Advanced Process Dynamics

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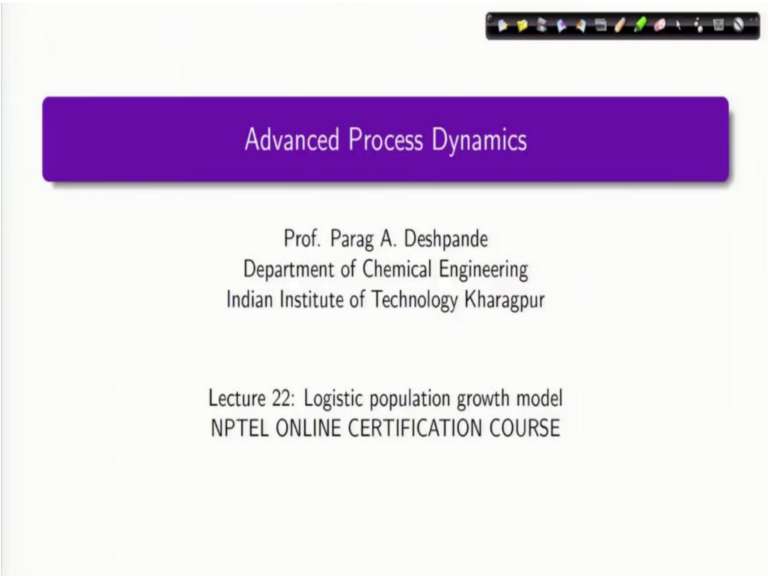
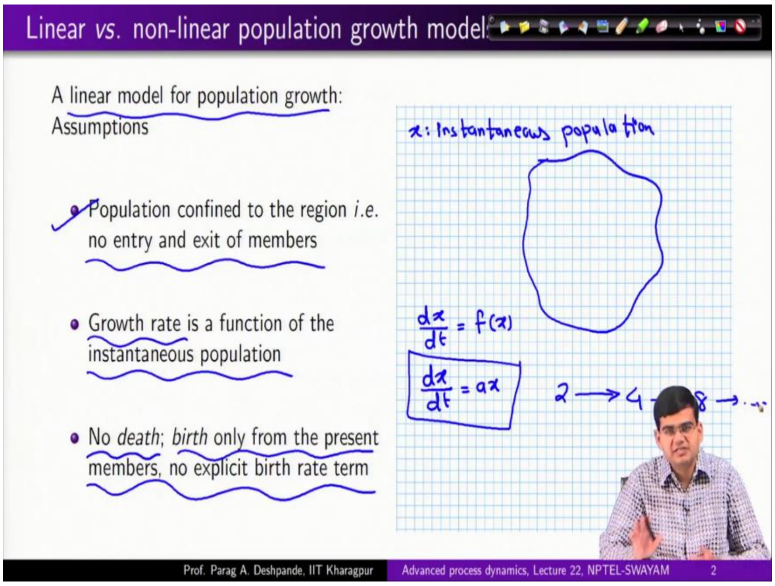
Lecture No. 22

Logistic Population Growth Model





Linear *vs.* non-linear population growth model

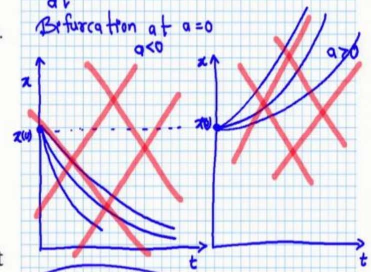
A linear model for population growth:
Assumptions

- Population confined to the region *i.e.* no entry and exit of members
- Growth rate is a function of the instantaneous population
- No *death*; *birth* only from the present members, no explicit birth rate term

Linear model

$$\frac{dx}{dt} = ax$$

Bifurcation at $a=0$

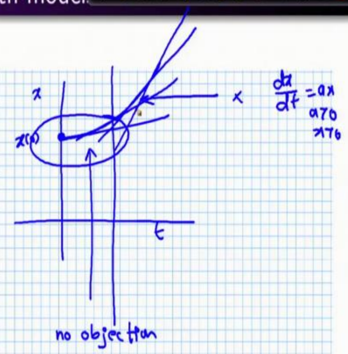


$$\lim_{t\to\infty} x(0)e^{at} \to \infty \quad \text{for } a>0$$

Linear *vs.* non-linear population growth model

A non-linear model for population growth:
Assumptions to overcome the issues of the linear model

