Advanced Process Dynamics

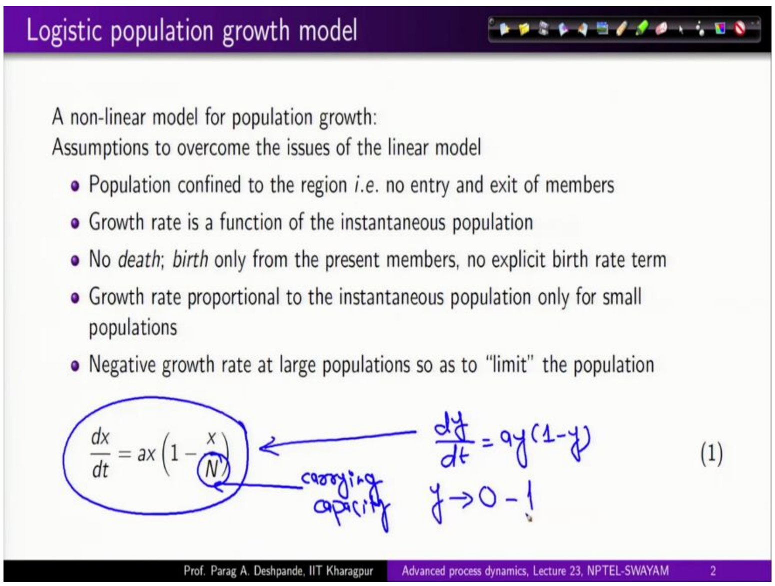
Professor Parag A. Deshpande

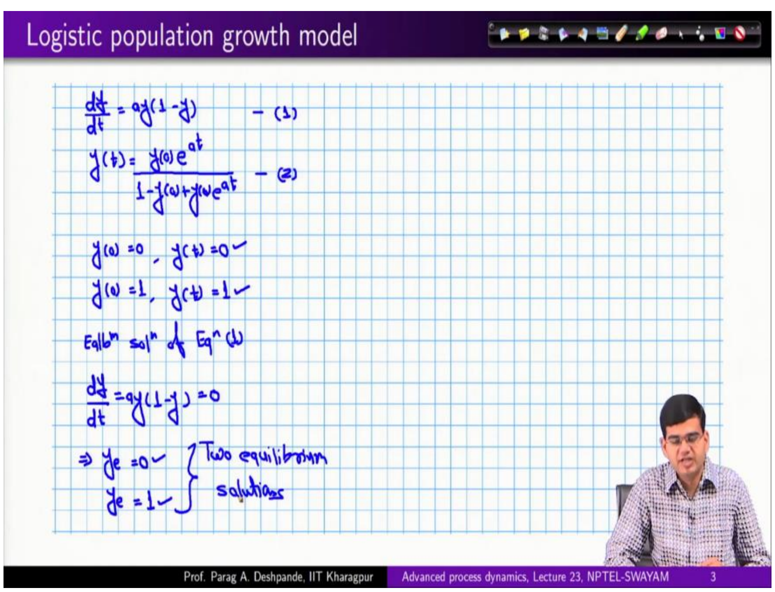
Department of Chemical Engineering

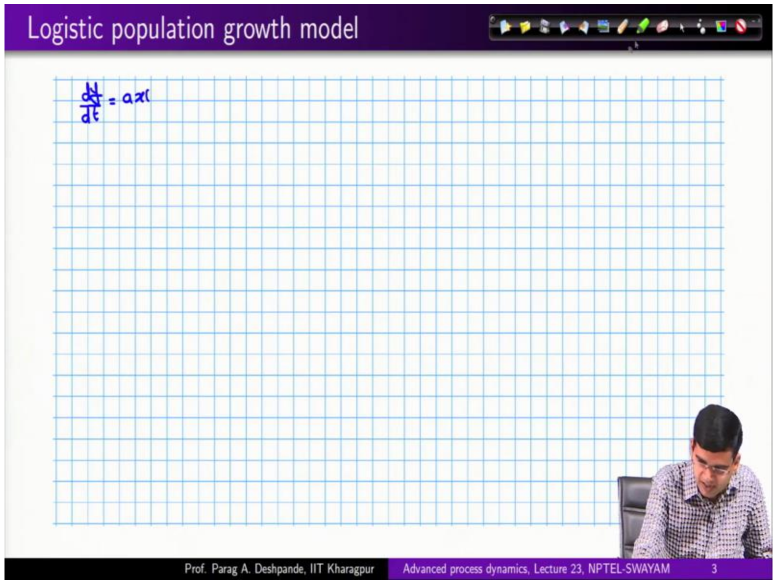
Indian Institute of Technology, Kharagpur

Lecture 23

Logistic Population Growth Model Continued...







Logistic population growth model
dy/dt = ax(
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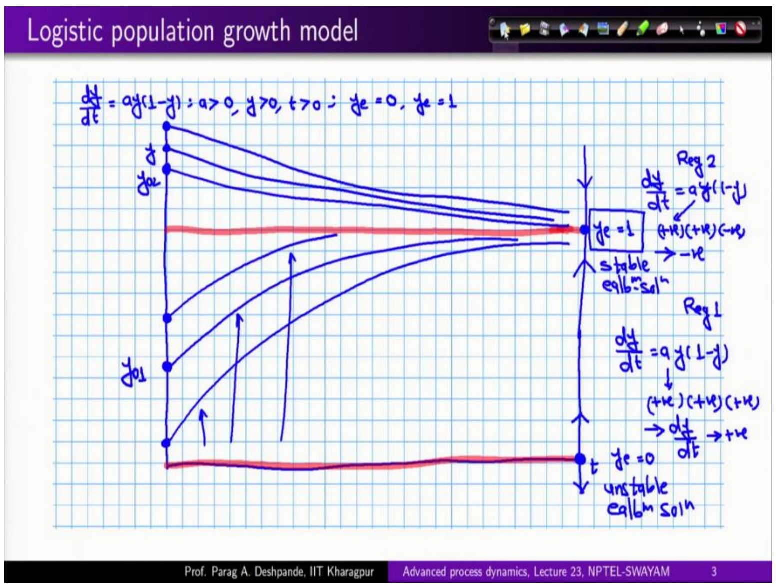

