**Advanced Process Dynamics**

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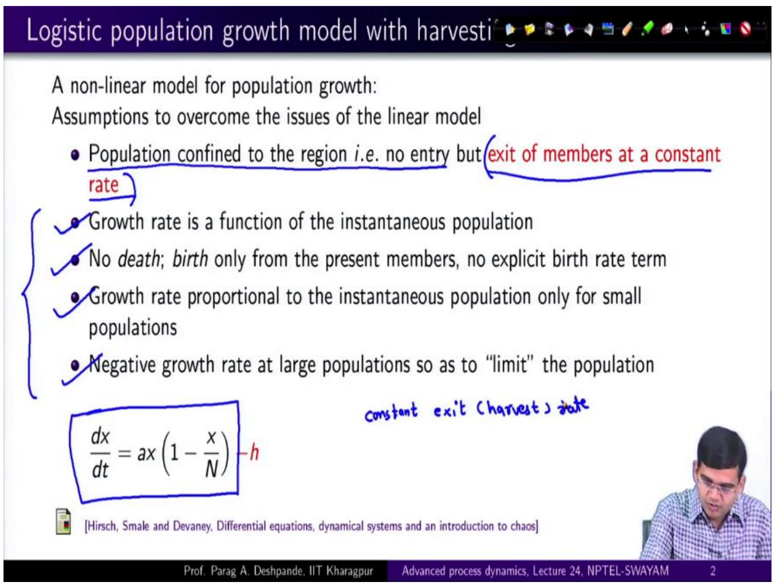
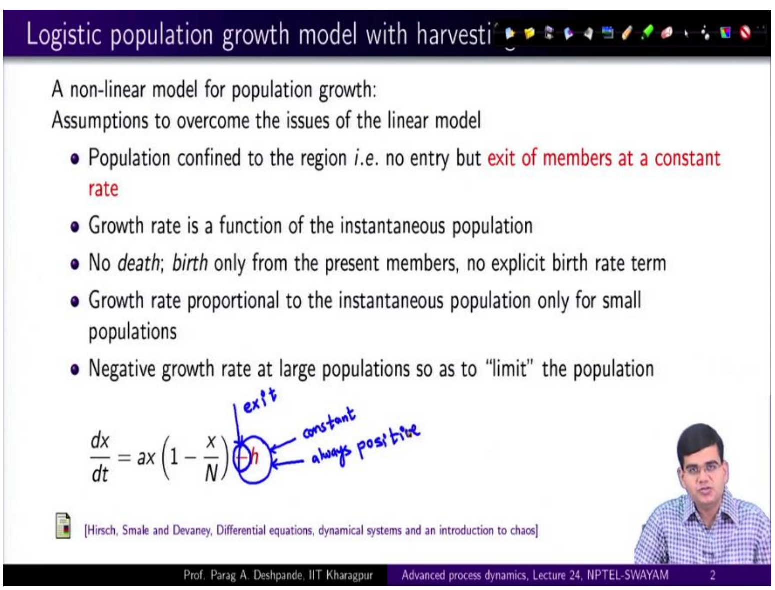
**Lecture 24**

**Logistic Population Growth with Harvesting**



## Logistic population growth model with harvesti[ng]

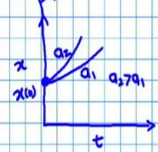
$$\frac{dx}{dt} = a x\left(1 - \frac{x}{N}\right) - h \quad \leftarrow \text{logistic growth with harvesting}$$

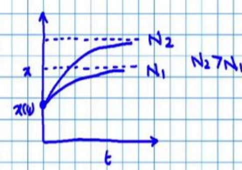
$a = 1$, $N = 1$

p1 — growth parameter ($a$)

p2 — carrying capacity ($N$)

p3 ← harvest rate ($h$)






