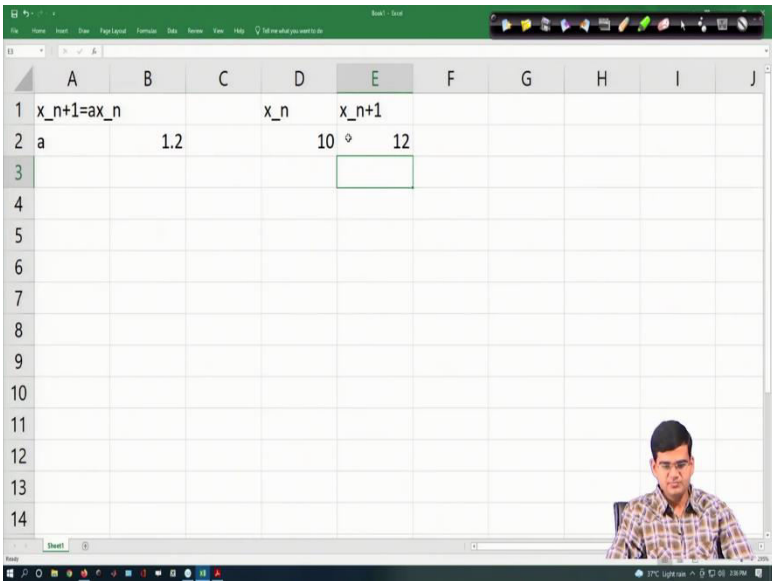

x_n+1=ax_n
x_n
x_n+1
a
1.2
10
=B2*D2
Sheet1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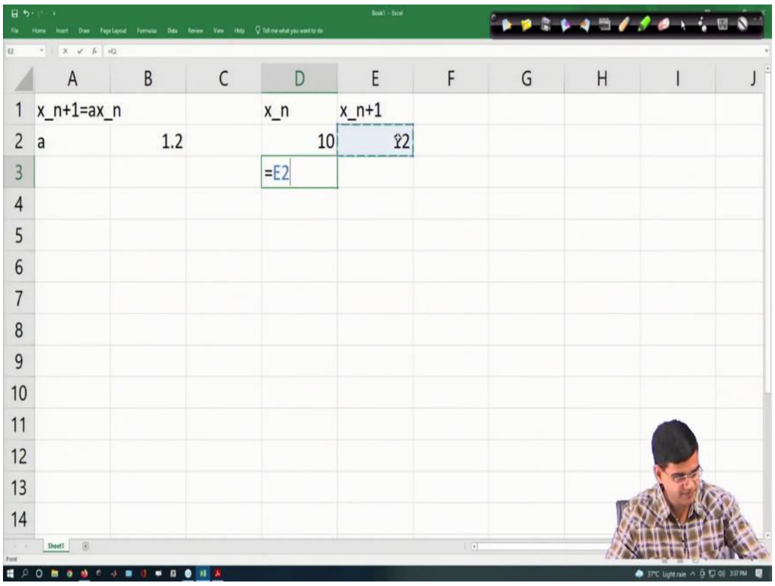
x_n
x_n+1
a
1.2
10
12
Sheet1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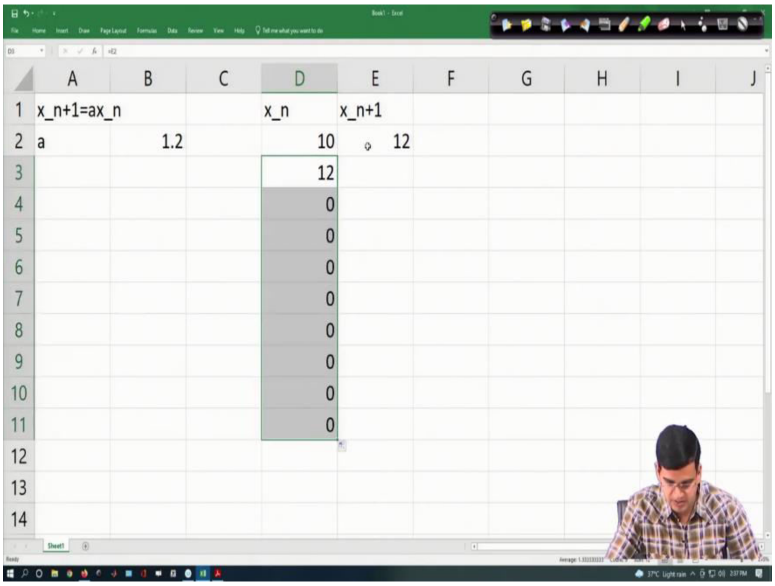