Logistic population growth model
dy/dt = ay(1-y) ; a>0, y>0, t>0 ; ye = 0, ye = 1
y
yoz
yo1
t
Reg 2
dy/dt = ay(1-y)
(+ve)(+ve)(-ve)
→ -ve
ye = 1
stable eqlbm soln
Reg 1
dy/dt = a y(1-y)
(+ve)(+ve)(+ve)
→ dy/dt → +ve
ye = 0
unstable eqlbm soln
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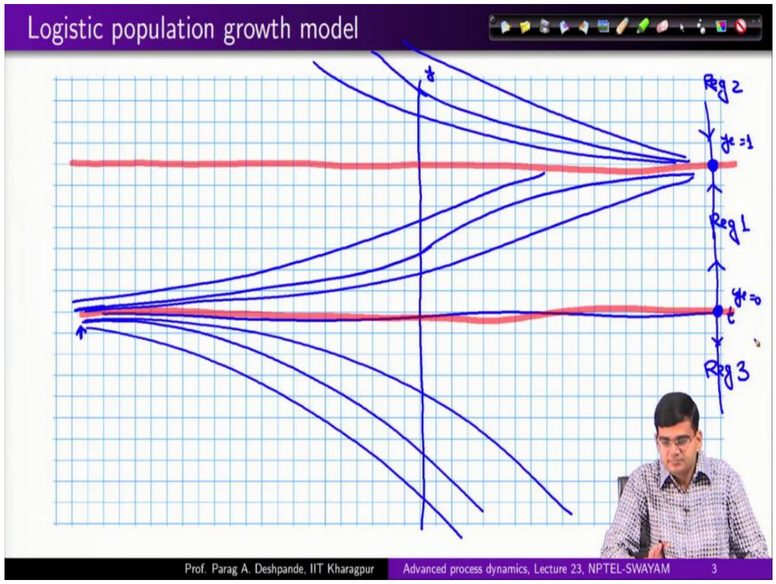
Logistic population growth model
Reg 2
ye =1
Reg 1
ye =0
Reg 3
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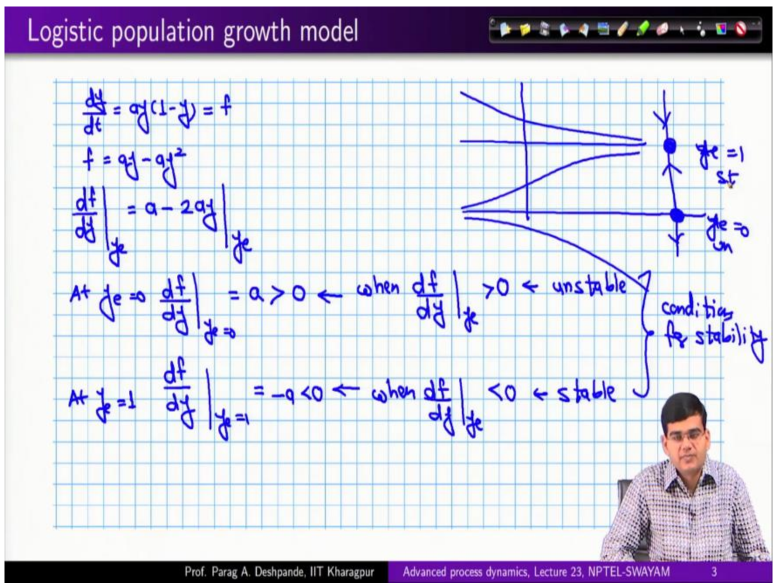
Logistic population growth model
dy/dt = ay(1-y) = f
f = ay - ay²
df/dy |ye = a - 2ay |ye
At ye = 0 df/dy |ye=0 = a > 0 ← when df/dy |ye > 0 ← unstable
At ye = 1 df/dy |ye=1 = -a < 0 ← when df/dy |ye < 0 ← stable
condition for stability
ye = 1 st
ye = 0 un
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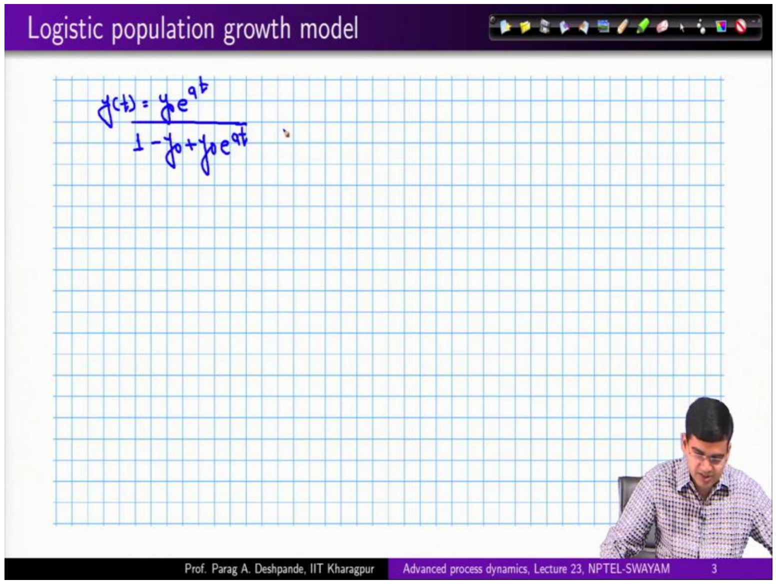
Logistic population growth model
$y(t) = \frac{y_0 e^{at}}{1 - y_0 + y_0 e^{at}}$
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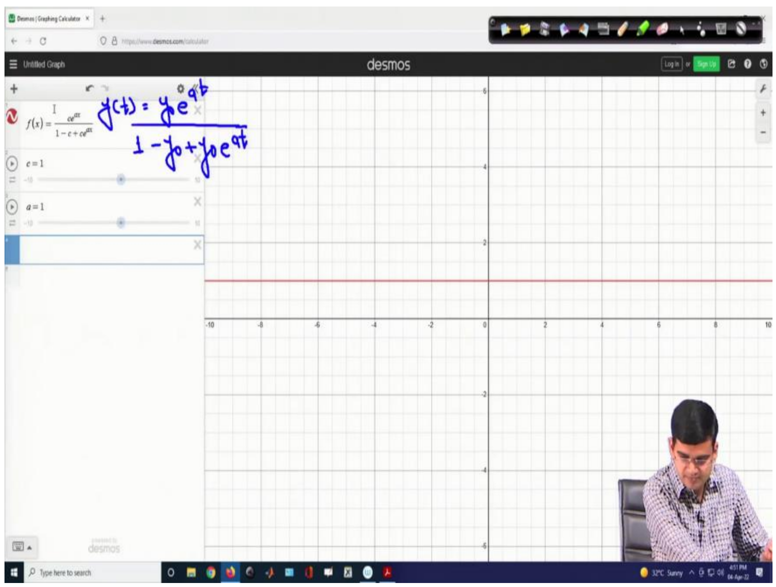
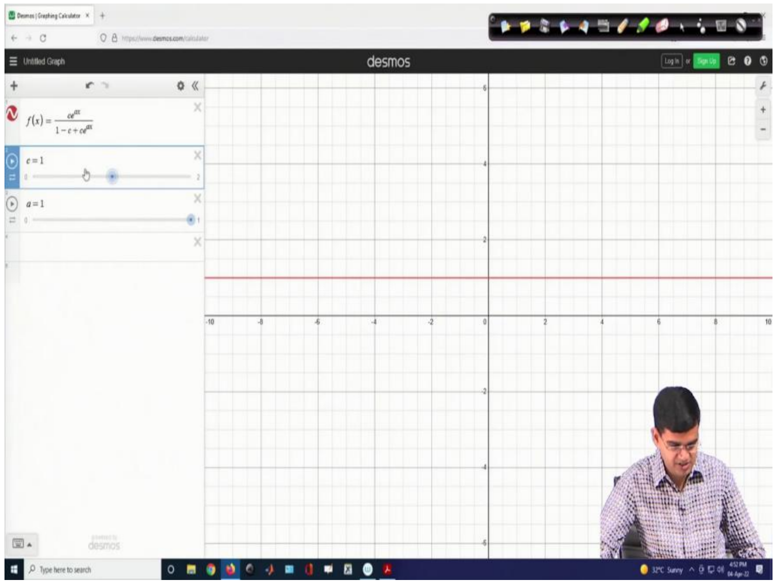
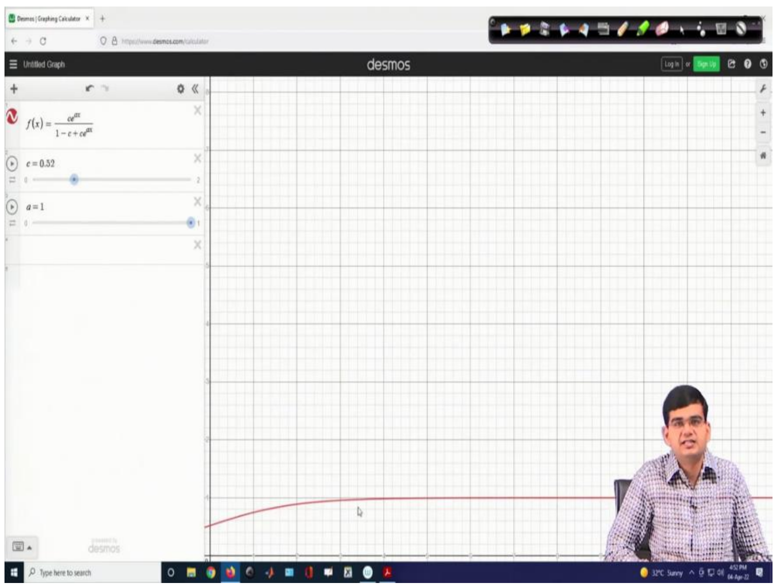
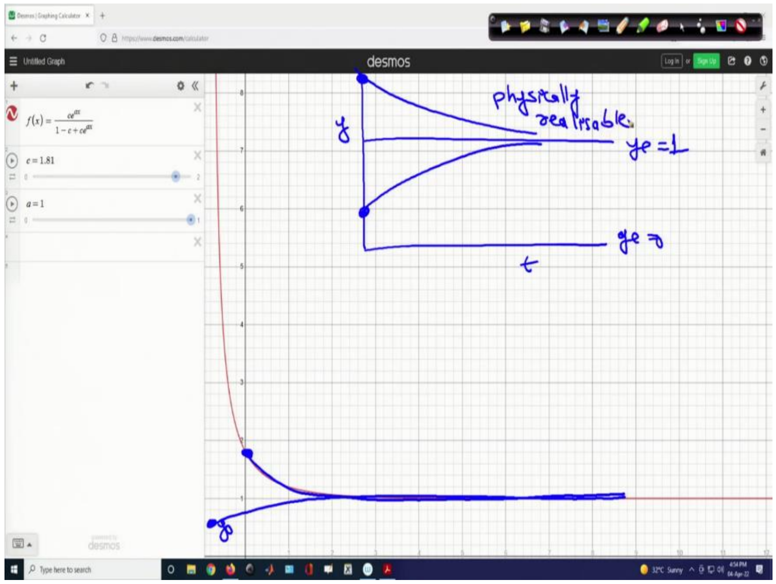
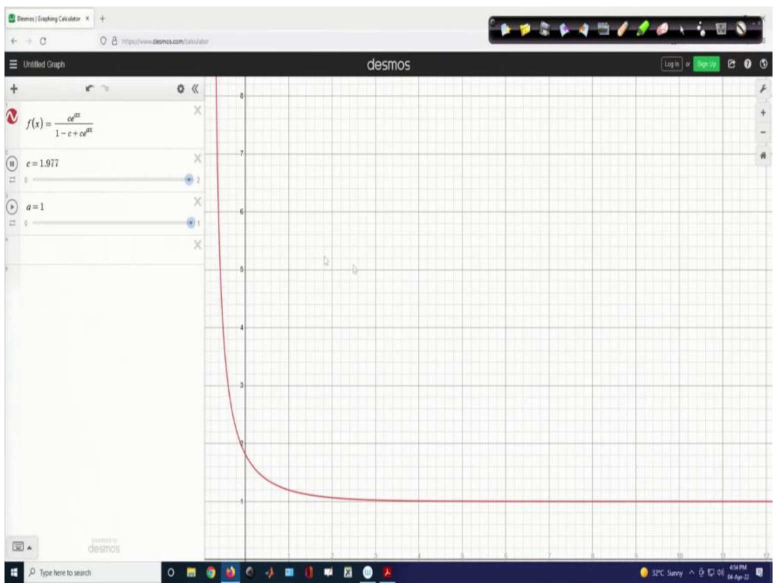