## Logistic population growth model with harvesti[ng]

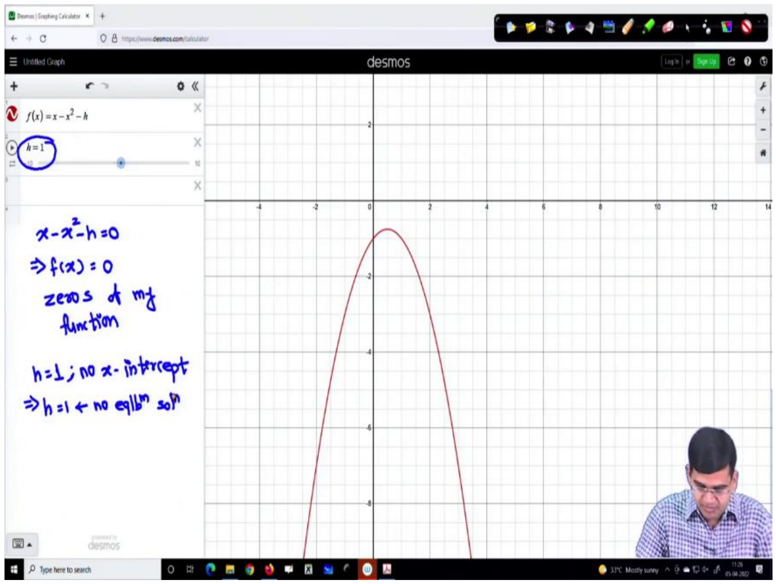
$$\frac{dx}{dt} = x(1-x) - h \qquad -(1)$$

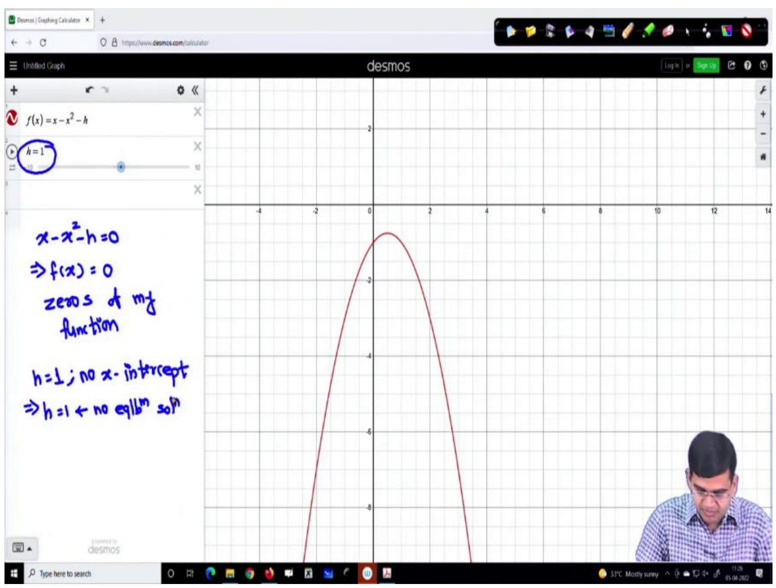
$$\frac{dx}{dt} = x(1-x)\,; \quad x_e = 0, 1$$

Do I always get equilibrium in the case of constant harvest?

$$x(1-x) - h = 0$$

$x - x^2 - h = 0$ ← equilibrium solutions exist if $x - x^2 - h = 0$ has real solution(s).