x_n
x_n+1
a
1.2
10
12
=E2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0
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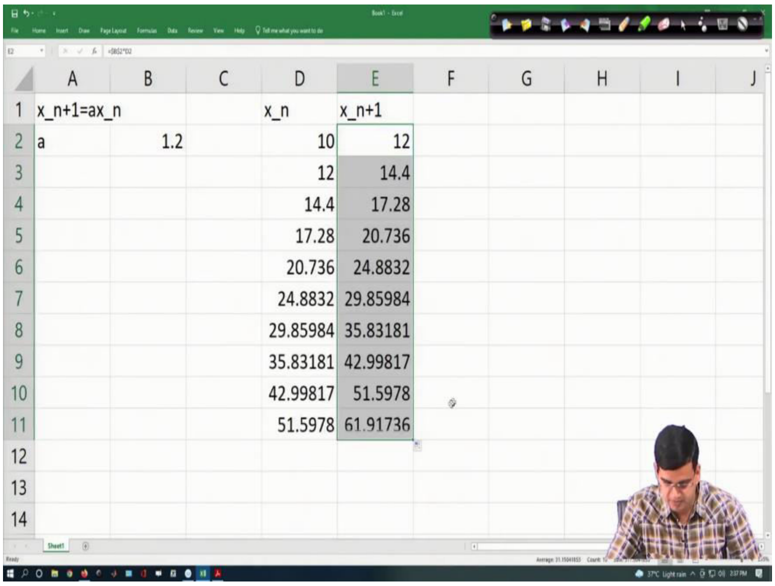
x_n+1=ax_n
x_n
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1.2