Untitled Graph
desmos
$f(x) = \frac{ce^{ax}}{1 - c + ce^{ax}}$
$c = 1$
$a = 1$
$y(t) = \frac{y_0 e^{at}}{1 - y_0 + y_0 e^{at}}$


Untitled Graph
desmos
f(x) = ce^{ax} / (1 − c + ce^{ax})
c = 1
a = 1

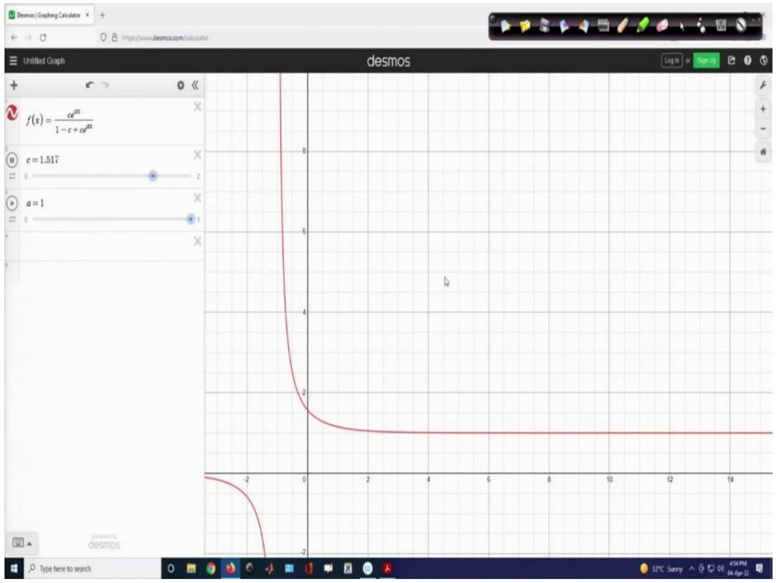

Untitled Graph
desmos
f(x) = ce^{ax} / (1 − c + ce^{ax})
c = 0.52
a = 1

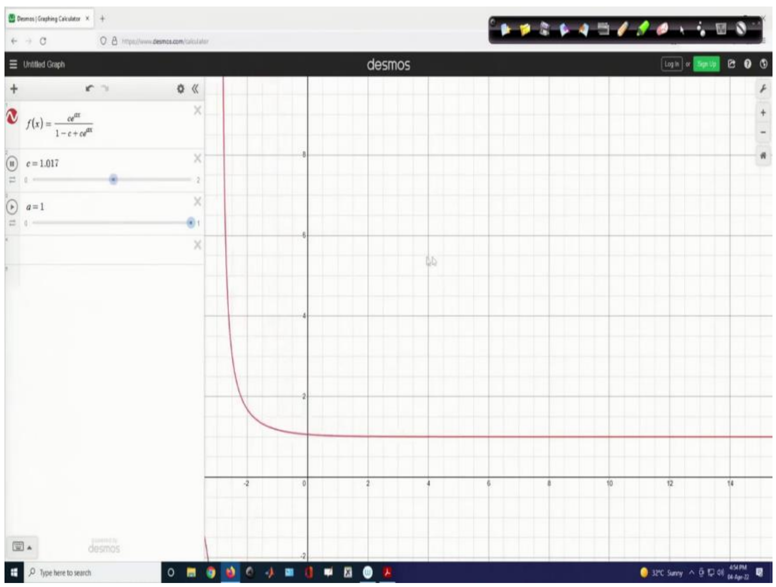


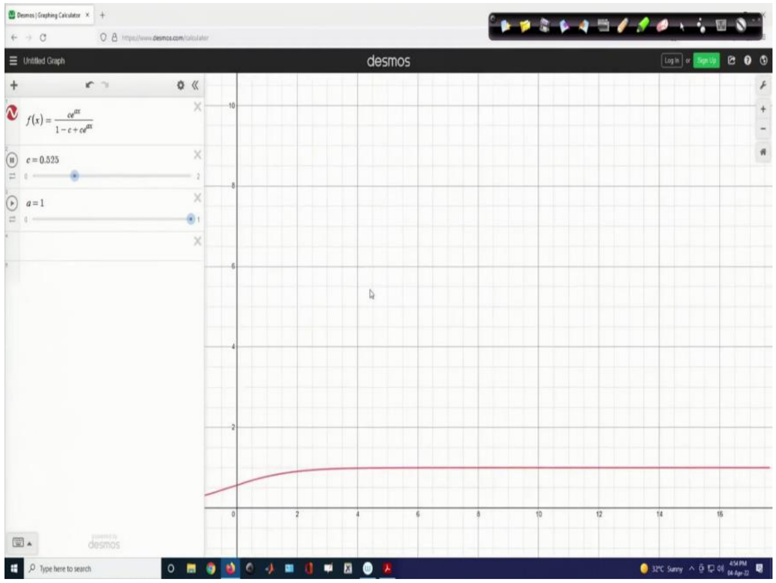
desmos
$f(x) = \frac{ce^{ax}}{1 - c + ce^{ax}}$
c = 1.81
a = 1
physically realisable
y
ye = 1
ye = 0
t

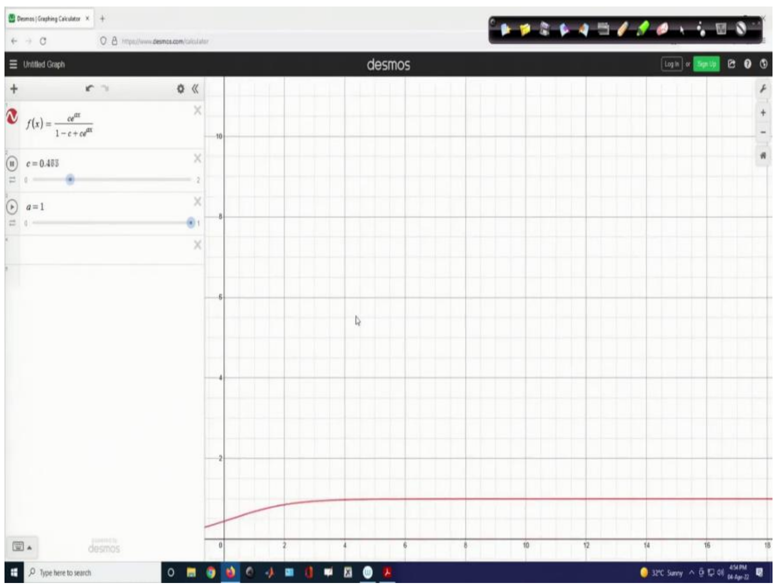


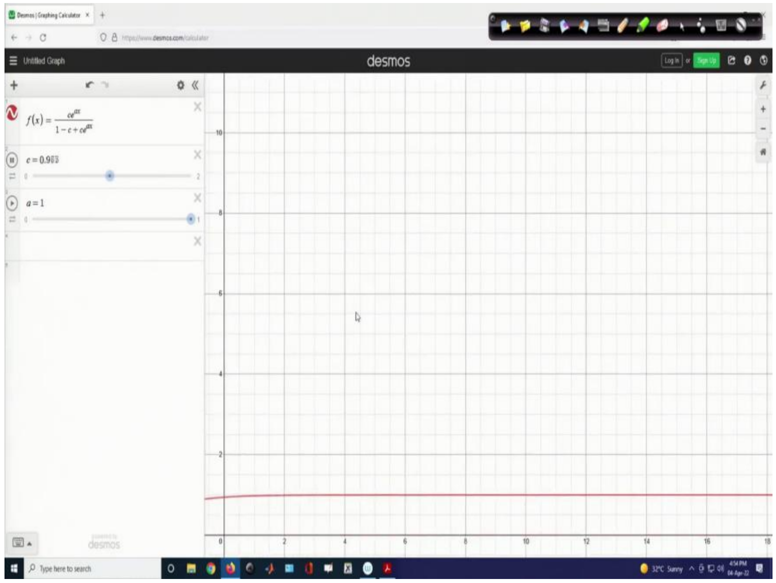
desmos
$f(x) = \frac{ce^{ax}}{1 - c + ce^{ax}}$
c = 1.977
a = 1

