Advanced Process Dynamics

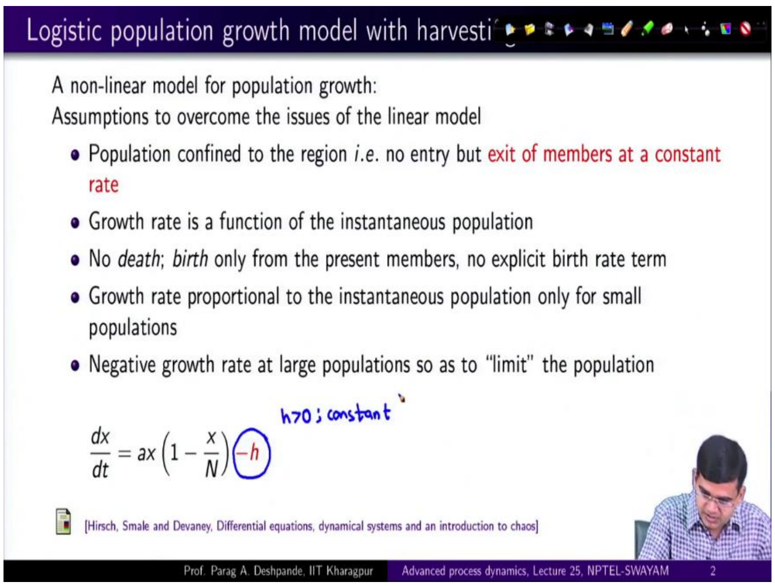
Professor Parag A. Deshpande

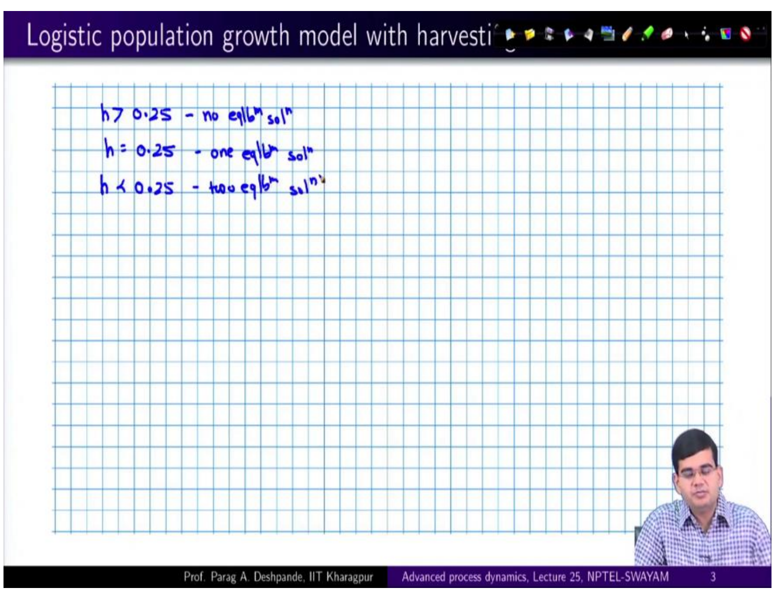
Department of Chemical Engineering

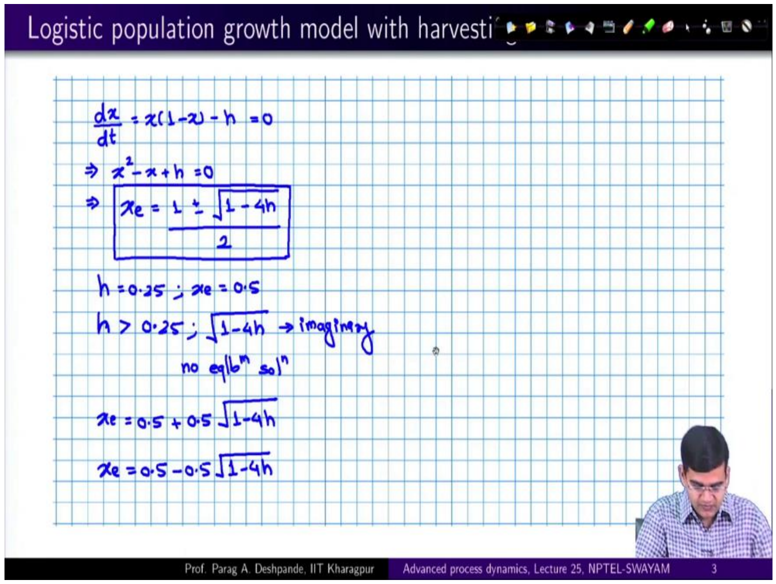
Indian Institute of Technology, Kharagpur