desmos
Untitled Graph
f(x) = x − x^2 − h
h = 1
x - x² - h = 0
⇒ f(x) = 0
zeros of my function
h = 1 ; no x - intercept
⇒ h = 1 ← no eqlb^m sol^n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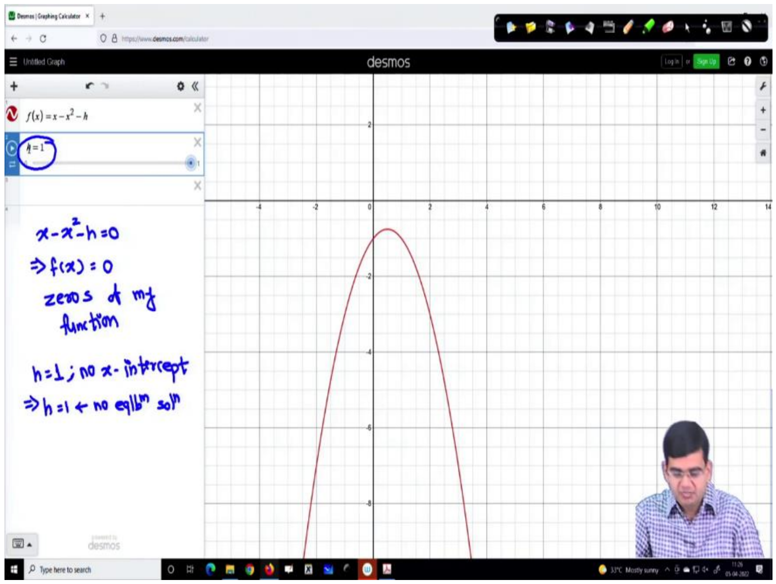
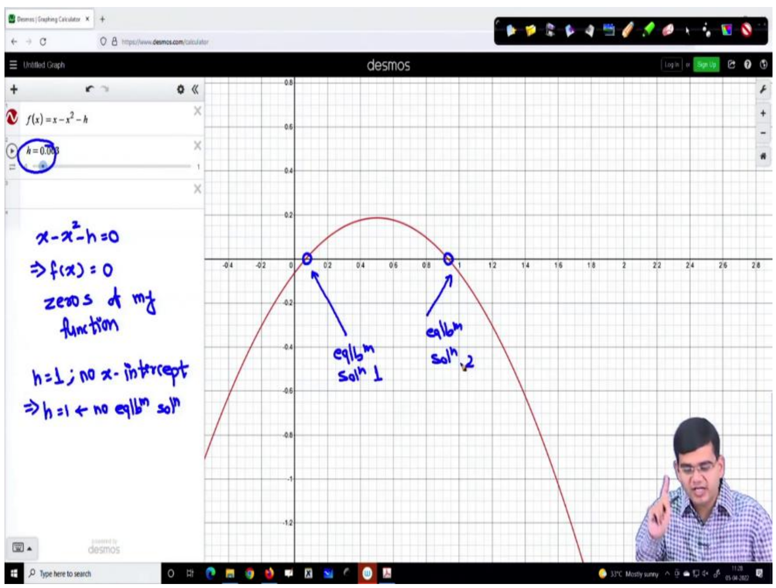
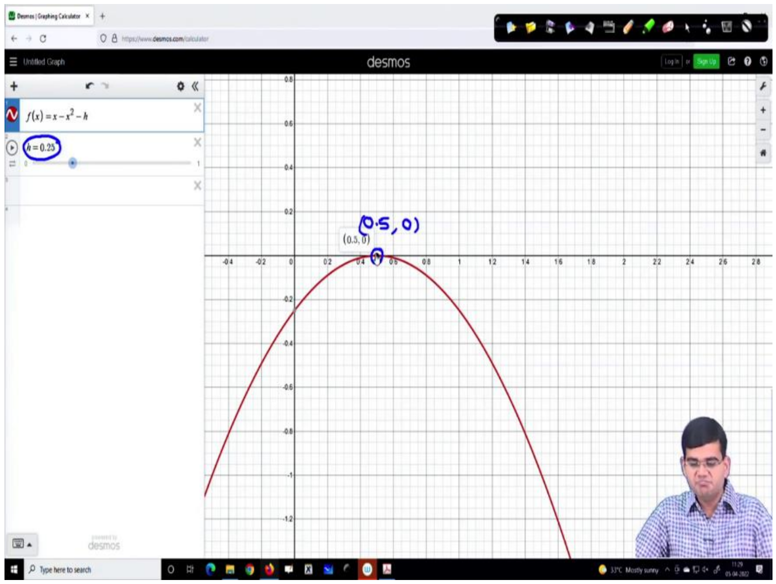
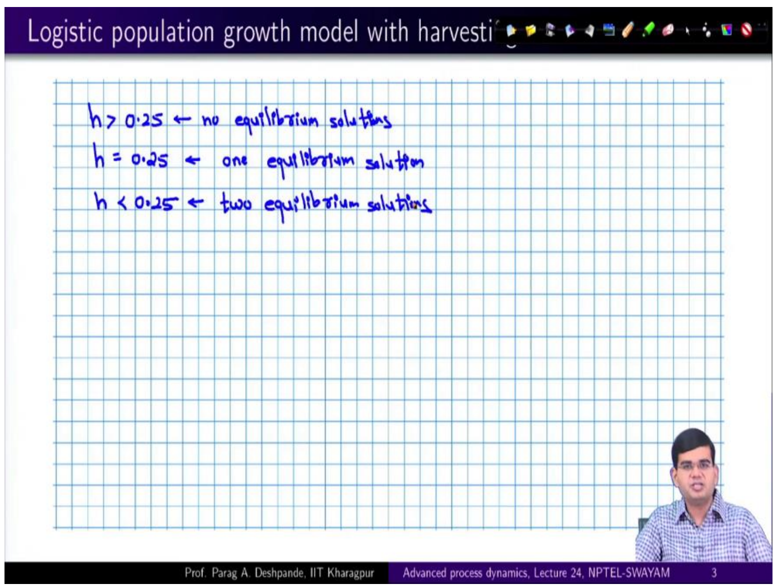