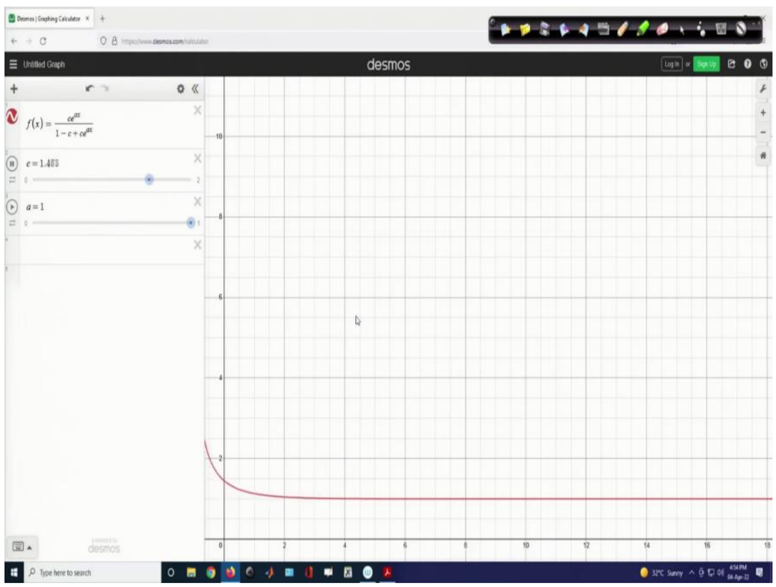

desmos
Untitled Graph
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
$c = 1.517$
$a = 1$

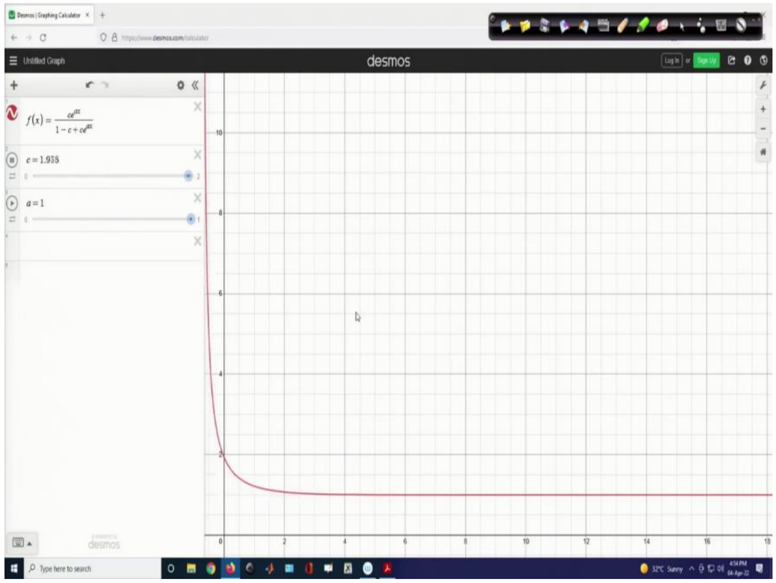

desmos
Untitled Graph
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
$c = 1.017$
$a = 1$

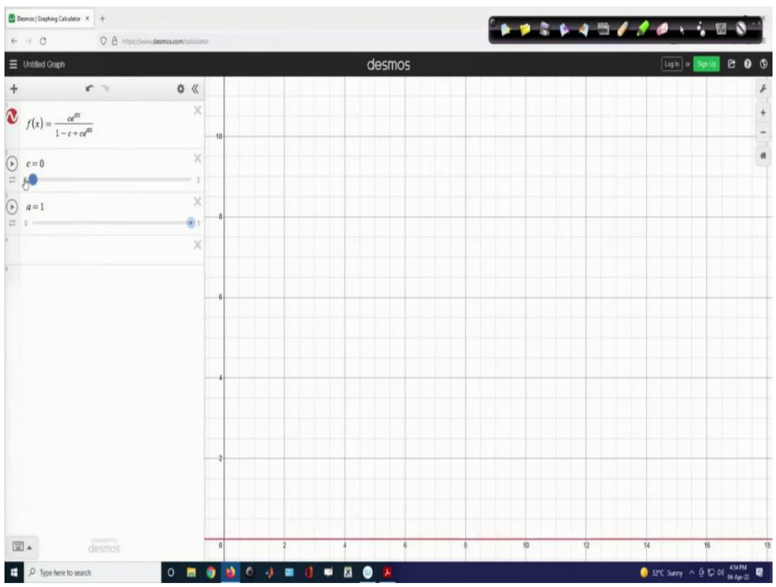

Untitled Graph
desmos
$f(x)=\frac{ce^{ax}}{1-c+ce^{ax}}$
$c=0.525$
$a=1$

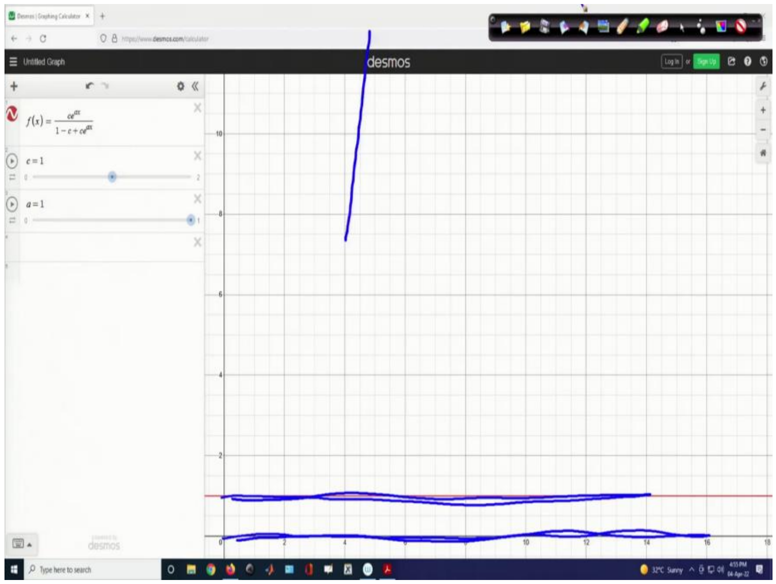

Untitled Graph
desmos
$f(x)=\frac{ce^{ax}}{1-c+ce^{ax}}$
$c=0.483$
$a=1$

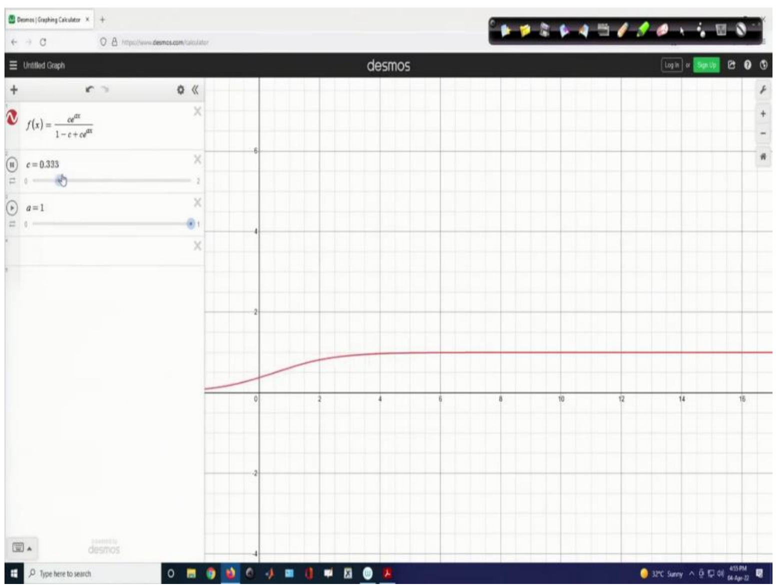

desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
$c = 0.983$
$a = 1$

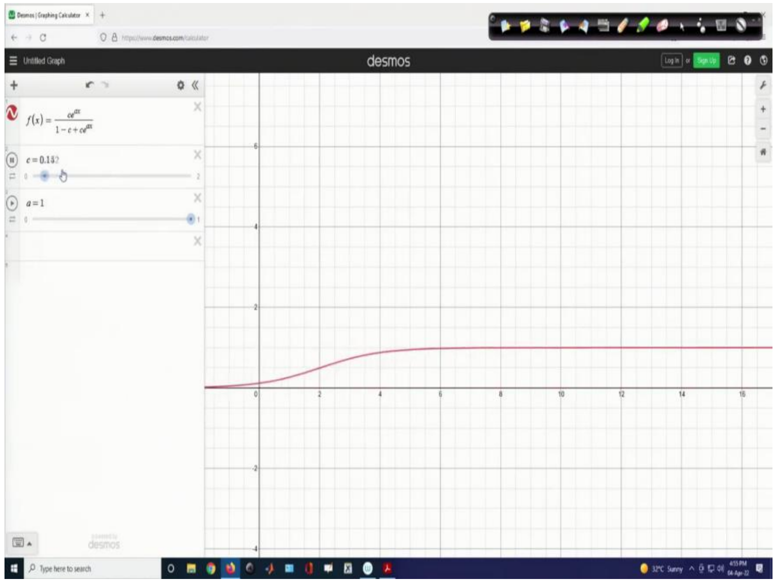

desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
$c = 1.483$
$a = 1$

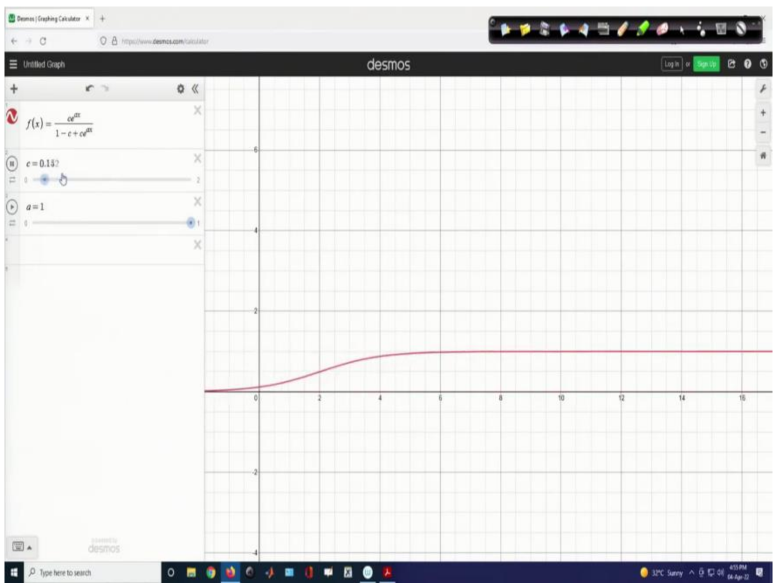

Untitled Graph
desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
c = 1.935
a = 1