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14.4
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20.736
24.8832
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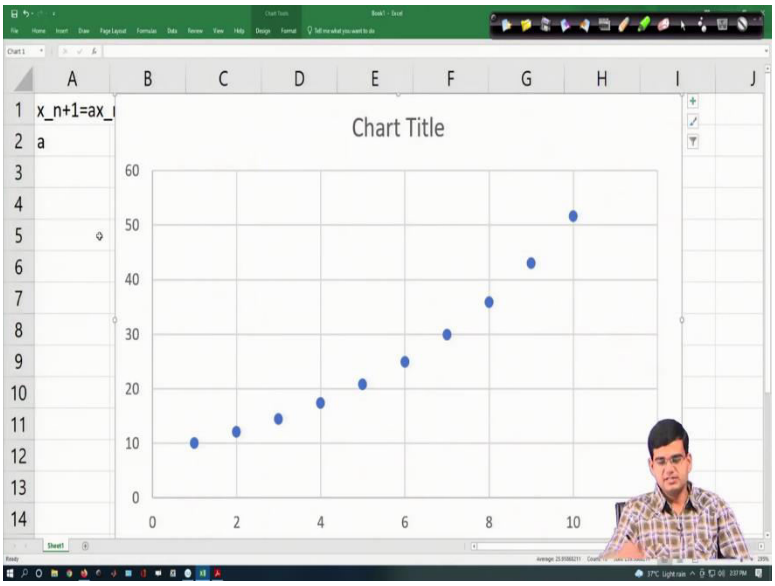
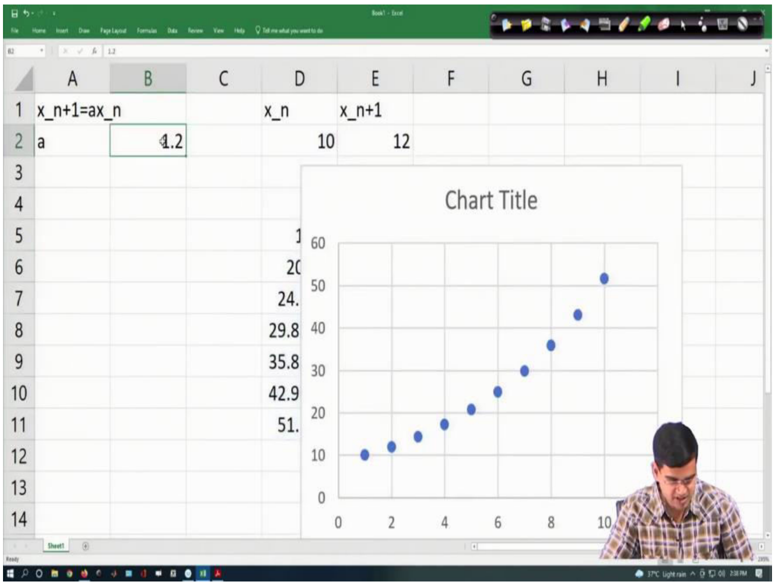
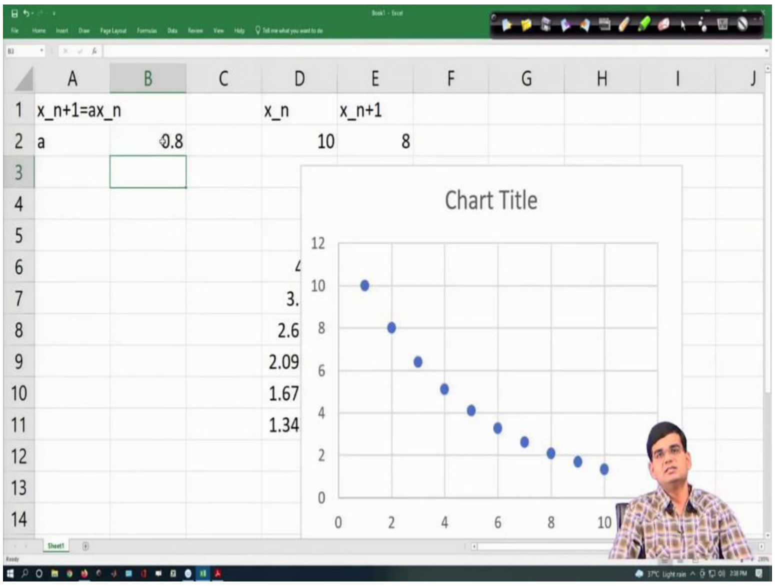