Lecture 25

Logistic Population Growth with Harvesting Continued…






Logistic population growth model with harvesti
dx/dt = x(1-x) - h = 0
⇒ x² - x + h = 0
⇒ xe = (1 ± √(1-4h)) / 2
h = 0.25 ; xe = 0.5
h > 0.25 ; √(1-4h) → imaginary
no eqlbm soln
xe = 0.5 + 0.5 √(1-4h)
xe = 0.5 - 0.5 √(1-4h)
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 25, NPTEL-SWAYAM
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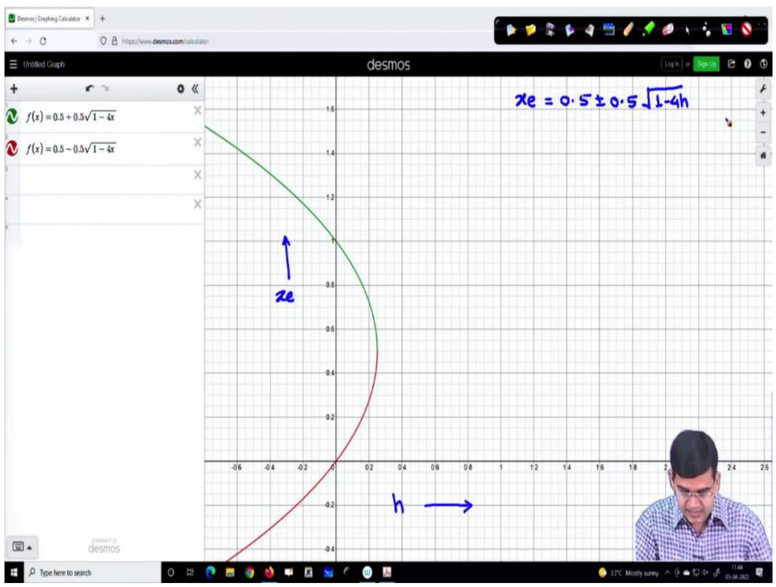
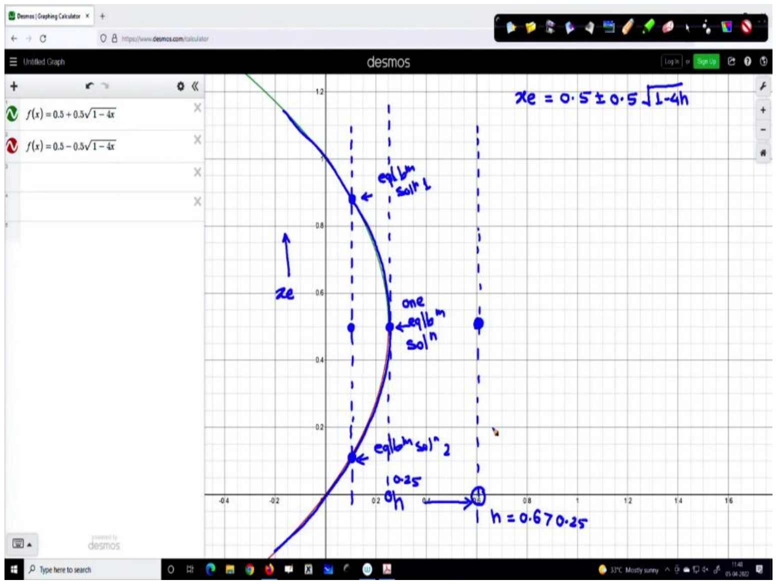
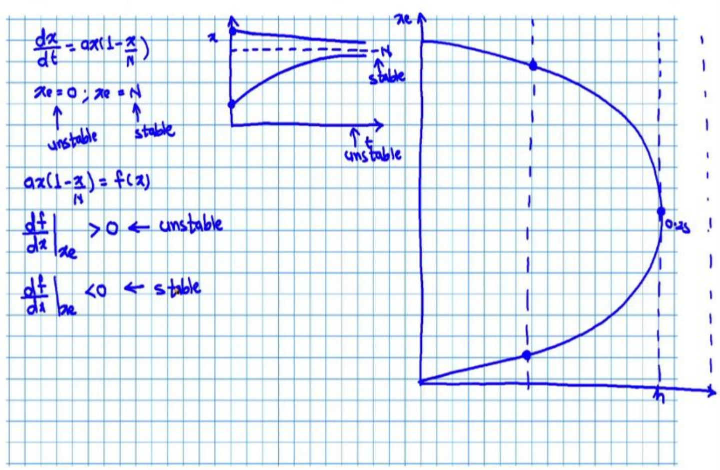
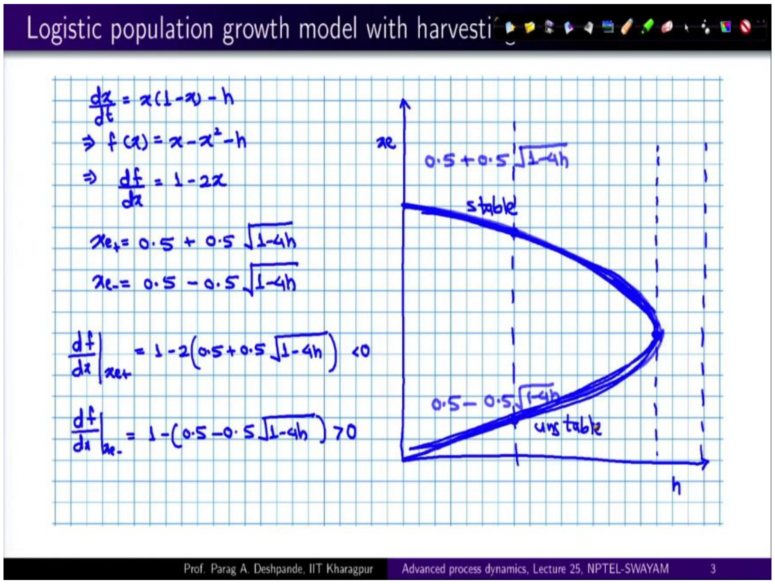
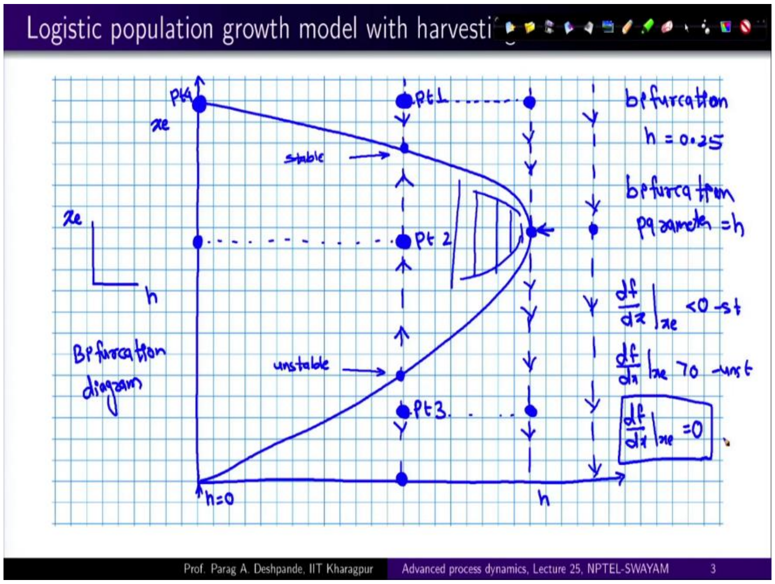