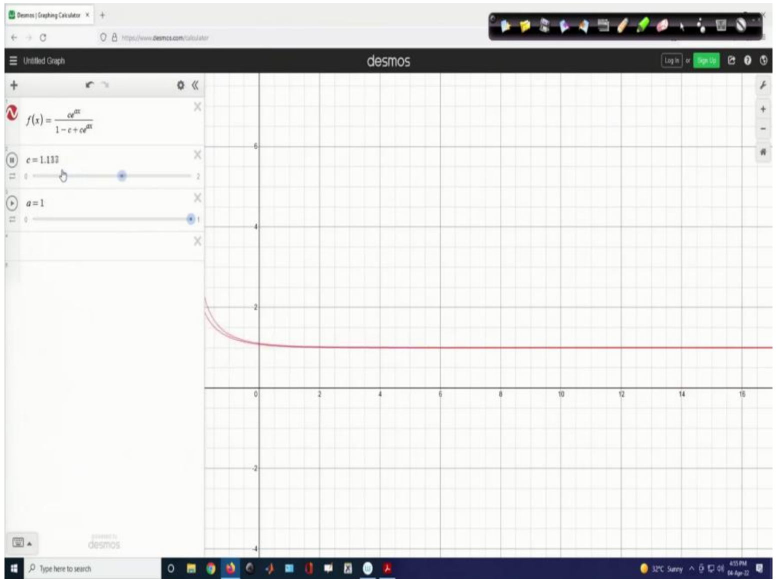

Untitled Graph
desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
c = 0
a = 1

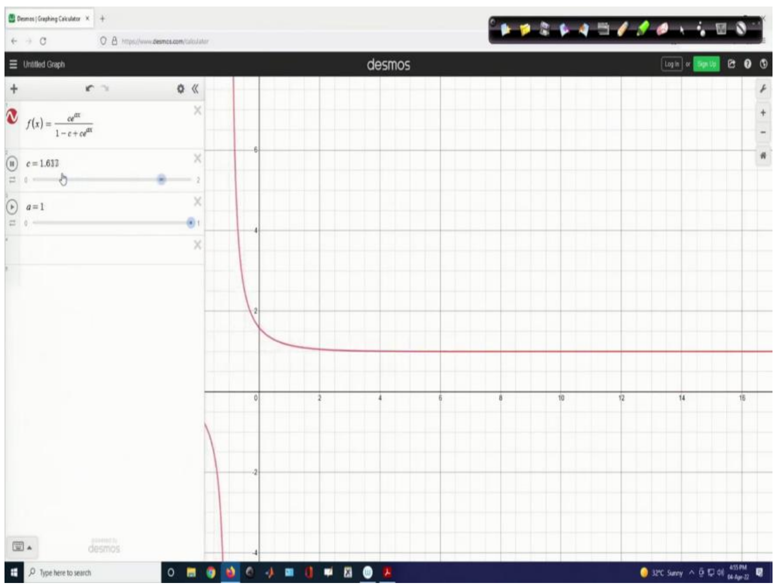

desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
$c = 1$
$a = 1$

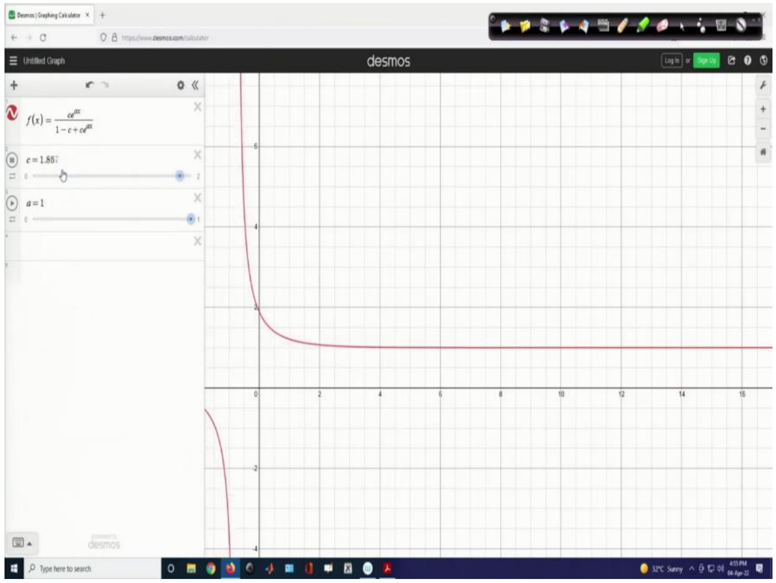

desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
$c = 0.333$
$a = 1$

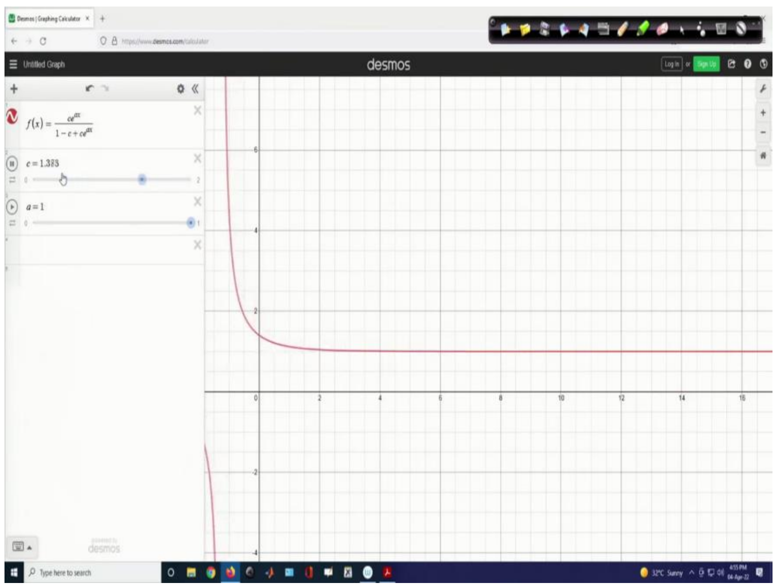

desmos
Untitled Graph
$f(x)=\frac{ce^{ax}}{1-c+ce^{ax}}$
$c=0.182$
$a=1$

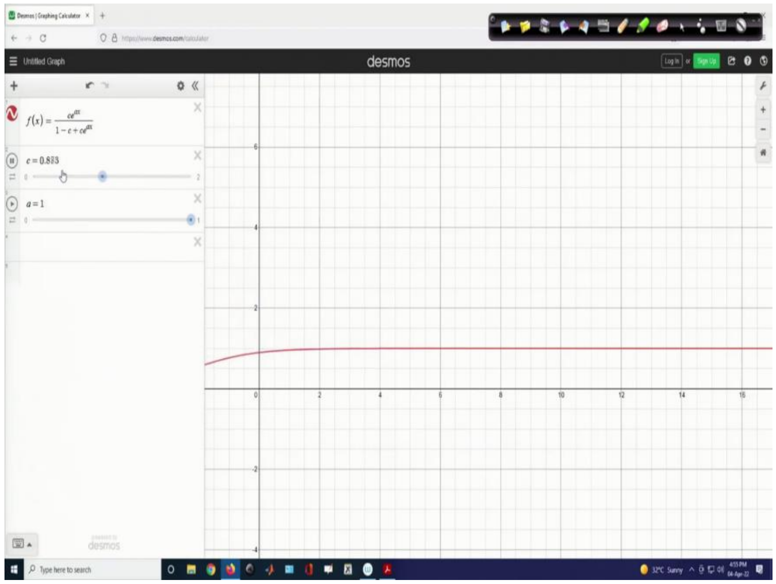

desmos
Untitled Graph
$f(x)=\frac{ce^{ax}}{1-c+ce^{ax}}$
$c=0.182$
$a=1$

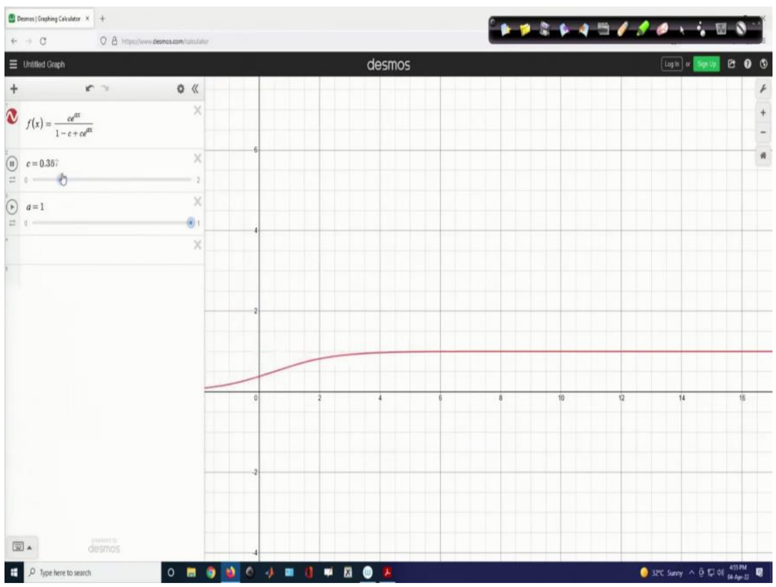

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x)=\frac{ce^{ax}}{1-c+ce^{ax}}$
$c=1.133$
$a=1$