x_n+1=ax_
a
Chart Title


x_n+1=ax_n
a
1.2
x_n
x_n+1
10
12
Chart Title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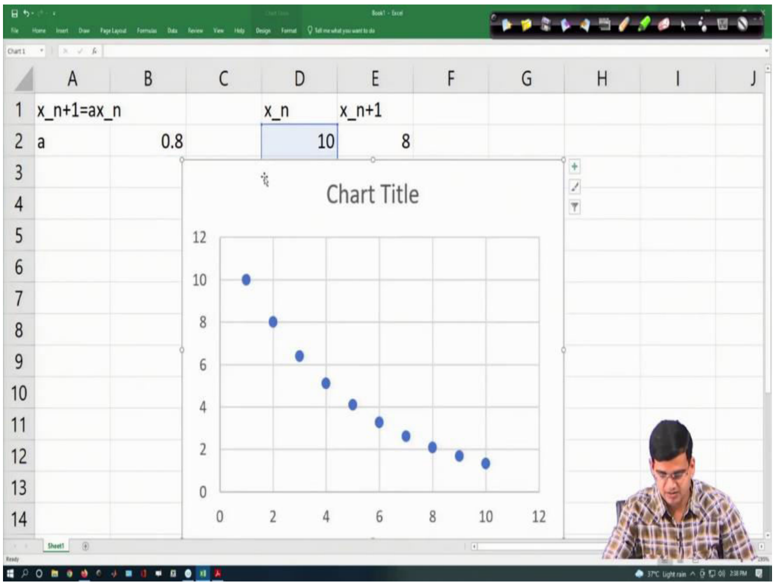
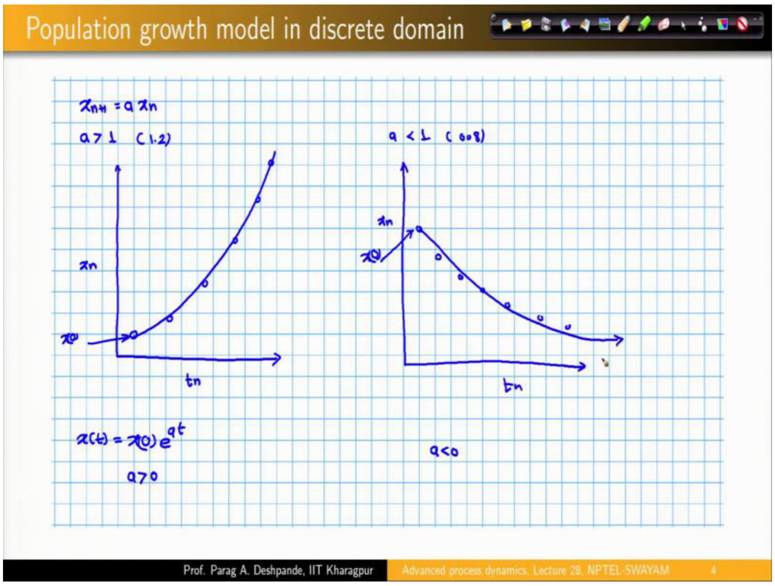
x_n
x_n+1
10
8
Chart Title





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Welcome back, we are currently discussing Population Dynamics, the various models that we considered till the last week were all continuous models. So, let us have a quick look into the models, which we looked at in the last week.

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We started with a linear model given as $\frac{dx}{dt} = ax$, we saw its limitations, we introduced carrying capacity parameter to give us logistic model then, to account for harvesting, we introduced another parameter h to give logistic growth with harvesting and finally, for certain species, where a threshold population is required for the species to thrive, we had logistic model with threshold population.

Now, if we consider any of these cases, then what we see is that all of these are continuous models. So, these are the models in continuous domain and what is the meaning of continuous in all of these models, the population *x* as well as time *t* are continuous in all of the cases, but, if we consider, let us for example, first two cases and try to determine the populations of species with some given initial population.

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So, I have this in front of me for example, I take the linear model. So, the solution of the linear model was given $x(t) = x_0e^{at}$. So, x_0 is the initial population, a is the growth factor logistic growth, $x(t) = x_0 e^{at} / (1- x_0 + x_0 e^{at})$. So, if I start with an initial population say $x_0 =$ 10 and I take a growth factor as 2. So, $a = 2$ in my case. So, the table on the right hand side gives you the evolution of the population in time.

So, what you see here is that 10 which is the initial population 2 which is the growth factor and time here t, they all so, x, t and e. So, in fact, not x, x_0, t and a they all are natural numbers, $x0$, t, a are natural numbers in this case, but even if these are natural numbers, which means that the inputs to your function growth function are natural numbers, the growth of population comes with numbers which are like this for example 73.89 and so on.

So, now, my question is that can I have the number of bacteria in a petri dish for example, equal to 73.89. The answer is no because that number has to be a natural number. So, in all these cases, which you see here are the following the linear model or nonlinear model, I did not bother about truncating or rounding off the decimal places to a fixed number to show you that, you know, none of these things actually make any sense because your population is a non-integer or it is a non-natural number. And in reality, you cannot have a non-natural population or non-natural number as your population.

So, there is a problem with these models. The problem is that even if your initial population your growth factor, your time are all natural numbers, your population predicted by the model may not be a natural number, and that seems to be a problem. But that actually is not a great problem, because let us see how do you actually get these populations.