desmos
$f(x) = x - x^2 - h$
$h = 1$
$x - x^2 - h = 0$
$\Rightarrow f(x) = 0$
zeros of my function
h=1; no x-intercept
⇒ h=1 ← no eqlbm soln


desmos
$f(x) = x - x^2 - h$
$x - x^2 - h = 0$
$\Rightarrow f(x) = 0$
zeros of my function
h=1; no x-intercept
⇒ h=1 ← no eqlbm soln
eqlbm soln 1
eqlbm soln 2




So, we were discussing population growth models. We started off with a linear model, we saw its limitations and then we took up the logistic growth model. The logistic growth model had the carrying capacity parameter N in it, which resulted in saturation of the population at long time. Now imagine that we have a different situation where you would like to harvest some of the population at regular intervals.

So, what kind of situation can this be? We all have heard about antibiotics. So, antibiotics are the chemicals which suppress the growth of bacteria. And in case of bacterial infection, antibiotics are prescribed. Now, antibiotics is the one, as the name suggests, which would suppress the growth of bacteria. We also have something called probiotics, which means which would support the growth of bacteria.

Now in certain situations, like for example, vomiting, diarrhea, et cetera, we have imbalance of certain bacteria in our body, which our body requires for normal physiology, and we may as well have the introduction of foreign bacteria which might be undesirable. So, this lack of balance of the desirable bacteria and the introduction of the foreign bacteria in our body can be addressed by introduction of probiotics to the body, to the human body.

Now what are probiotics? These are typically the cultures where you introduce bacteria and yeasts. And these populations of bacteria and yeast in the probiotic try to restore the balance of this microbiome in our human body. In fact, there are certain studies which state that the total number of microorganisms in human body, foreign microorganisms in human body, exceeds the number of the cells of human body itself.

They are so large in number, but since they are very small in size, we often do not realize that they are present in our body, and often, and in fact now at this stage of evolution, they have become an integral part of our entire human physiology. And therefore, any imbalance in the number in the human body causes troubles. So therefore, we need to restore it and if I want to do the production of these probiotics on a large scale, then what do? I need to do I need to start with some culture as I explained in the previous lecture.