- Population confined to the region *i.e.* no entry and exit of members
- Growth rate is a function of the instantaneous population
- No *death*; *birth* only from the present members, no explicit birth rate term
- Growth rate proportional to the instantaneous population only for small populations
- Negative growth rate at large populations so as to "limit" the population



$\frac{dx}{dt} = ax$, $a>0$, $x>0$

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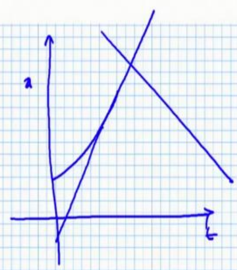

Linear *vs.* non-linear population growth model

A non-linear model for population growth: Assumptions to overcome the issues of the linear model

- Population confined to the region *i.e.* no entry and exit of members
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$$\frac{dx}{dt} = ax\left(1-\frac{x}{N}\right) \quad -(1)$$

Logistic population growth model

Linear *vs.* non-linear population growth model

A non-linear model for population growth: Assumptions to overcome the issues of the linear model

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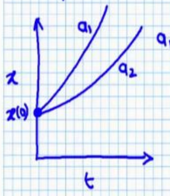
$$\frac{dx}{dt} = ax\left(1-\frac{x}{N}\right)$$

Non-linear model

growth parameter

carrying capacity of the system



Logistic population growth model

$$\frac{dx}{dt} = ax\left(1-\frac{x}{N}\right) \quad -(1)$$

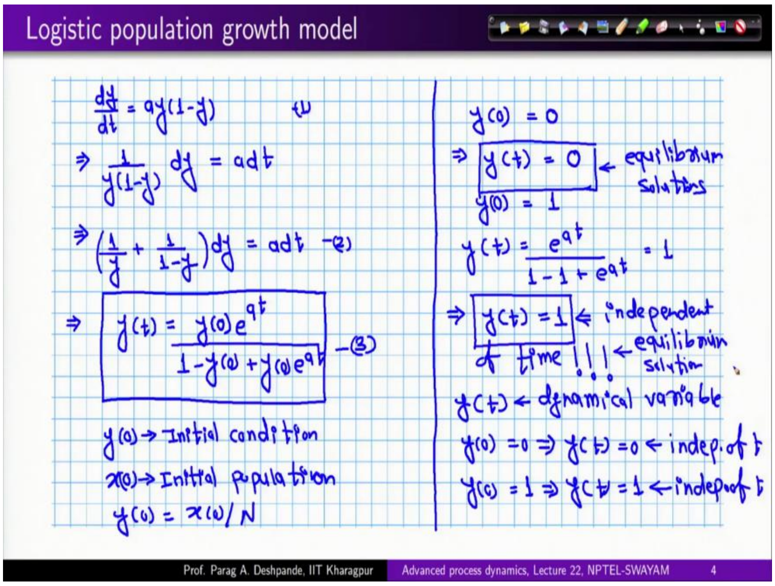
$$\text{Let } \frac{x}{N} = y \quad -(2)$$

$$\Rightarrow \frac{dx}{dt} = N\frac{dy}{dt} \quad -(3)$$

$$\cancel{N}\frac{dy}{dt} = a\cancel{N}y(1-y)$$

$$\Rightarrow \frac{dy}{dt} = ay(1-y) \quad -(4)$$

Max. population = 1



Hello, so we are currently discussing non-linear dynamics. In the previous lecture, we saw the method of linearization. But today we will take an example in which you do not need to linearize your system rather you can do the analysis of dynamics directly in the non-linear domain.

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The situation comes from the population growth of a biological species. The particular example or the model which we are taking is called the logistic growth model. But before we come to the logistic growth model.

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Let us first look into certain conditions under which we can develop a linear model for population growth. So, what we are currently doing is we are trying to understand the growth of a biological species. So, the biological species can be bacteria in a culture for example, it can be the population of deer in forest, for example, so, under certain conditions, let us say you want to develop the growth of this population with time and you can develop a linear model, but that linear model would be accompanied with certain assumptions.

