# Advanced Process Dynamics

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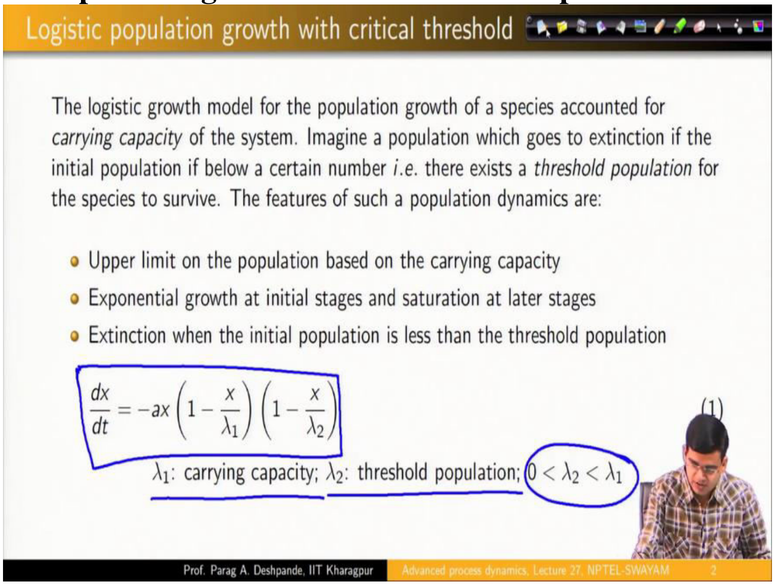
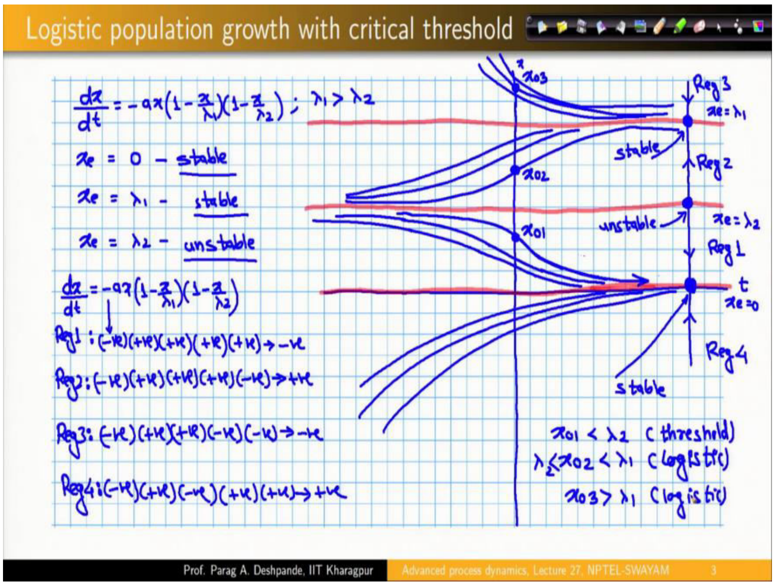
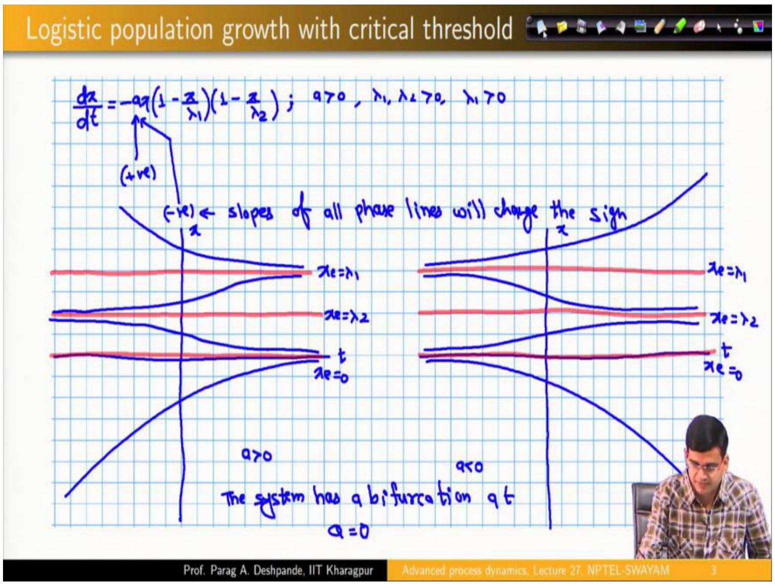
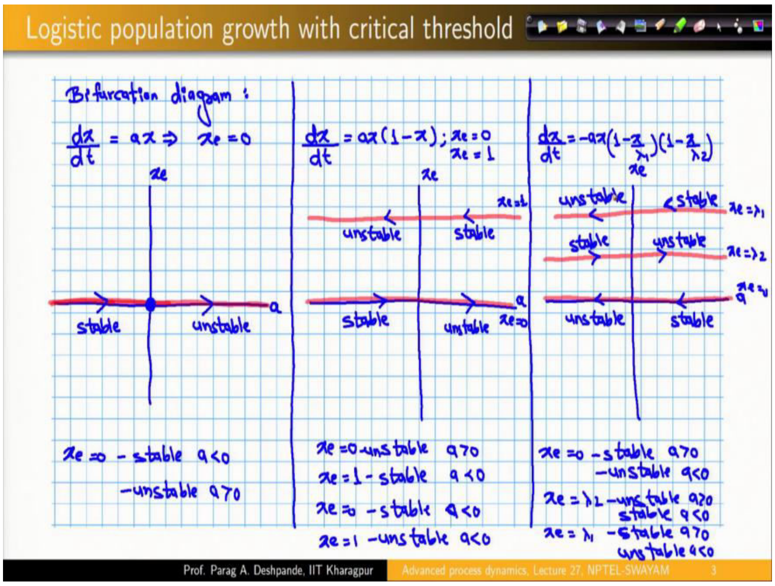
Lecture 27