Following the logistic model, I may expect the population of the bacteria in the culture to increase. Now, I do not simply want the population of the bacteria to increase but I want to utilize it. So that means I want an output from my production plant. So therefore, imagine that I want to take away some amount of bacteria from that culture at regular time intervals or at a constant rate. So, what is going on here?

Your population of the bacteria in the cell culture is increasing following the natural growth of the bacteria. But because you have introduced a harvesting function in the model, there would occur a balance between the growth of the bacterium, bacterial population and the rate at which you take the population away from the system. And therefore, you need to appropriately change the, or modify the logistic growth model to accomplish such a situation where you would like to harvest a part of the bacterial population for this current case at constant rate.

Let us take another example. Imagine that you have a fish pond or you have an aquarium and you introduce some number of fishes in it. The population of the fish in the pond is going to increase following the logistic model, which we saw yesterday. But apart from that, if I plan to take out some number of fishes at regular intervals, then what would happen to the dynamics, and why would I do such a thing?

Imagine that I have a fishery in which I want to sell the fish commercially, and I now encounter this supply chain management problem that I want to determine the rate at which I should take, I should keep taking the population of fish out of the of the pond so that I do not exhaust my fish in the pond, but at the same time, I make a viable commercial supply. Then what is going to happen?

If I do not take the fish out, I follow logistic model, the moment I introduce a function which allows me to account for taking the constant rate of fish out of the pond, then I modify my logistic growth model. Both of these two were the examples where you start with a logistic growth model where there is a natural growth because it is a biological species, whether you take bacteria or the fish, they grow naturally.

And given the amount of resources that you supply, in other words, the carrying capacity, the initial growth would be exponential followed by the saturation, saturation being determined by the carrying capacity. But then you can introduce a harvest function to the system to account for the harvesting that you want to do. So let us take this particular case in this lecture, and look into how we can change the model equation.

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So, what are the assumptions? The first assumption is that the population is confined to the region that is no entry. This is the same assumption that we took yesterday. But now, exit of the members at a constant rate. This is important. Now, members are allowed to exit, but the rate is constant.

Then, the later assumptions are the same assumptions for the logistic growth that the growth rate is a function of instantaneous population. Makes sense. No death and birth only from the members present. So, there is no explicit birth rate term. Fine. Growth rate proportional to instantaneous population only for small populations, again logistic growth. And finally, negative growth rate at large populations so as to limit the population.

So, these four are the same assumptions which you did for logistic growth. So how do I impose the condition which is given in red in assumption 1 on the otherwise logistic model? So, if you see equation one what I am putting in the rectangular box is simply the logistic growth model. And then what I do is I add a constant exit. In other words, harvest rate. It is important that this exit or harvest rate is a constant.

So, now what kind of the quantity would h be? So, I have equation (1).

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) - h \dots\dots\dots (1)$$

h would be a constant. And there is a certain rate at which you are taking the population out. You already have the negative sign here to account for the exit. Your process is such that you always take some population out. You are not adding the population. And since there is a negative sign here, I can say that h is always positive. So, h is a positive constant in my system. So, let us now try to understand the dynamics of the system.

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So, I have the equation as

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) - h$$

This is logistic growth with harvesting. And what I do now is to understand the significance of individual parameters. This is a three-parameter system. This is parameter 1, this is parameter 2 and this is parameter 3. Parameter 1 is the growth factor, parameter 2, which is N, is the saturation population, and parameter 3, which is h, is the constant harvest rate.

So, to understand the importance of individual parameters, let me simplify this equation a bit. I already know the importance of a. I have understood this from yesterday's discussion. This is a growth parameter, which means that if I draw two cases of x versus t with the same initial population x(0), then in one case initial, initial growth, if it is $a_1$, and in other case the initial growth is $a_2$, then I know that $a_2 > a_1$. This is the importance of the growth parameter. I have understood it.