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So, typically, what you will do is, you will do an experiment in which you will determine the variation of so, you will determine the variation of $x(t)$ with time. And suppose you do this experiment, then what you are going to get is, you are going to get a plot which looks like this. Let us say you did this experiment for a very small time. So, you are going to get this as t in the along the x axis x of t along the y axis, you will have some initial population x0 and then you will keep on having these observations.

And for small number of initial samples, let us say that these five are the points which you get after doing experiments, so, for initial growth, let us say you say that you find the linear model to work good enough. So, you have $\frac{dx}{dt} = ax$ in other words, you have $x(t) = x_0 e^{at}$. So, what you will try to do is you will try to determine the value of a determine the value of a and how are you going to do that, you are going to do that by curve fitting.

So, what you will do is you will assume some guess value of a and let us say that guess value of a gives you a curve which looks like this. So, you certainly know that that that value of x that value of a is not good. So, you will quantify this term not good by taking this reduced residuals, you will take the square of them some of you will sum then up and then you will change the value of a let us say you now get a quantity like this, you will repeat the procedure you will keep on doing this change of a till you get a curve which satisfies the condition that the Root Mean Squared Error is less than some number alpha, which you specify to the system.

Now, this alpha is determined by the user itself, which means that how accurately we would like to determine the value of a which means that if I determine the value of a as 2 then there is going to be an uncertainty, there is going to be an uncertainty into itself which means that a would be 2 plus or minus some quantity, say beta.

And if that is the case, then you actually cannot say that your population is 73.89 and so on, because this is this is the model predicted population, this is the model predicted population. So, therefore, there is an error bar associated with every entry and therefore, this puts me in a position to be a little confident that probably these, all of these things have no meaning and all the integral parts which you get which means these, these and so on are actually the real populations and the decimal part which I get they basically come from the error of estimation of a.

So, although setting up a set of parameters, which are natural numbers is giving me non-natural number population, I can always say that I can be confident that I am close to that population, which is the real population, which is actually going to be a natural number. So, this particular problem is not a really great problem of continuous domain model as it is thought and especially when the number n when the population x of t is a large number then compared to in this case for example, compared to 8886121105 compared to this, this is going to be a very small number and for all practical purposes you can you can consider that only the natural number closest to it is the population, you do not consider the decimal part.

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So, although this is a problem, this is not such a great problem, I have another issue related to these models and the issue is this that imagine that I have binary fission of bacterium of a bacterium, so, binary fission of bacteria and I want to now describe the dynamics of the population. So, I would have one bacterium and by binary fission, it would go to two bacteria. So, the population is 1 here the population becomes 2 here, now, each of the daughter bacterium will result into 2 further bacteria. So, I can further see this as this case. So, now, the population is 4, so, how is my system evolving? 1 goes to 2, 2 goes to 4, 4 goes to 8 and so on.

But as I am doing this experiment and noting down the evolution of the population is the evolution happening in continuous time? Which means that, will I have the observations which would be different populations of the bacteria in my petri dish at every instant of time, the answer would be no. Because I know that if you start with a bacterium, it would grow become mature, and then it would split into 2.

From the instant that it splits into 2, the daughter bacteria would start growing and they will again split into 2 once both of them become mature individually. So, if I consider that this happens over a period of maturity happens over a period of say, for example, 10 seconds, then what is going to happen at time t = 0 this is the number of bacteria which I have N of t so, at time t = 0, I have 1 bacteria. And then while going from one to two let us say it takes 10 seconds, so, this is 10 seconds. So, this is 0 seconds, this is 10 seconds.

So, now, this has become 2, the next one would be 4 but this observation will happen only after further 10 seconds. So, this is going to be 20 then what is going to happen is at 30 seconds my population will become 8. So, this is going to be 4, this is going to be 8. So, now I have a situation like this where this is let us say this is time 30 and this is my observation.

So, what kind of plot can I make? Can I join all these points with a continuous line say some something like this. Can I do this?

The answer is no, because between for example, 10 and 20 seconds, there was nothing which was going on as far as the population changing the population is concerned the bacteria was growing the number was not changing?