desmos
f(x) = 0.5 + 0.5√(1 − 4x)
f(x) = 0.5 − 0.5√(1 − 4x)
xe = 0.5 ± 0.5 √(1-4h)
xe
h



Logistic population growth model with harvesting







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So, we were looking at the population dynamics with harvesting. Let us look at our system. Our system was a population growth logistic model with harvesting. So, I have the constant harvest parameter h, h is always positive. This was an assumption. And h was a constant. This was also the model assumption.

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And we looked into the solutions of, rather the equilibrium solutions of this particular system in the previous lecture and we concluded that for $h > 0.25$, you have no equilibrium solutions. Makes sense. If you take very large amount from your pond or from your bioreactor, you are going to be left with nothing.

For $h = 0.25$, you have exactly one equilibrium solution. This also made sense to us because if you take some amount and there is a natural growth, you can expect a constant number of the population of the species in the pond or the bioreactor. The problem is that for $h < 0.25$, we got two equilibrium solutions. And we were intrigued to see that how can there be two equilibrium solutions.

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So, let us see how to understand this particular aspect. So, I have

$$\frac{dx}{dt} = x(1 - x) - h$$

And the equilibrium solutions were given by setting this as zero.

$$\frac{dx}{dt} = x(1 - x) - h = 0$$

which means I can write this

$$x^2 - x + h = 0$$

In other words, my equilibrium solution would be

$$x_e = \frac{1 \pm \sqrt{1-4h}}{2}$$

So, my equilibrium population is a function of the harvest rate.

So,

$$h = 0.25\ ;\ x_e = 0.5$$

And if you remember the plot which we made yesterday, the point which you got was actually (0.5,0). So, this is exactly 0.5. From here, from the analytical solution also you get the same thing. So, the case of one equilibrium solution makes sense here.

For, $h > 0.25$, what will happen is that the $\sqrt{1-4h}$ would become imaginary. And therefore, since the population has to be a real number, you say that there are no equilibrium solutions. But now, how does exactly $\frac{1 \pm \sqrt{1-4h}}{2}$ look like? What is the dependence?

So, let me plot this. So, I have

$$x_e = 0.5 + 0.5\sqrt{1-4h}$$

And

$$x_e = 0.5 - 0.5\sqrt{1-4h}$$

Mathematically, it makes sense. We previously got the fact that you would have two equilibrium solutions. From the analytical analysis also, you got two equilibrium solutions corresponding to the + and - sign. Let us see if we can plot the dependence of equilibrium solution on h and determine how the whole system would look like.

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So, I have

$$f(x) = 0.5 + 0.5\sqrt{1 - 4x}$$

So, can I see the plot here? Just let me clear it. So, the plot is this.

And now, the other solution

$$f(x) = 0.5 - 0.5\sqrt{1 - 4x}$$

See, this is how the variation of equilibrium solution with the parameter h looks like. Remember, in this particular plot, in this particular plot, I am plotting f(x) which is the equilibrium solution. So, let me write here. Along this axis, I have x_e and along the x-axis, I have the bifurcation parameter h, because I wrote $x_e = 0.5 \pm 0.5\sqrt{1 - 4h}$. This was my equation.

And therefore, along the x-axis I have the, I have the parameter h, along the y axis, I have the parameter x_e. So, let me analyze this plot carefully and see if I can make some conclusions. So let me zoom it in. During any operation, when you are operating your bioreactor or your fishery, you will keep the harvest rate constant.

So therefore, how would I draw a line, how would I draw a line which would correspond to a constant harvest rate? It would quite simply be a vertical line. So therefore, if I draw a vertical line on this plane, it would give me the dynamics of my system. So, if I take one point here, if I take one point here, then a vertical line would give me the dynamics of my system, how my system is evolving.

And this curve, which you can see here, is the equilibrium curve. This is the equilibrium curve, and the dotted vertical line is your operational line. So, the intersection of your operational line with the equilibrium curve would give you the equilibrium population of your species. In this particular case where I have drawn the point here, you do not see any intersection. And since you do not see any intersection, there is no equilibrium solution.

And see what is the value of h here. $h = 0.6 > 0.25$. So, this makes sense that if your harvest rate is large, you do not have an equilibrium population. Let me see what is the exact value of this point here. This point is (0.25, 0.5) which means that I have the harvest rate as 0.25.

Does it click anything? Yes, that is the harvest rate at which you have exactly one solution. And 0.5, and 0.5 was the equilibrium population which we determined previously.

So therefore, if I start with this point which is (0.25, 0.5) and draw a vertical line like this, then what is going to happen? I am going to encounter exactly one equilibrium solution or equilibrium population. So, I am currently doing an operation in which I have set my h = 0.25, and I will get only one solution. Makes sense. And then, I have an arbitrary point which is this. See this.

Now what happens? How would my dynamics look like? For this, I will have to draw a vertical line passing through this point, passing through this point, and what I observe is that I have two equilibrium solutions, equilibrium solution 1 and equilibrium solution 2. Again, the entire analysis looks to be in place. The only thing which is left for me to understand, the physical significance of the two solutions.

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So let me go back to the sheet and try to highlight the importance of the two points. So, I have here, this is h. So let me extend it further. This is h, this is x_e. And this is the point 0.25. I have, so let me draw a dotted line here, let me draw a dotted line here for h > 0.25, and again, a dotted line for h < 0.25.