N is the carrying capacity. So, I can again very quickly draw the phase, draw the dynamics. And if x(0) is my initial population, and I have two parameters, two values for the parameter, $N_1$ and $N_2$, then in one case I will saturate like this, in the other case, I will saturate like this. So, $N_2 > N_1$. So, I have understood the importance of the two parameters. I am yet to understand the importance of the harvest rate, and what is the effect of harvest rate on the dynamics.

So let me simplify the equation bit so that I convert this three-parameter system to a one parameter system without changing the dynamics. Let me set a = 1, and let me set N = 1. So, for unit growth parameter and unit carrying capacity, which means the saturation population would be unity, what would happen to my dynamics?

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So, I can write

$$\frac{dx}{dt} = x(1-x) - h \dots\dots\dots (1)$$

And in the previous logistic model, which was, the analogous logistic model will become

$$\frac{dx}{dt} = x(1-x);\ x_e = 0,1$$

which means that my system always had equilibrium, equilibrium in my system was guaranteed.

But what happens in the current case? So, my question is do I always get equilibrium in case of constant harvest? And for this particular case of constant harvest, how would I determine the equilibrium? Equilibrium would be determined by

$$x(1-x) - h = 0$$

So now this is a quadratic equation, so I have

$$x - x^2 - h = 0$$

This is a quadratic equation, and therefore, whether the equilibrium solutions would exist or not would depend upon the fact whether or not this quadratic equation have real solutions. Why real? Because my population is real. This equation will always have solutions, but I am interested in populations and populations are real.

So, therefore, as long as I can identify populations from this equation $x - x^2 - h = 0$, which are real, I can be assured that the populations, equilibrium populations would exist. So, therefore, equilibrium solutions exist if $x - x^2 - h = 0$ has real solution or solutions, in fact. At this point of time, we cannot say anything about the number of solutions.

So, let us see how do I know whether these, this equation has real solutions or not. We can always analytically solve it, but let us first try to make, to solve this graphically. So, we would use our graphical calculator. So let us plot this on the graphical calculator and try to analyze the situation.

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So, we have the graphing calculator in front of us, and the equation is $x - x^2 - h = 0$. So, I will write

$$f(x) = x - x^2 - h$$

So, let me get rid of the text here. And now what I see is that I get a curve which looks like this. What is the importance of the x-intercept of this curve? If I have the equation which is

$$x - x^2 - h = 0$$

that means I have

$$f(x) = 0$$

So, $f(x) = 0$ means I am trying to determine the zeros of my function. The values of x at which my function becomes zero. So, in the current case, does the curve intersect the x-axis at all? No, it does not. And what is the value of h here? The value of h here is 1. So, for h = 1, there is no x-intercept. And since there is no x-intercept, what is going to happen? My $f(x) = x - x^2 - h$ will never become zero.

And if it would never become zero, this means that I will never have an equilibrium solution, which means that for h = 1, there is no equilibrium solution. Now, I have set the value of the harvest rate arbitrarily at h = 1. What happens if I change the value of h? So let me do this, so let me increase the value of h, let me increase the harvest rate.

So, when I increase the harvest rate, what I see is that in fact my curve starts going down, which means there would not be any situation at which I would have ever any equilibrium solution. So, as I increase my h to a larger value, the, the chance of having my equilibrium solution, in fact, there is no chance of having my equilibrium solution.

And I know that harvest rate has to be positive, so let us just make it between 0, and we have already tested, 1. But what happens if I reduce the harvest rate? And by reducing the harvest rate, I mean that I am taking less number of bacteria from my culture. So then in such a case, does the equilibrium solution exist?

So let me do this. I am reducing it, and there you go, the curve starts going up. And therefore, what I see is that if I reduce it to a very small value, you can see here that now I have two intercepts, two x-intercepts. x-intercept 1, and x-intercept 2, which means that I have two equilibrium solutions, equilibrium solution 1 and equilibrium solution 2.