So, therefore, what I should do is, I should do something like this that initially I have a population which = 1 and then at 10 So, from 0 to 10 seconds the population remains a single bacterium it does not change and then it suddenly jumps to this value 2 again from 10 to 20, it remains constant and then it suddenly jumps to 4 between 20 and 30 it remains constant, then again it jumps to 8 and so, on.

So, therefore, the time is no more continuous, but it has become discrete, you do not have observations at all instances of time, but you have observations available only at certain instances of time and therefore, such domain is called discrete time domain because you do not have observations available at all in this particular domain. So, what now can be done for this particular analysis.

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So, let us see, now, I would like to know the population of the bacteria in time. So, I know that time t is now going to be like time t_0, t_1, t_2, and so, on. These are discrete quantities. Further what I see is that when I had one bacterium from one bacterium, I got two bacteria. So, I had 1 bacterium I got 2 bacteria. So, why did I get 2 bacteria, which means, that the population at t1 and I can consider this as population at t_0.

So, why did I get population at t_1 as x_t because I had certain population at t_0 you can appreciate it better by looking at this particular stage of your system that now, you have this that you have the total number of bacteria is 4 and you have x t_2. So, now, you get x t_2 because you had certain population at x certain population x t_1 if instead of 2 if this x t_1 were 3, now, you would have got 6 which means that x_{n+1} is for this particular case 2 times $x(n)$, x_n where n n plus 1 and so on indicate the instances of time that discretize time and since the time is discretized and you are making observations at those discrete times only your population also has been discretized.

So, now, what you see is that your model equation or your dynamical equation will $x_{n+1} = 2$ x_n not very difficult to see because I can simply write the series here 1, 2, 4, 8, 16, and so on. So, for any of this x_n if the x_n plus 1 would be obtained just by multiplying x_n by 2 that means, that the only equation which is possible for discrete time domain would be $x_{n+1} = 2$ times x_n, is this correct?

The answer is no, because, when you have large number of bacteria and you introduce a large number of bacteria in a traditional culture, it is not guaranteed that every single bacterium in that culture would be at the same stage of maturity with reference to its cell division.

So, we said in the previous we made a plot in the previous discussion today that you have a plot like this and so, on. So, this was xn, this was t let us say n. So, when you drew this plot, you started with 1 bacterium and you did not observe any division till 10 seconds in our previous example, this was 10, in the previous example, this was 10.

So, between 0 and 10 there was no division but that was because that bacterium which you started was only 1 and if you consider 10 bacteria, then if you assume that all of the bacteria are at the same level of maturity with reference to the its division, then you would see that at the same instant all of the bacteria get undergo binary fission that may not be the case different bacterium bacteria might be at different stages.

So, they might be undergoing cell divisions at different instances of time. And therefore, this number 2 can be changed to some other number a which basically would tell the fraction of the bacteria which undergo binary fission at any given instant of time. And therefore, let us say you had 50 number of initial number of bacteria and 10 of them are at a stage that at 1 second they undergo fission.

So, instead of having a blank reading from 0 to 10, at 1 second you will have some observations similarly, it is 2 seconds you will have some observations and so on. So, therefore, these portions of the dynamical plot will also be populated, which means that ultimately this continuous behavior which you got can be approximated even if the reality is something like this, what I am plotting as the lines vertical lines is the reality.

But then if you have large number of bacteria, which are at different stages of their maturity, then the interval of time between which these events, events of division take place, they will become smaller and smaller and smaller. And therefore, you can approximate a discrete event by a continuous event, in in case and it is very large.

But even if, let us say you do want to model this particular system, explicitly using discrete domain, so, this is why we say that continued why do continuous models work, in spite of the fact that is a fundamental flaw in the science behind science, which they follow for modeling the system? Why do they work? They work because *n* is generally a large number and therefore, Δt is small and since Δt small in the limit Δt tends to 0 the continuous observation and discrete observation they would merge.

But if I do want to model my system explicitly using continue using discrete domain, then let us see what I can do I will just make use of this particular observation I will make use of this observation.

