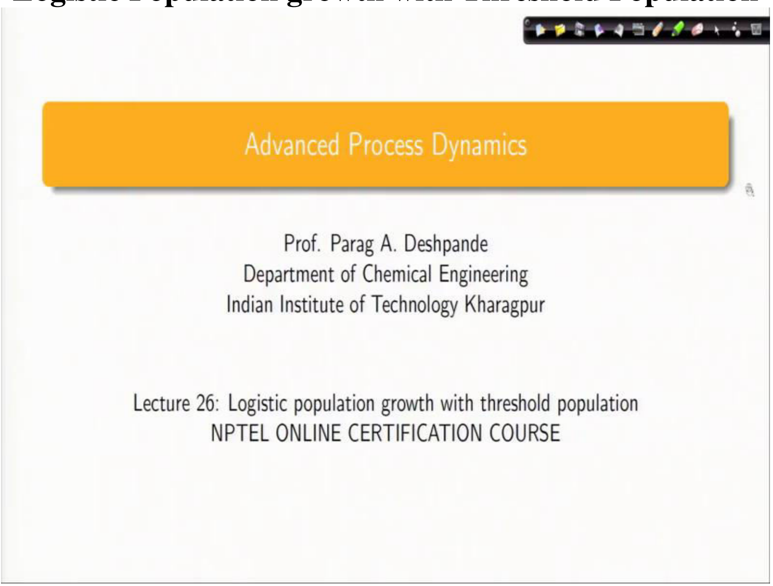
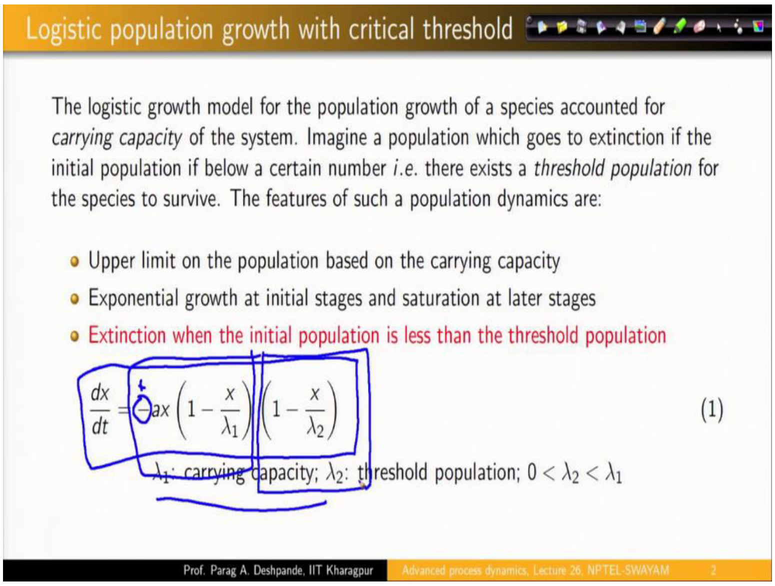
Advanced Process Dynamics

Professor Parag A. Deshpande Sharma
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture 26

Logistic Population growth with Threshold Population





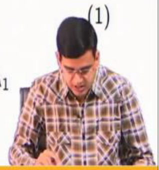
Logistic population growth with critical threshold

The logistic growth model for the population growth of a species accounted for *carrying capacity* of the system. Imagine a population which goes to extinction if the initial population if below a certain number *i.e.* there exists a *threshold population* for the species to survive. The features of such a population dynamics are:

- Upper limit on the population based on the carrying capacity
- Exponential growth at initial stages and saturation at later stages
- Extinction when the initial population is less than the threshold population

$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right) \quad (1)$$

λ_1: carrying capacity; λ_2: threshold population; $0<\lambda_2<\lambda_1$



Logistic population growth with critical threshold

$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right); \quad a>0,\ \lambda_1>0,\ \lambda_2>0,\ \lambda_1>\lambda_2$$

$$\Rightarrow \frac{dx}{dt} = \frac{-ax}{\lambda_1\lambda_2}(\lambda_1-x)(\lambda_2-x)$$

$$\Rightarrow \frac{dx}{dt} = \left(\frac{-a}{\lambda_1\lambda_2}\right)x(\lambda_1-x)(\lambda_2-x)$$