So, at low harvest rates, you have two equilibrium solutions. So, no equilibrium solution or zero equilibrium solutions and two equilibrium solutions. Can have the case of one equilibrium solutions? In fact, I do, and let me see what is that case. Let me keep on adjusting the value of h such that at $h = 0.25$, here, $h = 0.25$, what I get is this.

So let me see here what is the value. See, what you get is this. This point. Yes, the x-intercept is 0.5 and the y-intercept is 0...... (0.5, 0). So, let me cross check. This point is, see, (0.5, 0) at $h = 0.25$. So, this means that I have three situations now with me. The first situation is that there is no equilibrium solution at all. So, let me write down.

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So, I have the case where, so everything is a function of h. So, for $h > 0.25$, there are no equilibrium solutions. For $h = 0.25$, there is one equilibrium solution. And then for $h < 0.25$, there are two equilibrium solutions, which means that the equilibrium population will depend upon the harvest rate.

How do I physically make any sense out of it? So, first of all what is the meaning of no equilibrium solution? No equilibrium solution simply means that you have an either ever increasing or ever decreasing value of the dynamical variable. See, at equilibrium solution the value of the dynamical variable becomes constant.

So, if you are at that value of equilibrium solution, the slope is zero, the gradients are zero, and at that equilibrium solution, this system would continue to have that value. That is the meaning of equilibrium solution. So, when you do not have equilibrium solution means that the system has no value of the dynamical variable, where the gradient would become zero and it would continue that way.

Which means that since the gradient is non-zero, if, if the gradient is positive, the value of the dynamical variable will keep on increasing and if the gradient is negative the value of the dynamical variable will keep on decreasing. In this particular case, you will find that the value of the dynamical variable will keep on decreasing. And mathematically, it would go to $-\infty$. That is the meaning of no equilibrium solution.

Now mathematically, it would go to $-\infty$, but practically the minimum that you can have in any population is zero. So, what is the meaning of this mathematical observation that for harvest rate $h > 0.25$, there does not exist an equilibrium population? This simply means that the population would go $-\infty$. And since $-\infty$ is only mathematically possible, physically you have always some finite population, the minimum that you can have is zero population.

In other words, this is a devastating situation for your case because you would end up with no fish in the pond which is not a good situation for, from the commercial purpose because you need to have some fish which would grow naturally, and you can keep on harvesting. So, $h > 0.25$ is not a recommended harvest rate.

For the case of bacteria, this basically means that you have taken out all the bacteria from your culture and no bacteria is left. Again, you can, this particular mode of operation of such high harvest rate is not a commercially viable option to be explored. Now, $h = 0.25$ means, I have one equilibrium population. And what is the meaning of this?

The meaning of this is that if I maintain a constant harvest rate which is 0.25, then my system will continue to have an increase in population following the logistic model, and I will keep on taking the bacteria or fish away from the pond at a constant rate, which I can subsequently sell or commercialize.

And therefore, I will have a unique value, I know that this is, this much of the bacteria or this much of the fish will be produced every day, let us say every day, from my commercial plant. This also can be understood and appreciated that I have a constant rate at which I am taking out the biological species.

The problem is that you have two equilibrium solutions when harvest rate is less than 0.25 ($h < 0.25$). So, let us try to understand what is the meaning of this. The meaning of this is that now you may not have one population of the biological species in the system but you may have two populations.

Now, this sounds physically quite unrealistic. I mean either, let us say, I have, I would have 100 fishes or I would have, say, 25 fishes, but the number of fish in the tank or in the pond would be a unique number. They cannot be two numbers. If the num, if the count of fish is 100, it is going to be 100. It cannot be either 100 or 25.

So, then what is the meaning of this last mathematical observation that I have two equilibrium solutions? We will need to look into this meaning in greater details, and we will take that up in the next lecture. Till then, good bye.