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So, the observation is that x of n plus 1 is some quantity a some quantity a times x, some quantity a times x and this is my dynamical equation in discrete domain and what was my solution for the dynamical equation in continuous domain, $x(t) = x_0 e^{at}$ and my plot for this equation was this, $x(t)$, t if this is x_0 then this would be your $x(t) = x_0 e^{at}$, growth factor has been considered to be positive. So, therefore, you will see a growth or increasing behavior with time.

Now, it should not matter whether you model your system in discrete time domain or continuous time domain as long as your assumptions and the underlying physics is correct, you should get the same model prediction. So, therefore, if a is some parameter disk concerning the growth and your model equation is given as x_{n+1} = n times x_n then this also equation also should show an increasing behavior increasing population with time, let us see if this happens.

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So, let me plot this here. So, I have $x_{n+1} = a\, x_n$ this is what I want to plot. Now, let us say that a has some value and for the present case, let me consider it as 1.2. So, I need to start with some initial population. So, that initial population is at any instant of time is say 10. So, I start with the number of bacteria.

So, now, x_{n+1} is going to be what? The population at next instant of time. So, how can I get that. This will be equal to the initial population for a times the current population and I will make a $ here for the sake of the syntax, if you know how to run Excel, well, you would have understood why did I put a $.

Now, then in the next instant of time n plus 1 would act as the initial population. So, this is the meaning of the equation $x_{n+1} = x_n$ and you are evolving in terms of n. So, in the next step x_{n+1} will become x_n so, therefore, this is equal to this quantity and now, this way I can determine the evolution of my system. So, let me determine the evolution and what you can see here as xn is the evolution of the population.

So, let me see if I can draw the plot and see this is the evolution of your population you start with an initial population and you actually diverge you can see that your population is diverging following the trend, which is very similar to x_0e^{at}. So, whether you do it in continuous domain or discrete domain, the observation is the same it does start with a population and your population tends to infinity I can go on increasing the values of this time axis and you will see that the population increases.

Now, I do one thing I make this parameter from 1.2 to 2.8 and see what happens, this is this you will see is going is what is happening here, you start with an initial population and the population comes down.

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So, what is going on here is something which can be understood like this. So, I have $x_{n+1} = a$ x_n and when a was greater than 1 greater than 1 I use in my example 1.2 then the plot which I got was this x_n versus discretize time t_n the plot which I got was like this and when x when $a < 1$ in my particular case I used 0.8 the plot that I got x_n versus t_n was something like this.

So, what is going on? Well, we have seen this in case of continuous domain modeling I have x of t = x_0 what is x_0 this is your x_0 this is your x_0, the population at initial instants of time is equal to e^{ant} when $a > 0$, what was the trend that you got? So, on average increasing population.

And when *a* was less than 0 you saw that the population would converge to 10 to 0, this is what we all also see from here. So, what we have learned in this particular lesson is that, in fact it is possible to model your population dynamics in discrete domain your population modeling comes from the fact that population at any instant of time come from the population which was at the previous instant of time.

So, the simplest case which we saw corresponding to corresponded to the linear model of population growth, which we saw in the continuous domain modeling, where you can simply write the dynamical equation in the discrete form as $x_n + 1 = a$ times x_n.

So, if x_n is the population of your system at any given instant of time, then the population of your system at next instant of time would be given by multiplication of the population at that instant of time by a factor *a*. Now, if the value of the factor a is greater than 1, then you see that your system is unstable your system diverges and as time tends to infinity, you can expect the system to go to infinity, very similar to qualitatively very similar to the behavior which you saw for continuous system also, when the solution was given as d given as $x_t = x_0e^{ar}$ and *a* was greater than 0.

Now, your population would the dynamical variable would converge to 0. When your equation dynamical discrete equation is $x_{n+1} = a\ x_n$ with $a < 1$, again, behavior qualitatively very similar to the behavior $x_t = x_0$ it is our at with $a < 0$. We will continue our discussion on further analysis of such systems in the next lecture. Thank you.