$$\Rightarrow \frac{dx}{x(\lambda_1-x)(\lambda_2-x)} = \left(\frac{-a}{\lambda_1\lambda_2}\right)dt$$

$$\frac{1}{x(\lambda_1-x)(\lambda_2-x)} = \frac{A}{x}+\frac{B}{\lambda_1-x}+\frac{C}{\lambda_2-x}$$

$$\Rightarrow A(\lambda_1-x)(\lambda_2-x)+Bx(\lambda_2-x)+Cx(\lambda_1-x) = 1$$

when $x=0$, $A=\frac{1}{\lambda_1\lambda_2}$

when $x=\lambda_1$, $B=\frac{1}{\lambda_2-\lambda_1}$

when $x=\lambda_2$, $C=\frac{1}{\lambda_1-\lambda_2}$

$$\Rightarrow \left(\frac{1}{\lambda_1\lambda_2}\frac{1}{x}+\left(\frac{1}{\lambda_2-\lambda_1}\right)\frac{1}{\lambda_1-x}+\left(\frac{1}{\lambda_1-\lambda_2}\right)\frac{1}{\lambda_2-x}\right)dx = \left(\frac{-a}{\lambda_1\lambda_2}\right)dt$$

$$\Rightarrow \frac{1}{\lambda_1\lambda_2}\ln x-\left(\frac{1}{\lambda_2-\lambda_1}\right)\ln(\lambda_1-x)-\left(\frac{1}{\lambda_1-\lambda_2}\right)(\lambda_2-x) = \frac{-at}{\lambda_1\lambda_2}+C$$

$$\Rightarrow \left(x^{1/\lambda_1\lambda_2}\right)(\lambda_1-x)^{-\left(\frac{1}{\lambda_2-\lambda_1}\right)}(\lambda_2-x)^{-\left(\frac{1}{\lambda_1-\lambda_2}\right)} = f(t) \quad -(1)$$

Explicit representation of x as a function of t is difficult

Logistic population growth with critical threshold

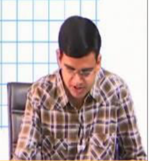
$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$$

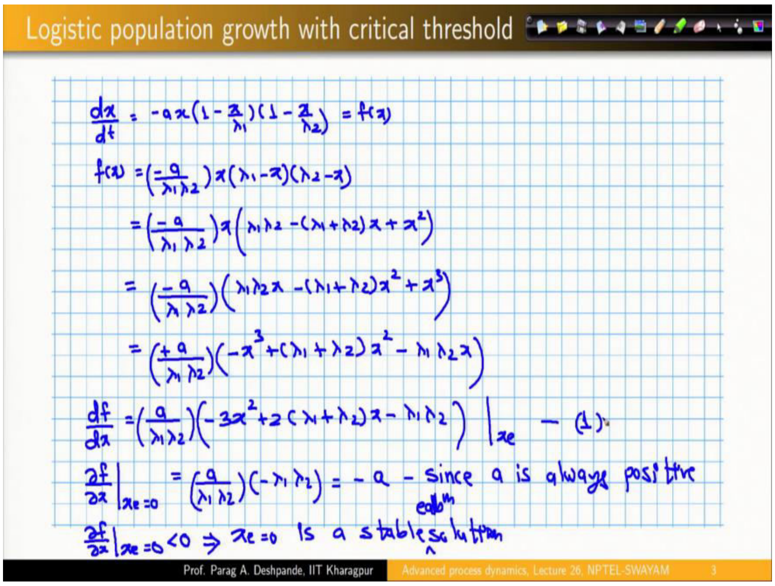
Equilibrium solutions: $\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)=0$

$\Rightarrow x_e=0;\ x_e=\lambda_1;\ x_e=\lambda_2$ ← three eqlbm soln

$\frac{dx}{dt} = ax\left(1-\frac{x}{N}\right) \Rightarrow x_e=0;\ x_e=N\ (\equiv\lambda_1)$ ← two eqlbm soln

$\frac{dx}{dt} = f(x) \rightarrow \left.\frac{df}{dx}\right|_{x_e}$ $\begin{cases} >0 & \text{- unstable} \\ <0 & \text{- stable}\end{cases}$





Welcome back, we are currently studying non-linear dynamics, and for understanding non-linear Dynamics we are taking the examples from population dynamics, how the population of a species varies with time? We have taken several examples and for the case of logistic growth, we saw that there is a upper saturation which means that, if you start with the population which is lesser than the carrying capacity of the system, then the population increases and saturates to that value. And, if the initial population is more than that value, then the population decreases and settles again to that carrying capacity of saturation population, it was quite evident by various various phase portraits that we developed.