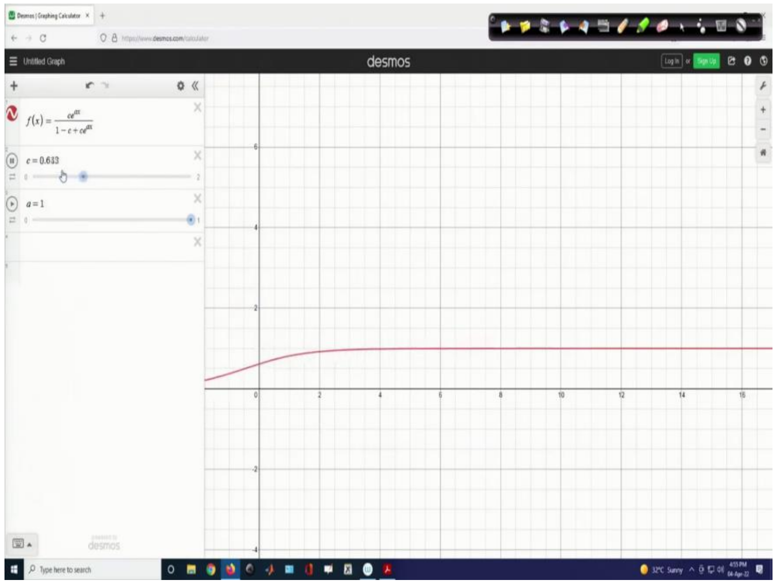

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x)=\frac{ce^{ax}}{1-c+ce^{ax}}$
$c=1.633$
$a=1$

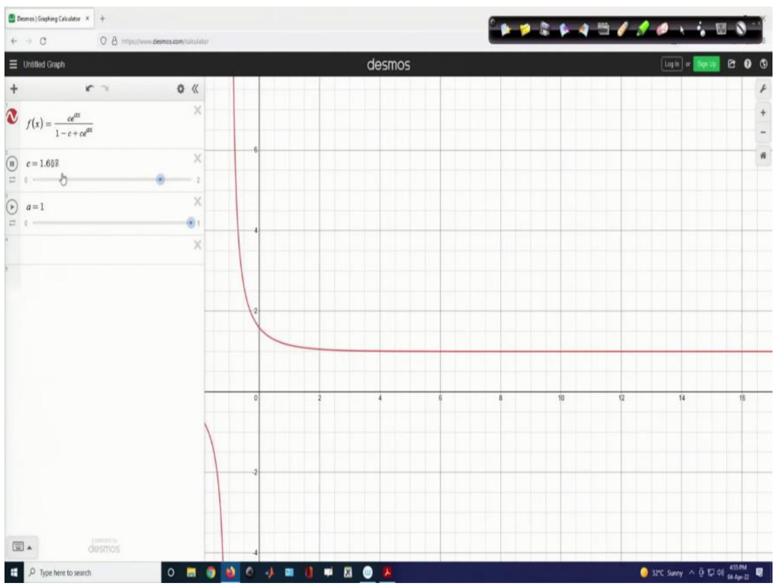

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
c = 1.887
a = 1

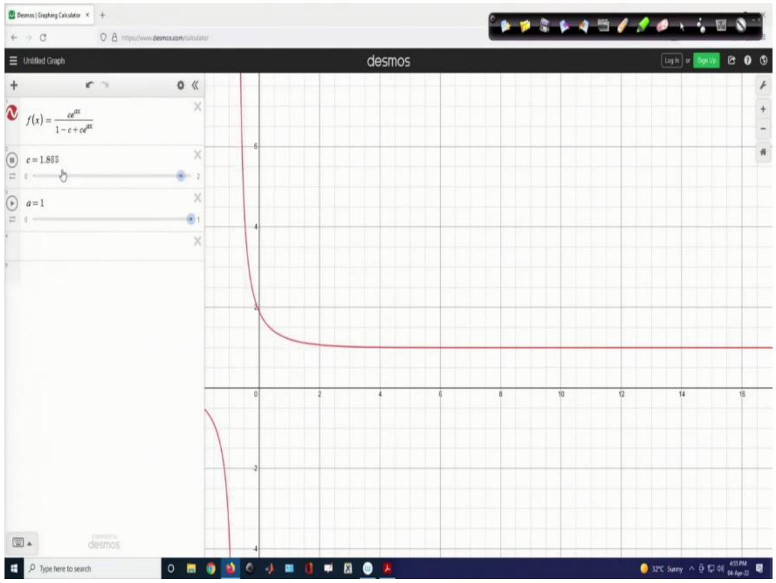

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
c = 1.383
a = 1

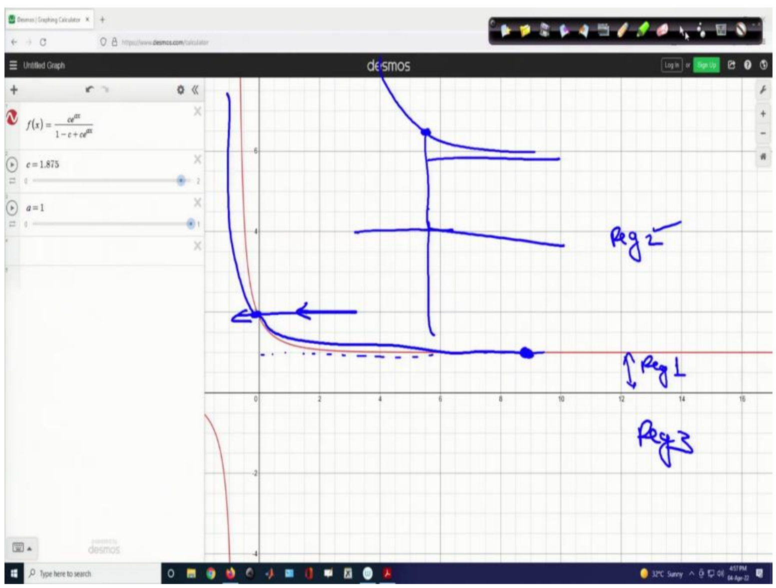

Desmos | Graphing Calculator
Untitled Graph
desmos
f(x) = ce^{ax} / (1 - c + ce^{ax})
c = 0.833
a = 1

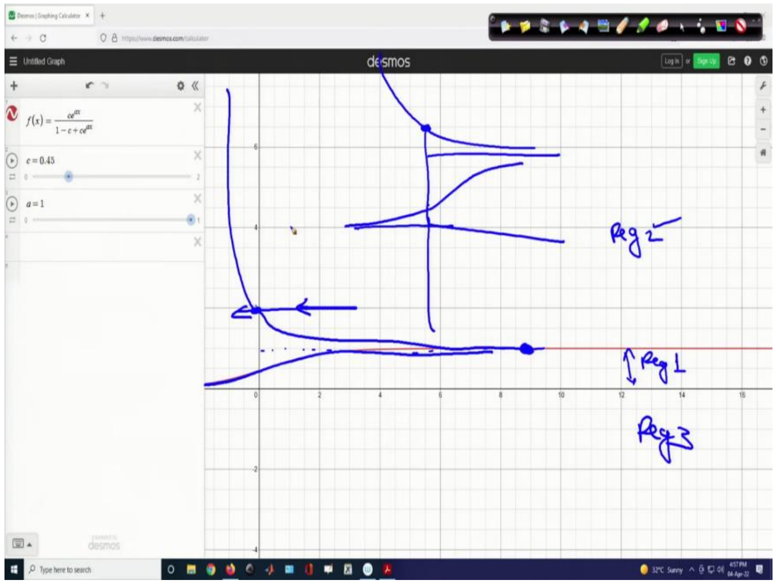

Desmos | Graphing Calculator
Untitled Graph
desmos
f(x) = ce^{ax} / (1 - c + ce^{ax})
c = 0.387
a = 1