So, I have a linear model for population growth, and what are the assumptions, the population is confined to a region. So, I have a region, let us say, I have a Petri dish, and I am putting my culture in the Petri dish. I am not taking anything out from the dish, I am not also adding anything to dish. I have a forest I have a confined region there deers are not allowed to escape the forest or no additional deer comes from anywhere any other forest for example. So, the population is confined to the region no entry or no exit of these biological members.

Growth rate is a function of instantaneous population. This is important, growth rate, so let us say x is my instantaneous population, population at any point of time t. So,

$$\frac{dx}{dt} = f(x)$$

Does this make sense? In fact, it does. If you have more number of deers, it is expected that the further reproduction rate of reproduction would be higher and therefore, the rate of change of population at higher populations would be higher compared to the rate at lower populations.

So therefore,

$$\frac{dx}{dt} = f(x)$$

and to make the model linear, I have no option but to set it as

$$\frac{dx}{dt} = ax$$

This is the only way that the population growth would be, the model would be linear. And then finally, no death. Later on, we will see much more complex and realistic in fact, population growth models, which account the death rate the birth rate and so on.

But as the first population growth model, here we say that there is no death in the community and birth only from the present members, no explicit birth rate term, there is no explicit birth rate term, I have just so I have for example, if I have two bacteria, then by binary fission, they will become 4, they will become 8, and so on. So, this is basically the mechanism of population growth.

If this is the case, let us see what happens. So, I have currently my linear model. So, this is a linear model. My linear model is $\frac{dx}{dt} = ax$. Now, after understanding the dynamics of linear systems, so well can I look into this equation and then try to understand what can be the possible problems with this particular model. Well, in fact I do so, I will draw the face portrait, I know the system has a bifurcation at a = 0.

So, therefore, I would need to draw two phase portraits, so, here would be t...... here would be x..... here would be t....... here would be x. And now, if I have the same initial population x(0). I know what is that it has to be some initial population for the species to grow, there has to be some species, some number of members of the species have to be present.

So, if that is the case, for a < 0, what is going to happen is that one of my phase lines would look like this. In fact, you can draw several of these phase lines depending upon the value of a, this is how it would look like. And what is the problem with this particular situation, this particular situation describes only the decay in the population how the population goes down to zero with time.

But that is not we are trying to basically model we are trying to model the growth of the population and there is no death, this is the assumption on the left-hand side you can see So, if there is no death, there is no way that the population of the species in the confined area or confined region can decrease with time. So, therefore, this is absolutely not possible, in my current model, fine.

So, then the second case is when a > 0, a is positive. And now, I have these different solution curves, which means that if I start with some population with time, my population is going to increase makes sense, right? If you start with some amount of some number of bacteria in a culture, the number of bacteria is going to, in fact increase with time, but then there is a problem.

The problem is that

$$\lim_{t \to \infty} x(0)e^{at} \to \infty \,; a > 0$$

So, imagine that you have a population of deer in a forest and you start with some number of deers, what is going to happen at large time interval at large times. Well, the population a will become very, very, very large. That is the meaning of infinity.

Do we ever see this? It does not happen. You your population cannot tend to infinity. So, therefore, this also is practically not possible. So, if this is practically not possible, then what is the problem with our model equation? The problem with our model equation is that my population tends to infinity, as $t \to \infty$. I need to overcome this problem. I need to control my population at larger time intervals.

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So, that is what gives rise to the need of the non-linear model. So, the non-linear model has the following assumptions. The population is confined to a region is the same assumption, which was saw in the linear model. The growth rate is a function of instantaneous population exactly the same, assumption, no death, birth only from the present members no explicit birth rate term this also we saw from the previous assumption, no growth rate proportional to instantaneous population only for small populations this is important.

What you have done is you have said that growth rate is proportional to the instantaneous population only for small populations that means, only when the populations are small your growth rate would be proportional to the instantaneous population, because, if we remember our previous phase portrait this was t.... this was x...... this was x(0) and say this is one of the phase lines.