## Logistic Population growth with Threshold Population continued…



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So, we were looking into logistic population growth with threshold population, so let us continue our discussion to quickly remind ourselves of our system. The dynamical equation was

$$\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right)$$

and $\lambda_1$ is the carrying capacity $\lambda_2$ is the threshold population. The condition for the inequality is provided here.

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So, we were trying to have a qualitative analysis of this equation, so as to understand the dynamical behaviour, what we saw in the previous lecture was this, that if $\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right)$ is your dynamical equation, then your $x_e = 0$, is an equilibrium solution, $x_e = \lambda_1$ is an equilibrium solution, and $x_e = \lambda_2$ is an equilibrium solution, the condition that we had was that $\lambda_1 > \lambda_2$, $\lambda_1$ is the carrying capacity $\lambda_2$ is the threshold population.

Yesterday's analysis we actually proved that x = 0 was a stable equilibrium solution, I am confident that you must have shown this at home, that $x_e = \lambda_1$ is an unstable solution, and $x_e = \lambda_1$ is a stable solution, and $x_e = \lambda_2$ is an unstable solution. How you must have done this? You define, dx/dt = f(x), take the derivative of f(x) and calculate the value of the derivative at these 2 points, at these 3 points in fact, x = 0 we already proved, determined, x = $\lambda_1$, and x = $\lambda_2$, you must have done.

For x = $\lambda_1$, you must have found that the derivative is negative always. For x = $\lambda_2$, you must have found that derivative is positive always. So, if I have this these results in front of me, then what I can do is, I can try to draw this x y plane where the x-axis is time, the y-axis is the population. My equilibrium solutions are x = 0, x = $\lambda_1$ and x = $\lambda_2$, so let me try to draw, rather highlight these populations here, this is xe = 0, this would be $x_e = \lambda_1$ , so this is $x_e = 0$, this is $x_e = \lambda_2$, the smaller of $\lambda_1$ and $\lambda_2$, and this is $x_e = \lambda_1$, the carrying capacity.