Now, what I need to do is, I need to understand if that if there are two equilibrium solutions, then what is the meaning of the two equilibrium solutions? Did I have equilibrium solutions which were more than one previously? In fact, I did. Let us see,

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)$$

in the previous case. And the equilibrium solutions were

$$x_e = 0\ ;\ x_e = N$$

The difference between the two populations, equilibrium populations was that this was unstable, and this was stable. How did I ascertain that the lower was unstable and the top one was stable? So, you can do, for that particular case, you could do the analysis very simply. You have t, you have x, you have N and for a starting initial population, you had this, and then you had this.

So, since all of the solution lines were converging at N, you said this is stable. And for the lower one you said, since this was, the solutions were moving away, you said this was this as unstable. Further, you also did one thing. You assigned

$$ax\left(1-\frac{x}{N}\right)=f(x)$$

This is what you did. Then, what you did was you determined

$$\left.\frac{df}{dx}\right|_{x_e}$$

These all are the steps which we did previously.

And when

$$\left.\frac{df}{dx}\right|_{x_e}>0, \qquad unstable$$

And in fact, the equilibrium solution was unstable, and when

$$\left.\frac{df}{dx}\right|_{x_e}<0, \qquad stable$$

So, now you have two equilibrium solutions in your system. So, for the current case also you can do one thing. You can assess whether the solutions are stable or unstable.

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So, let us do that. I have

$$\frac{dx}{dt} = x(1-x) - h$$

implies that

$$f(x) = x - x^2 - h$$

From here, I can write

$$\frac{df}{dx} = 1 - 2x$$

Fine? And what are my equilibrium solutions? My equilibrium solutions are

$$x_e = 0.5 + 0.5\sqrt{1-4h}$$

and

$$x_e = 0.5 - 0.5\sqrt{1-4h}$$

And let me remind you that in this plot, x_e versus h, you have this point. So, drop a vertical, one solution, I will drop another vertical, no solution, and I will drop two verticals, two solutions. In this particular diagram, the upper curve, I hope you remember, corresponds to $0.5 + 0.5\sqrt{1-4h}$ and the lower curve corresponds to $0.5 - 0.5\sqrt{1-4h}$.

So, what is the next step? What the next step is I will determine

$$\left.\frac{df}{dx}\right|_{x_e^+} = 1 - 2\left(0.5 + 0.5\sqrt{1-4h}\right) < 0$$

And you will find that

$$\left.\frac{df}{dx}\right|_{x_e^-} = 1 - 2\left(0.5 - 0.5\sqrt{1-4h}\right) > 0$$

And what I can learn from here? What I can learn from here is, this entire top portion this entire top portion is the stable part. And this entire down portion, entire down portion is the unstable part. Now, things make sense. What I have seen is that although I have two equilibrium solutions, one of them is a stable solution, the other one is an unstable solution.

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So, what exactly goes on during the dynamics? What goes on during the dynamics is something like this. So let me draw this diagram where on the y axis you have x_e, on the x-axis you have the parameter h. This is, the first one is the case for one solution, the second one is the case for no solution. And then let me draw a vertical line again to take an intermediate case.

Now, what I see is that I have the upper point which is stable and the lower point which is unstable. And what happens to the solutions in the proximity of a stable equilibrium point? The solutions converge to all those points, which means my dynamics would be, if I can use a notation involving arrows in this diagram, then I would say that the population tends to come to this point.

My population will come to, tend to come to this point. And similarly, from the unstable point, then my population will tend to diverge away. So, I can draw this curve. So, this is, you can consider the arrows as the directions of time. So, what do I understand from this one particular vertical line?

If I have a population, if I have an initial, so let me take three cases. So, point 1, point 2 and point 3. So, I have three cases. If my initial population conforms to point 1, which means I have way too species in my system, which is more than what the equilibrium can support and my population will reduce to which equilibrium population, the upper one or the lower one? The upper one is the stable one the lower one is the unstable one.

So, the population would reduce, and it would come down and settle to the upper stable equilibrium point. So, this is quite simply the effect of both logistic growth as well as the harvest parameter because, If your initial population is more than the carrying capacity, following the logistic growth itself, your population will come down.

And on top of it, you are also taking some number of members of the biological species away. So therefore, your population is going to come down. But it could have come down to the lower equilibrium solution. That does not happen because then the moment you start coming down to the lower unstable solution, logistic growth kicks in and your population goes to the upper carrying capacity domain.

Now, when you are in point 2, when you are in point 2, which means you are between the upper equilibrium point and the lower equilibrium point, what is going to happen? What is going to happen is that following the logistic growth, your population will increase. Now, your population will not be equal to this population here.