Now, imagine that we have a situation where there is a threshold population for a species to survive, what is the meaning of this? This means that if the population of the species is lesser than the threshold population, then the population must die out to 0. So, we have a situation in which now I have 2 populations which need to be considered, the highest limit which is the carrying capacity or the saturation population, an intermediate new limit, which means that if your population is between the threshold and the saturation limit, then the population will thrive and it would move towards the saturation population, but when the population becomes lesser than the threshold population, then it would die out to 0.

Can we imagine such a case where this would be applicable? We call human beings as social animals, so what exactly do we mean by social animals? Well in fact at today's age, we need to survive in groups, we need to, we need to survive in societies, we need to live in Societies in order to survive. How can we imagine this particular case? Imagine that we have very,

very, very, few farmers or imagine that we have a human population on Earth which is very small, and proportionately there would be very small number of farmers in the society, so therefore the crops that are grown might not be sufficient to sustain this population, to feed this population, you need some farmers to grow these crops, even if the land is available.

Now, imagine the situation that you know we in fact those number of farmers are in fact able to grow sufficient number of, sufficient amount of crops, then these crops have to be distributed among the entire popular, groups of populations which would be divided in different geographical locations, how would that distribution take place? What would be the mode of transportation? Who would do that transportation? Who would produce oil for transportation of these vehicles? Who would manufacture those vehicles? Or in absence of human species or in absence of sufficient number of members of human species, all of these problems would arise.

If you want to manufacture vehicles on a commercial scale, so that transportation of crops is possible. If you want to produce petroleum oils on a commercial scale, you need, you would need investment, so you need rich people who can make investments in large numbers, and so on and so forth, if you do not have the number of human beings in sufficient numbers, then in absence of this entire Supply Chain management, the population will not thrive.

You may not have sufficient number of doctors available, and therefore if you do not have doctors then what would happen is that in case any of the disease, even minor spreads, it would be difficult to be contained or to be addressed or to be treated, so therefore today human species need certain minimum number of population to thrive or to survive. And therefore, probably, it is not a bad idea to consider the concept of threshold population for population Dynamics, can we have other examples? We see wolves always in packs, in groups, again, the evolutionary reason is the same that you need to be in group and if you, if a particular wolf is left alone in the forest, survival becomes difficult.

You see the prides of lions, they also move in groups, in fact there is an interesting debate which always goes on, whether lion is a, lion is more powerful or a tiger is more powerful? We generally think that lions are the most powerful animals, currently alive on Earth. But, physically if you see this is not true, a tiger is physically more well-built, it is heavier, it is larger in size, it has better strength of the teeth, it is more agile.

So, from the viewpoint of survival as a lone member of the species, tiger is much more well-fitted to survive compared to lions, lions are much smaller in size compared to Tigers, they are not as strong, they are not as agile, but then how do lions then survive? They survive because they are present in groups, they hunt in groups. Tiger can hunt alone. It is not possible for a lion to hunt alone, and therefore, they need to come up with a strategy to do hunting. So that they do not die of hunger.

Again, if you have 1 single lion left in the forest, probably the species would not survive, that particular member will go to, will die. Do we have other examples? Well yes, these days during this pandemic you must have come across this term very often, serological survey, so what does this serological survey give? What it, what is done is that you take the sample blood sample and determine the presence of antibodies, which act against the virus in the present case novel coronavirus.

So now, how is this particular example relevant to our case? Serological survey tells whether the person has been exposed to the viral load previously or not. Now, if the person has been exposed to the viral load, you would expect that the person would have been infected, and he would show symptoms, but that does not happen. This particular case happens because the viral load in the human, in that particular human body is very small.

So, once the virus enters the human body, human's immune system kicks in, starts making antibodies. The process of making antibodies follows certain dynamics which means that the concentration of the antibodies in the body would rise with time, it would follow certain dynamics. And if the dynamics of the increase in the number of viral, number of virus in viruses in the human body is larger, than you would expect that you the viral load would grow at least following logistic model, so there would be an upper saturation, so virus will not become will go to Infinity, the number will not go to infinity, there would be certain viral load, but that viral load may be good enough to cause Health damage.