Now, I would like to draw, now these are the equilibrium solutions, now I would like to draw the phase lines, what I did previously for an analogous system was, that I divided this entire phase portrait into different regions, did the analysis of the gradients, and then try to draw the phase lines.

Let me repeat the same procedure here, so I have region 1, I have region 2, I have region 3, and I have region 4, in fact, it corresponds to negative population, but as we have been always doing for the sake of mathematical completeness, I will definitely populate this region with curves, you may not physically observe that, but we never know, tomorrow we may come across the system where this particular regime is observable.

So, region 1, so my equation is $\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right)$. So, my region 1 goes like this, negative sign is negative, a was always positive, so positive. x in region 1 is always positive, so this is positive. Now, x in region 1 is between 0, and $\lambda_2$ and $\lambda_2$ itself is less than $\lambda_1$, so 1 –

$x/\lambda_1$ is going to be positive, and so would be $1 - x/\lambda_2$. So, this is also going to be positive, so overall I see that this should be negative.

So, if I have any initial population, which is this, how should I draw a curve which shows the bottom red curve as an asymptote? This was my condition, so let us see if you agree that this can be one of the curves. So, to make it look like really curved so that we do not confuse it against a straight line, let me try to make it a little more curve and asymptotically you should reach this.

Now, let me go to the negative region, if I go to negative time, this curve will continue, and it would continue such that the slope is always positive, and now asymptote which it can see is $x_e = \lambda_1$, right. So, therefore, I can draw a solution curve, like this and you can draw multiple curves, which look or like this, the asymptotes would be x, $\lambda_2$ and 0, $\lambda_2$ for negative time, 0 for positive time.

So now, let me do the analysis for region 2, region 2, region, in region 2, x is between $\lambda_1$ and $\lambda_2$, right, $x > \lambda_2$, $x < \lambda_1$, so therefore the negative sign is negative, a is positive, x is positive, $1 - x / \lambda_1$ is positive, but since is $x > \lambda_2$, $1 - x/\lambda_2$ would become negative, so negative and negative will become positive.

By now, it might not be very difficult to see that what is going to happen is that I will have an initial population with always a positive slope, so you can and the asymptote being $\lambda_1$, that is the upper asymptote it can see, so it would be like this and therefore I will come here. This would be the 1 of the phase lines, so I can draw several other phase lines, you shall go like this.

What about region 3? In region 3, $x > \lambda_1$, okay, so negative sign, negative a, positive x, always positive, so positive, but $1 - x / \lambda_1$ is going to be now negative, so will be $1 - x/\lambda_2$, so overall this is going to be negative. So now, if I have an initial population which is like this, such that the curve has solution curve has always a negative sign and the asymptote is $\lambda_1$, so I can draw a solution curve like this, and I can draw several curves, and so on, fine.

And finally, region four for the sake of mathematical completeness, negative sign is negative, but x in this case, a is always positive, x in this particular case is negative, x is negative, and then $1 - x / \lambda_1$ is going to be positive, $1 - x / \lambda_2$ is also going to be positive, so what is going to happen, the slope is going to be always positive. So, how should I draw a positive slope curve such that the asymptote is $x = 0$, perhaps these would be the curves.

So now, as I said that $x_e = 0$ is a stable equilibrium solution, $x = \lambda_1$ is a stable equilibrium solution, and $x_e = \lambda_2$ is an unstable equilibrium solution. Can I have a look into this phase portrait and confirm the same thing? Let us do this, so let me draw a vertical line here, these are the equilibrium solutions, and I can see that all the equilibrium solutions tend towards $x_e = 0$, all the phase lines they are all coming to x is equal to 0, so therefore this is stable, which I also see from here, stable.