So, this is an important point, the point which is marked here, let us say point 4, is an is an interesting point because this corresponds to $h = 0$, means no harvest. So, if there were no harvest, at any, so, this point would horizontally translate to this point. And it would go to the carrying capacity. It cannot go to the upper carrying capacity value because you are taking some amount of, some number of biological species away from your system.

So, you are increasing in number following the logistic model but you are not going as far as the carrying capacity for a simple reason that you have harvest in your system. Now why are you not going down? You are not going down because the lower point is the unsteady state, is the unstable equilibrium point, and therefore, following the logistic model growth, you should go up.

Now, what about point 3? Point 3 is an unsteady, sorry, point 3 is an unstable equilibrium point, which means solutions would move away. So, what is the meaning of this? The significance of point 3 is that if you start with a very small population, the population which is lesser in number than the lower equilibrium unstable population, then you see, you can follow the direction of the arrow and you will realize that your population will go to zero.

So mathematically, it would go to $-\infty$. $-\infty$ does not exist, so therefore, your population will become zero. So now, if you analyze the curve along the points, point 1, point 2, point 3, what you will realize is that if your initial population is larger than the point which lies on the equilibrium curve, then the, then the population would come down. It would come down to the upper equilibrium state.

If it is between these two, it would not go down, it would go up. Following natural logistic growth, the maximum that it can go would be the carrying capacity but it will not reach the carrying capacity because you are harvesting the system. And if your initial population goes below it, what is going to happen? Then what is going to happen is that your population will go to zero.

So therefore, it is important that when you do this business of fishery or when you run this bioreactor, you have to make sure of two things. First thing is that you must always operate at a harvest rate which is less than 0.25 ($h < 0.25$). If you go beyond that, you will not, you will end up with nothing.

And at any given moment, your population of the species must be larger than the lower equilibrium unstable population. If you go below lower unstable equilibrium population for any reason, and that any reason can be an external disturbance for example, then your species would die down to zero.

So, now we can analyze the curve corresponding to $h = 0.25$. So, you have only one solution, and if I want to draw an arrow, then I know that I, I can shrink this entire parabolic curve to one single point. So therefore, the effect of this intermediate region which is here is shrinking down. And here, at this point, so what is the effect of this parabola?

The effect of the parabola is that within the parabola, or inside the parabola, you will go up. Now that effect has shrunk to zero, which means that if I translate this point here, I will go down. It is quite obvious that if you have very large population, it may not be sustained, and your population will come down. And then if I translate this point here, this will also come down and there is no region within the parabola, so everywhere, the population would come down.

So, if your initial population is here, you would come to here, otherwise you will come down. And if you go beyond that, now on the third curve for as greater than zero now, I can follow the directions of the arrow and then you will see that you will simply have irrespective of initial population the population going down to zero.

So, what happens at h = 0.25? You observe a phenomenon which we call as bifurcation. So, the system has bifurcation at h = 0.25. And then, since I have, so my bifurcation, my bifurcation parameter is h, bifurcation happens at h = 0.25 and h is my bifurcation parameter, and since I have drawn equilibrium solution versus the bifurcation parameter h, this entire plot which you see here, is called bifurcation diagram.

This is called bifurcation diagram. And finally, for stability, what you did was you did

$$\left.\frac{df}{dx}\right|_{x_e} > 0, \qquad unstable \;\&\; \left.\frac{df}{dx}\right|_{x_e} < 0, \qquad stable$$

What happens at bifurcation? Well at bifurcation, you see the

$$\left.\frac{df}{dx}\right|_{x_e} = 0, \qquad bifurcation$$

So, this is the condition for bifurcation.

So, we took a variant of population growth model today, and what we saw is that the parameters which appear in the equations, in fact, affect the dynamics of your system. And therefore, you can come up with what is called the bifurcation plot. So, in a bifurcation plot, you plot the variation of the equilibrium solutions with the bifurcation parameter.

Remember, in the phase portraits you plotted, in case of one-dimensional system or first order system, you plotted the solutions as a function, or time in case of higher dimensions, you plotted the variation of all the solutions. But in case of bifurcation plot, what you need to do is you need to consider the equilibrium solutions and not just the solutions. We are interested to know how do the equilibrium solutions vary as a function of bifurcation parameters.

So, we will take up more on bifurcation parameters by taking more examples in the weeks to come. In this particular week, we started off with non-linear models. We will continue with non-linear models in the future weeks as well. And population dynamics, as you saw, pose a very good example for the analysis of non-linear systems. So, we will meet again next week. Till then, good bye.