But, if you have a situation where the viral load is very small, so at initial stages, the growth of the virus is such that the number of viruses in the body is small, and you can make more number of antibodies, the bodies can body can make more number of antibodies, so as to counter that viral load, then what will happen the population of the viruses in the body would go down or in other words the wire load would go down. So, therefore, you need certain minimum viral load in the body for the viruses to thrive in the body, and do any specific damage, so again this is a case where you have threshold population.

So, how can we take into account the autonomous growth, which means that initial exponential growth followed by saturation, but at the same time you have a factor accounting for the decay of population, when your population is lesser than the threshold population. This is what we are going to study today.

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So, let us look into you know today's topic and the equations given like this, the equation is given like this. You have the equation as

$$\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right)$$

where λ_1 is the carrying capacity, so if you just consider this part without the negative sign, without the negative sign, then this is simply the logistic equation. Now, to account for the threshold population, what you do is, you add this extra term. We will see how adding this extra term helps in determine in governing the equation to give the correct dynamics, but let us see what is our system.

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So, our system is like this, that the logistic growth model that we considered previously, accounted for the carrying capacity, so the upper saturation was captured properly. Now, if you imagine a population which goes to extinction if the initial population is below certain number, which means that there exists a threshold population, then what kind of features should the population have or the model have.

Upper limit on the population based on the carrying capacity which is basically the logistic model. Exponential growth at initial stages, and then saturation again, your logistic model, but extinction when the initial population is less than the threshold operation. So, this is my governing equation, 1 / λ_1, sorry λ_1 is the carrying capacity not very difficult to imagine, what λ_1 is, and λ_2 is the threshold population so population as a minimum population which is required for the population to thrive. So let us analyse this equation, let us see how we look into it.

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So, our equation is,

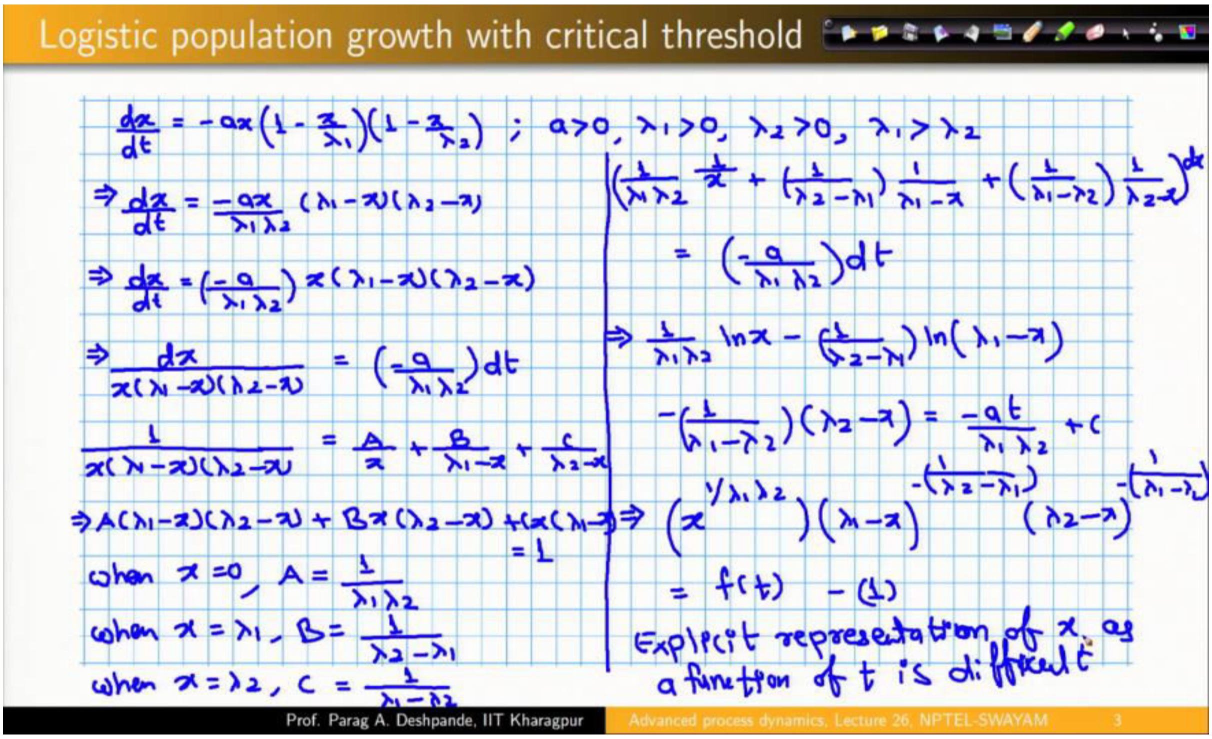
$$\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right)$$