Now, at $x = \lambda_2$ the solution lines or the phase lines are going away, so therefore this is unstable. And finally, for $x_e = \lambda_1$, the solution lines converge all to $x = \lambda_1$, so this is stable, and you will see here stable, $\lambda_2$ is unstable. So, this is the phase portrait of the system which we developed without solving the equation explicitly, and now let us try to test whether physically this phase portrait makes sense.

So, let me take 3 conditions, 3 initial conditions, this is x01, the point x01, this is the point $x_0$, and this is the point $x_0 < \lambda_2$, and $\lambda_2$ is the threshold population, okay, $\lambda_2$ is the threshold population. And what did our model assumptions say? Our model assumption said that if your initial population is lesser than the threshold population, then it should extinct the population should extinct to 0.

So, therefore if I start with $x = x_0$, 1 as the initial population then you see that your population is going to 0. So, if your initial population is lesser than the threshold population, your population would become 0 as time tends to infinity, this was in conformation with our model assumption.

Now $x_{02}$ is between $\lambda_1$ and $\lambda_2$, it is between $\lambda_1$ and $\lambda_2$, and the kind of behaviour you see between $\lambda_1$ and $\lambda_2$ is exactly the same phase portrait which you saw in the previous lectures, initial exponential rise but followed by it you would see a saturation, saturation at the carrying capacity, the carrying capacity being $\lambda_1$ in this case, so therefore you have logistic, this is in confirmation.

Now, what also is in conformation with logistic model is that if your population is larger than the carrying capacity then the population should come down, and that is the case when x03 is the initial population, $x_{03} > \lambda_1$, again this is logistic, okay, that you start with an initial population which is greater than the carrying capacity of the system itself and in that case also you would die down to 0, okay, you would die down to 0. So therefore, its feels like whatever we have drawn is physically what makes sense.

Now, obviously you have to do this analysis only in the first quadrant where the time is positive and the population also is positive. The other 3 quadrants exist mathematically and therefore for such models where these particular quadrants are accessible you, would analyze this system in those quadrants, for the population dynamics you need to analyze only the first quadrant. So, this is what we saw and now what we would also like to do is to do an analysis of the bifurcation in the system. Does the system have a bifurcation? so you have a parameter a and we said that a is always greater than 0.

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So, let me write the equation $\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right)$, the condition was a > 0, $\lambda_1$, $\lambda_2$, greater than 0, $\lambda_1$ > 0, these were the conditions. And when I drew the phase lines, the equilibrium solution you would see is independent of a, it does not matter what a is, whether it is positive or negative, the equilibrium solutions remain. But the stability of the equilibrium solutions was determined on the basis of the signs of various quantities, and in all those cases we considered a as positive.

Now, the moment you make a negative, what is going to happen is, the slopes of all phase lines will change the sign, because negative sign is negative, the sign of x whether a is positive or negative will remain, the same the sign of 1 - x / $\lambda_1$, and all those quantities will remain the same, the only sign which will change is the sign of *a*, and therefore all the slopes will get inverted.

So, let me quickly draw the conditions here, I have t, I have x, I have t, I have x, my equilibrium solutions for this 1, 2, 3, 1, 2, 3, so the equilibrium solutions were $x_e = 0$, $x_e = \lambda_2$, $x_e = \lambda_1$, here also $x_e = 0$, $x_e = \lambda_2$, $x_e = \lambda_1$, and now for a greater than 0 the curves which we saw were like this, this is what we drew, few moments back, for a less than 0, what would happen there would occur an exact opposite sign for the derivative, so therefore you would see that, your face portrait would look like this.

Again physically, this is absolutely not which can happen but what you know is from this analysis that the system, the system has a bifurcation at $a = 0$, at $a = 0$, the system will offer a bifurcation. So, the way we develop the bifurcation diagram for the case of population dynamics with harvesting, can we draw the bifurcation diagram in this case also? let us see.

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So, bifurcation diagram. So, let us remind ourselves of what a bifurcation diagram is, a bifurcation diagram plots the bifurcation parameter along the x-axis and the equilibrium solution for solutions on the y-axis. So, let us in fact take various models and draw the bifurcation diagrams for all of them. So, the simplest model was dx by dt, the linear model is equal to ax, this was our linear model what was our equilibrium solution, for here my equilibrium solution was 0.