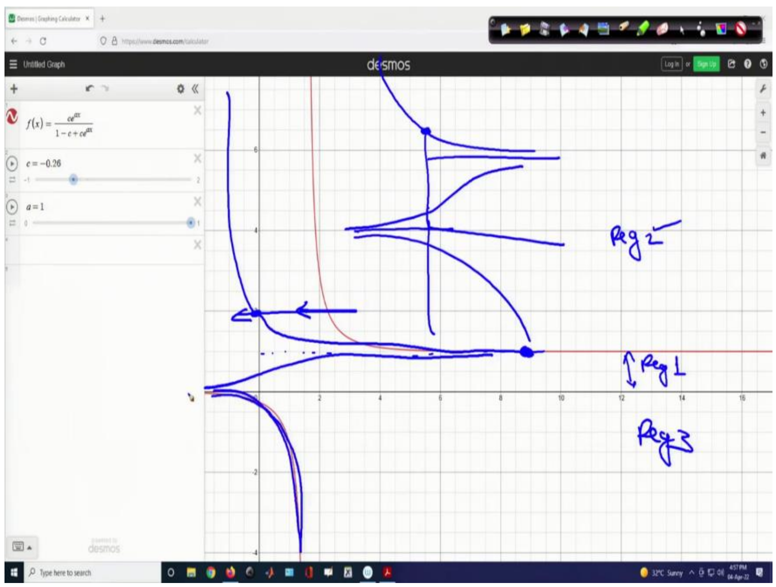

Untitled Graph
desmos
f(x) = ce^{ax} / (1 − c + ce^{ax})
c = 0.613
a = 1

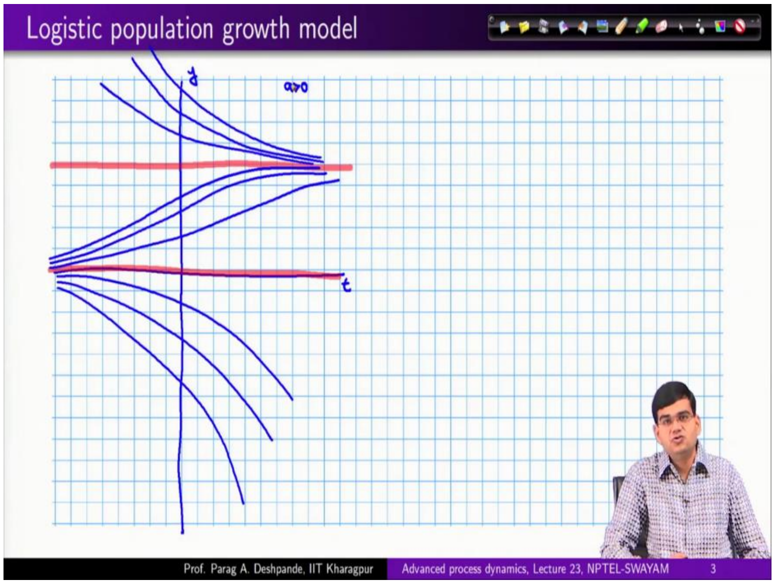

Untitled Graph
desmos
f(x) = ce^{ax} / (1 − c + ce^{ax})
c = 1.613
a = 1


Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
c = 1.865
a = 1
Type here to search


Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
c = 1.875
a = 1
Reg 2
Reg 1
Reg 3
Type here to search


Untitled Graph
desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
c = 0.45
a = 1
Reg 2
Reg 1
Reg 3


Untitled Graph
desmos
$f(x) = \frac{ce^{ax}}{1-c+ce^{ax}}$
c = −0.26
a = 1
Reg 2
Reg 1
Reg 3


Logistic population growth model
y
a>0
t
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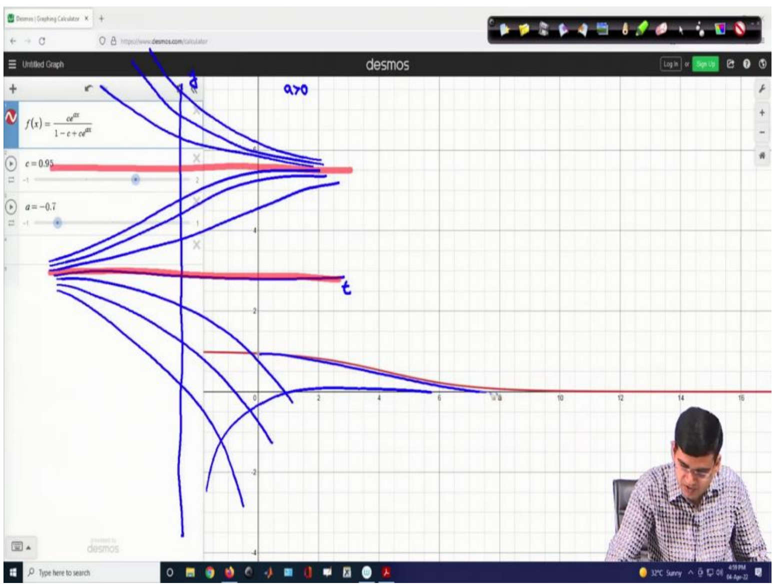
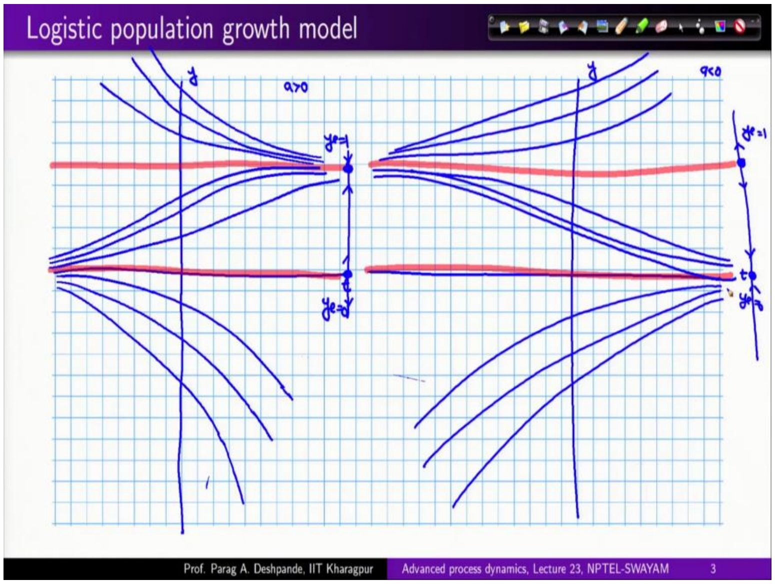

desmos
Untitled Graph
f(x) = ce^{ax} / (1 − c + ce^{ax})
c = 0.95
a = −0.7
y
a>0
t



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So now let us continue our analysis of logistic population growth model. The model is in front of us. This is nothing but

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)$$

where we identified N as the carrying capacity. Physically, we saw that maximum number of members of any biological species which the ecosystem provided by, provided to it can sustain.

So now, let us continue our analysis. We had converted this into normalized form as

$$\frac{dy}{dt} = ay(1 - y) \dots\dots\dots (1)$$

This was our normalized form. So instead of N, my y is expected to go between 0 and 1. This is the only difference, which would happen once you change x to y.

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So, let us see. We had written a model equation as

$$\frac{dy}{dt} = ay(1 - y)$$

and we had written the solution as

$$y(t) = \frac{y(0)e^{at}}{1 - y(0) + y(0)e^{at}} \dots \dots \dots (2)$$

And what we saw is that

$$y(0) = 0, y(t) = 0$$

which means that the population becomes independent of time, and

$$y(0) = 1, y(t) = 1$$

which again meant that the population became independent of time.

Now, what is the physical meaning of this? This means that as time progresses, nothing is changing in your system. And when does this happen? This happens when you have, when you have reached equilibrium state. So let me try to determine the equilibrium solutions of the system.

So, equilibrium solution of equation (1), What would be the equilibrium solution of equation (1)? I will determine it by setting

$$\frac{dy}{dt} = ay(1 - y) = 0$$

which means

$$y_e = 0 \;\; \& \;\; y_e = 1$$

This simply means that you have two equilibrium populations in your system.

And what does physically the two equilibrium populations signify? They simply signify that if you have no population, which means $y_e = 0$, then you cannot expect the population to increase. So therefore, population will remain zero forever. Now, your carrying capacity for your system is one in these normalized coordinates.

So therefore, when you have reached the population which is equal to the carrying capacity you will not expect the population to increase any further. So again, you have $y_e = 1$. So, you have two equilibrium, two equilibrium solutions. And when you have two equilibrium solutions, you would like to know their stabilities. But before we comment about their stabilities, let us try to develop the phase portrait for the system.

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So, we are developing phase portraits for $\frac{dy}{dt} = ay(1-y)$. I am considering the growth phenomenon. So therefore, I know that a > 0. So, my phase portrait corresponds to a > 0, my population y is always a positive quantity (y > 0), and time physically is always greater than zero (t > 0). So therefore, I am drawing the coordinates like this.

Now, given that this is the plane in which I need to draw all the phase lines which are the solutions to $\frac{dy}{dt} = ay(1-y)$, what do I need to do? First thing which I always do is I mark the equilibrium solutions. So, I know that $y_e = 0$ and $y_e = 1$. So, I will mark them. So, this is $y_e = 0$, and this is $y_e = 1$.

I have marked the two equilibrium solutions. Now, what did I do previously for this particular analysis? I divided my phase portrait into various regions. Let me do that here also. I have a Region 2; I have Region 1. In Region 1, the initial population see here the initial population $y_{01} < y_e$. In Region 1, the initial population or population at any instant of time for that matter, would be less than the equilibrium population.