and a few things which can be written about this equation is rather the parameters of this equation is that I know that a must be greater than 0, because a is the growth parameter. My λ_1, and λ_2 are populations, the saturation population of the carrying capacity and the threshold population, so λ_1 must be greater than 0, and λ_2 must be greater than 0, and what should be the inequality which must be satisfied for λ_1 and λ_2? λ_1 must be greater than λ_2, because λ_1 is the carrying capacity, and Lambda which is the threshold population, so Lambda 1 must be greater than λ_2.

So, let us try to First solve this equation explicitly and let us see if we can get an idea about the population Evolution with time If we are not able to solve it explicitly then we would also like to see that without solving explicitly can we comment upon the, at least the qualitative dynamical behaviour. So, to solve this, what I will do is, I will do a little simplification, so

$$\frac{dx}{dt} = -\frac{ax}{\lambda_1\lambda_2}(\lambda_1 - x)(\lambda_2 - x)$$

So, I can do a small rearrangement again, d$\frac{dx}{dt} = -\frac{ax}{\lambda_1\lambda_2}$ because all of these are parameters, they can just be combined into 1 parameter multiplied by x, $\lambda_1 - x$, $\lambda_2 - x$, fine, and how do I solve this equation? I will solve this equation, at least this particular equation can be solved easily by splitting the right hand term, right hand side term, in term by partial fractions.



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Now, by looking at equation number 1, I can see that explicit representation of x as a function of time is difficult. So, explicit representation of x as a function of t is difficult, it is not a simple expression as we used to get previously, like x is equal $x(t) = x_0 e^{at}$ and so on.

So, therefore, although we have a solutions method, we had a solution strategy, and the solution strategy was not very difficult in fact, it started off with rearranging the terms, we did partial fractions, got the logs or the arrange the logs, so solution is not very difficult, but is a little difficult is to get x as an explicit function of t, that proves to be a little difficult. So, can we have a different way of analysing this problem, so that we actually solve this problem without explicitly getting the final functional form or in other words, can we have the qualitative behaviour of this particular problem.

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So, let us see if we can have a qualitative analysis. So, I have

$$\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right)$$

This is the equation which I have. Now, I would like to know the qualitative behaviour, and the first thing which I can do is, I can determine the equilibrium solutions, equilibrium solutions. How do I get the equilibrium solutions? I get the equilibrium solutions by setting

$$\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right) = 0$$

Now, since this is a cubic equation, I have in fact have 3 equilibrium solutions, x equilibrium is equal to 0, $x_e = \lambda_1$, and $x_e = \lambda_2$, okay. Let us compare this against our logistic equation, our logistic equation was $\frac{dx}{dt} = -ax\left(1 - \frac{x}{N}\right)$ and my equilibrium solutions in this case where x = 0, and e is equal to n, which in the present case is nothing but λ_1 is the carrying capacity of the system.

So, we had 2 equilibrium solutions in logistic model, and in this model, we have 3 equilibrium solutions. So, now what we previously did was that once we had the equilibrium solutions, we try to find out whether the solutions were stable or unstable, so let us try to find out from this equation whether the equations are stable or unstable.

So, how do I determine whether the equations are stable or unstable? if I write dx / dt = f(x), then what I do is I determine df/dx at equilibrium solution, and this is greater than 0 means unstable system solution, and less than 0 means stable system, so we need to check this for $x_e = 0$, $x_e = \lambda_1$ and $x_e = \lambda_2$, so let us check this.