So, what I can do is, I can draw the bifurcation diagram here, so on the x-axis I will have the bifurcation parameter, in this case it is a, on the y-axis I will have x equilibrium, and I see that x equilibrium is simply 0, so in the, irrespective of the value of the bifurcation parameter, my x equilibrium is 0. Now, can I do some more markings on this, bifurcation parameter, bifurcation diagram, so as to get some more information. What I see, what I saw was that $x_e = 0$ was stable for $a < 0$, and unstable for $a > 0$.

So, how do I indicate this fact on this diagram? Whenever a > 0, your solutions would move away from your equilibrium solution, and what is going to happen is that, now you can say that for $a > 0$, your solutions would move away. And for $a < 0$ your solution would come towards $x_e = 0$, so I can draw this, so this is unstable, and this is stable. I am putting arrows just to show to indicate stability or lack of stability.

So, arrows pointing away means unstable, arrows pointing towards it means stability. So, for a greater than 0, that particular portion of your curve is unstable from minus in negative, all minus infinity to 0 a less than 0 is stable 1. Then, I have dx by dt is equal to ax, 1 minus x, let us consider the carrying capacity as unity, so I can draw here, the curves which look like this, so I have the bifurcation parameter a, I have the equilibrium solution e, and now we have 2 equilibrium solutions, $x_e = 0$, and $x_e = 1$.

So now, we know that $x_e = 1$ is a stable solution, and $x_e = 0$ is an unstable solution, for $a > 0$, so let me first make those plots, right, so this is $x_e= 0$, and $x_e = 1$, so $x_e= 0$ is stable for $a > 0$, and $x_e = 1$, sorry, for is unstable, is unstable and $x_e = 1$ is stable for $a < 0$. So, how do I indicate this, just an indication, $a > 0$, $x_e = 0$ is unstable, so I will write this as unstable, and I will make here as stable.

Now, for $x_e = 0$ becomes stable for $a < 0$, and $x_e = 1$ becomes unstable for $a < 0$, please refer to the previous lectures and you will get this conclusion. So, for $x_e = 0$ and $a < 1$ you have a stable part, and you have an unstable part here, fine. Now, we can do this analysis for the final model $\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right)$, so this is a, this is $x_e$, I have 3 equilibrium solutions, 1, 2, 3, the equilibrium solutions were $x_e = 0$, $x_e = \lambda_2$, $x_e = \lambda_1$.

And now, for so $x_e = 0$ was stable, right, for $a > 0$, so let me immediately write this, that this is stable, and it was unstable for $a < 0$, we saw it few moments back, so I can write this unstable. Now, $x_e = \lambda_2$ was unstable, for $a > 0$ and stable, for $a < 0$, so I can write this as this unstable, and stable, and finally $x_e = \lambda_1$ was stable for $a > 0$ and unstable, for $a < 0$, which means I can write here, stable and unstable.

So now, what we saw in these seven lectures is that there are various population models which can explain the dynamics of growth of a biological species in a region. Depending upon the model assumptions, there are various solutions, various stabilities, there are various instabilities which you can observe in the system, but what is pretty important to be noted is that, not all the time do you need to explicitly solve the problem. For certain cases, it is possible to solve the problem to get a qualitative idea about the problem without actually solving the problem.

So, if that be the case, what can, what we can do is we can develop the phase portrait, the phase portrait can be developed with the help of first determining the equilibrium solutions, determining whether those equilibrium solutions are stable or unstable, and then with the help of the derivative which is basically the definition of the dynamical equation, what you can do is, you can determine the derivative or the sign of the derivative of the quantities in different regions, and then you can draw the phase lines, so as to complete the phase portrait.

So, we will stop here today, and we will take a new topic of the dynamics of discrete systems from the next lecture onwards, thank you.