So, I will do the analysis for Region 1. In Region 1, I have to do the analysis for $\frac{\mathrm{dy}}{\mathrm{dt}} = ay(1-y)$.

So, a is positive (a > 0), I know. This y is between 0 and 1. So this is positive. And (1-y) for y between 0 and 1 would be positive. So therefore, overall $\frac{\mathrm{dy}}{\mathrm{dt}}$ is going to be positive.

Now since y_e is my equilibrium solution, I know from my previous concepts that y_e would act as the asymptote to the system. So how do I draw a line which starts from y_{01}, has always a positive slope and has an asymptote as $y_e = 1$? Let me draw one curve and let us see if you agree that this will be the case. This is going to be the phase line. And therefore, for different initial populations, you will have different phase lines.

What is the physical meaning of this? The physical meaning of this is that if your initial population, if your initial population is less than the equilibrium population or less than the carrying capacity of your system, then what is going to happen? Then what is going to happen is that your population will rise till it reaches the value $y_e = 1$, or in x-coordinates, $y_e = N$. This is the meaning of this.

Now, in Region 2, my initial population, $y_{02} > y_e$. How would that happen? Well, this would happen if you introduce suddenly a very large number of members in the region which, with the number which is larger than the amount, with the number which can be sustained by the, by the ecosystem provided by the system.

So, in that case how do I do this analysis? $\frac{\mathrm{dy}}{\mathrm{dt}} = ay(1-y)$. a is positive (a > 0), y > 1, so it is going to be positive. But (1-y) for y > 1 is negative. So therefore, the overall derivative would be negative in Region 2. That means, I have to start with y_{02}, always maintain negative slope, and reach $y_e = 1$ asymptotically. So not very difficult to see then this is going to be one phase line.

And therefore, you can draw several of such phase lines. And what is the physical meaning of these phase lines in this region? It physically means that if you have, at an instant, population which is larger than the, than the equilibrium population of the system, then the population must go down, population must reduce.

It must reduce till what extent? Till the carrying capacity. So now what you saw is something like this, that you had two equilibrium populations, and all the solutions, all the solutions they in fact merged or had a tendency to come asymptotically towards $y_e = 1$.

So therefore, I can write this, I can draw these arrows, which means $y_e = 1$ is a stable equilibrium solution. And then I can extend this. I see that from here, in this region, everything is moving away from y = 0. So therefore, this one, the arrows would be pointing outward. And this one would be an unstable equilibrium solution. Physically, all of these things make, makes sense.

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And now, again, for the sake of mathematical completeness, what I can do is, I can draw this complete phase portrait. So, this is t, this is y. I have two solutions, equilibrium solutions. This entire line is a solution. Let me emphasize that I am drawing this complete phase portrait for the sake of mathematical completeness so that you can appreciate that everything is in place, mathematically.

There is no problem. Physically, you have to worry, in this particular problem, only about the first quadrant. So now $y_e = 0$, $y_e = 1$. So, Region 2, Region 1. We saw that in Region 1, the derivative has to be positive and then we drew this curve, for example. Now, as I go towards negative time, as I go to negative time, $y_e = 0$ also has to be an asymptote.

So, when I am here, and I have a positive slope, and this region is an asymptote which means it would be something like this, how do I join the curve? Well, I will do something like this. And therefore, again, draw various phase lines like this. And then in this region, I had drawn this, I will go up, this, this, and for negative populations, only mathematically, I can continue with the same analysis.

You can do this analysis in Region 3. Same, $\frac{dy}{dt} = ay(1-y)$, a positive (a > 0), y negative, so, you will find that the slope must be negative. And this would be the phase lines. And finally, what you saw is that here, all the solutions converge, here, all the solutions diverge. So, the lower state is the unstable state, the top state is the stable state.

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Can we establish this stability using an analytical condition? So, you see here I have

$$\frac{\mathrm{dy}}{\mathrm{dt}} = ay(1-y) = f$$

So, what is f?

$$f = ay - ay^2$$

To determine whether the equilibrium solution is a stable solution or an unstable solution, what I will do is, I will determine $\left.\frac{df}{dy}\right|_{y_e}$

So,

$$\left.\frac{df}{dy}\right|_{y_e} = a - 2ay|_{y_e}$$

So,

$$\left.\frac{df}{dy}\right|_{y_e=0} = a > 0$$

So, when

$$\left.\frac{df}{dy}\right|_{y_e} > 0, \qquad unstable$$

Fine, let us take the other thing.

$$\left.\frac{df}{dy}\right|_{y_e=1} = -a < 0$$

So, when

$$\left.\frac{df}{dy}\right|_{y_e} < 0, \qquad stable$$

These are the conditions for stability, analytical conditions. And this, in fact, is what we also saw in the phase portrait. See here. $y_e = 0$, unstable and $y_e = 1$, stable. Analytically as well as graphically, we get the same answer.

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Now finally, when we wrote the expression

$$y(t) = \frac{y(0)e^{at}}{1 - y(0) + y(0)e^{at}}$$

as the solution, and without solving or plotting the solutions, we got the phase portrait, we now need to see whether the phase portraits match. So let us go and see whether the four phase portraits in the two cases match.

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So, now I can write here,

$$f(x) = \frac{c\ e^{ax}}{1 - c + ce^{ax}}$$

c is my initial condition. So, let me get rid of this, and then let me tweak around. So, a is always positive. So let me write this in this small interval. And my initial population, is always positive. So again, between 0 and 2.

We concluded previously, yesterday as well as today, that if your initial population itself is 1, it will remain 1. So, you see here, I have used c = 1, which means my initial population is 1. And then you see here, nothing changes. Now, I reduce my initial population and see the nature of the curve. It is exactly the same as we drew. So, can we quickly draw what we drew previously?

We had drawn this. This is y, this is t. $y_e = 0$, $y_e = 1$ and you see asymptotically, this curve starts from y_0, it starts from y_0 and asymptotically reaches 1. Then we saw this portion, when your initial population is larger than the equilibrium population. When your initial population is larger than equilibrium population, what do you need to do?

Well, you simply increase this here, and you see what do you get here. Now, my initial population was larger than the equilibrium population, in which case, I come down asymptotically to this limit. So, this phase portrait which I drew for physically realizable region, in fact, is absolutely correct. Now, what I will do is, I will try to see the effect of various parameters.

So, let me animate the effect of initial condition. You see here. We also saw that if the initial condition is simply that you have c = 0, you see here, you are at zero. We saw this previously. We are at zero. If the initial population is zero, it will remain at zero forever. Then, if your initial population itself is 1, at c = 1, again, you see here that you would remain forever at 1.

And then, between these two regions, between these two regions, if I animate, what I will get is that as long as my initial population is between 0 and 1, I will see a growth, and if my initial population is larger, then the equilibrium population, upper stable equilibrium population, I will see a decay. This is the decay.

Then, let us confirm the rest of the parts of the phase portrait. So, this was Region 1, this was Region 2. So, this is Region 1, this is Region 2. So, Region 2 holds true. For negative time, here, this is also true, because what we drew was this. And then what we did was we extended it like this. Here you see, you start, go up, and then you go up in negative direction.

Fine. Now, in Region 3, before we take Region 3, let us see Region 2. So, for looking at Region 2, let me make the $c < 1$. This is what we saw in Region 2. We had to draw like this. So, it came like this. But then I said that zero has to be an asymptote for negative time, and therefore this is what you are seeing here. Exactly the same behavior.

Now Region 3, so what you do is you go for negative initial condition. You see here. As said that we would draw this, and this is exactly what you got. So let me quickly redraw all of the regions of the space portrait neatly.

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So, I will have t, I will have y, I will have two equilibrium solutions. And the phase lines would look like this. The phase lines are in front of you. Try to convince yourselves of the physical meaning of each of this line. And then, you would realize that this is true only for $a > 0$.

But what about a particular model and a particular case for which $a < 0$? Not very difficult to see. You would, you can simply make here $a < 0$. So let me make it $a = -1$. And when I do this, you see here all of them would get reversed. Now, I have this going up. And for this, you see, going down, going down and so on.

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So therefore, what I can do is I can do this. I can draw another set of phase lines. This is t, this is y, this is one equilibrium solution, this is another equilibrium solution, and the gradients, because a has changed, will become exactly the opposite. So, now you have these as the phase lines.

You would have these as the phase lines, and you would have these as the phase lines, exactly the opposite for $a < 0$. And final thing which I would do is, I know this is $y_e = 0$, ye = 1, I know that this is stable, I know this is unstable, and you would see that, here, $y_e = 1$.

This will become unstable. And $y_e = 0$, this will become stable. Why would this happen? Well, this will happen because, quite simply, if you determine $\left.\frac{df}{dy}\right|_{y_e}$ with $a < 0$, then you will see that the situation has been exactly reversed.

So, we will stop here today. And I hope you understood as well as appreciated the importance of non-linear models in developing the population dynamics. We started with a linear model, solved the problems associated with linear models, and then continued to develop a non-linear model, which is called the logistic model. We will discuss these things, this particular model further, with a variant of it in the lectures to come. Thank you.