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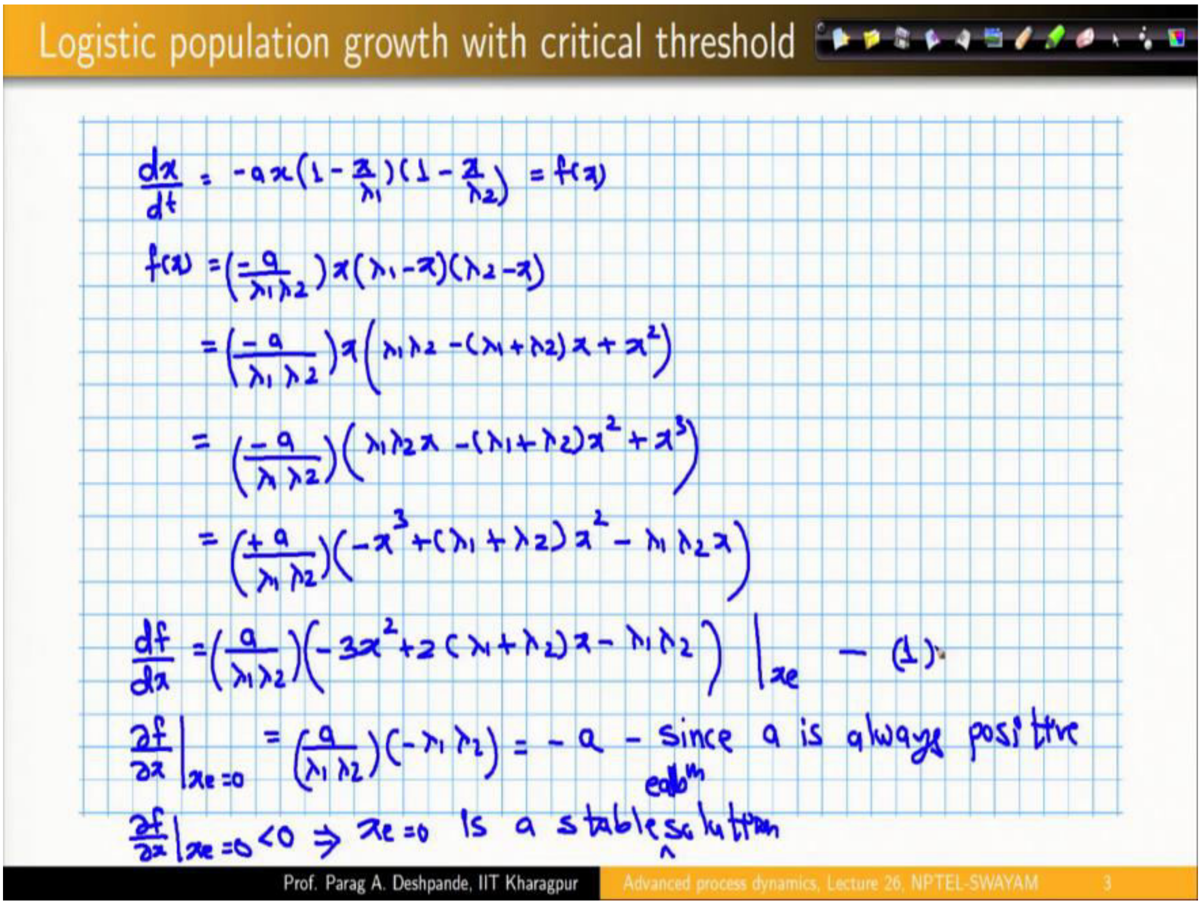

Now, what you will have to do is, you will have to repeat this exact same procedure by considering $x_e = \lambda_1$, and $x_e = \lambda_2$.

I encourage you to substitute the value of Lambda $x_e = \lambda_1$ in equation 1, and try to find out whether df / dx is less than 0 or greater than 0 and decide whether that particular, so equilibrium solution is stable or unstable similarly you will substitute the value of $x_e = \lambda_2$ and determine whether this solution is stable or unstable. Now, based upon this fact whether the solution is stable or unstable, what you can do is you can further think of determining the phase portrait of the system.

So, what we will do is, we will stop here today and leave this determination of stability of $x_e = \lambda_1$ and $x_e = \lambda_2$ as an exercise for you to work it out at home, and tomorrow we will continue from this point onward to develop the phase portrait, so that we can have a qualitative idea about the dynamical behaviour of the population dynamics equation with threshold population, bye.