So, our objection was not in this region, we have no objection in this region as far as the population grows, the problem is here, we cannot accept this that the population goes to infinity. So, therefore, the growth rate should be proportional to the instantaneous population only in small populations. When t is close, close to zero and then negative growth rate at large population, as population grows, the growth rate becomes negative.

So, in this particular portrait, you see, $\frac{dx}{dt}$ is always positive..... everywhere $\frac{dx}{dt}$ is positive. And why it is positive? Because you know

$$\frac{dx}{dt} = ax; \quad a > 0 \;\&\; x > 0$$

So, therefore, it will always be positive. Now, what you introduce is that you have negative growth rate at large population so, as to limit the population we want we know that the population cannot go to infinity, so, there has to be an upper limit.

So, therefore, if I somehow am able to invert this positive gradient to a negative gradient to a negative gradient this is t..... and this is x...... then what I expect is that the negative gradient would reduce the population the positive gradient part would push the population up and there would be some balance so, that the population goes to saturation, this is my expectation.

And the model equation which satisfies all of these conditions all of these goes like this

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) \ldots \ldots \ldots (1)$$

This is called the logistic population growth model. So, let us try to understand the meanings of various quantities.

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So, I have $\frac{dx}{dt}$ which is the rate of change of population and now, it is to reveal to you that you have a dynamical system. What is a? a is the same factor which appeared in the linear model this is the growth parameter. So, in the previous case when you have x and t and you start with some population x(0). What is the significance of a? a is a growth parameter.

So, when you have a_1 and a_2, the larger the growth parameter the faster the rate. a_1 > a2. This is the growth parameter which appears in a linear model and it appears here also. What is N? N is called the carrying capacity of the system. Now, this needs attention. What is the meaning of carrying capacity of the system?

Well, if you take a Petri dish and add certain initial amount of bacteria and your purpose is to grow bacteria in a typical setup, let us say in a bioreactor then why would bacteria grow, bacteria would grow only if you provide the system with the nutrients for the bio chemical processes in the within the bacteria to happen.

So, you will provide it with certain amount of nutrients, your assumption says that the population is confined nothing can come in the population cannot go out and if you start with certain amount of nutrients, once the nutrients are exhausted, the bacteria cannot grow anymore. So, therefore, depending upon the amount of nutrients which you have provided to the culture there is an upper limit to which the bacteria and the bacterial population grow.

Similarly, imagine that you have deer, in population of deer in a forest and they feed upon leaves. So, there would be certain fixed number of trees in that region and therefore, why would why can the population not grow indefinitely because then there is some of the deer population would starve to death because there would be no trees or no leaves left for them to feed on. So, therefore, every system every ecosystem has an upper limit which is called the carrying capacity and that parameter here is N.

So, N is the carrying capacity of the system. And now, this is a non-linear model the way we did the analysis for data mining linearity or non-linearity in the previous lecture, I would encourage you to do this and convince yourself that this in fact is a non-linear model. So, let us try to understand the dynamics of this non-linear model.

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Before we do that, let us try to first understand one thing

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) \dots \dots \dots (1)$$

So, this is a non-linear system. So, not every non-linear system can be solved trivially or easily in a few steps, but this particular equation can be. So, we will solve this equation and try to understand the dynamics before we solve this equation, let us do some rearrangements and some simplifications.

So, let me say that let

$$\frac{x}{N} = y \dots \dots \dots (2)$$

So, what is the physical significance of y, you will see that $\frac{x}{N}$ is the normalized population. So, if a region can support say 1000 members of the species, then at any instant of time what is the fraction of the species fraction what is the fraction what is that number as a fraction of the total which is present at in that region. So, this is this is given by y.

So, if this is the case then I know that

$$\frac{dx}{dt} = N\frac{dy}{dt} \ldots\ldots\ldots (3)$$

This is (1) this is (2) and this is (3). So, let me substitute equation (3) in equation (1) and write this as

$$N\frac{dy}{dt} = aNy(1-y)$$

So, I will write

$$\frac{dy}{dt} = ay(1-y) \ldots\ldots\ldots (4)$$

So, what do I get? In this particular case, I have just got rid of the parameter N because N is a constant.

And now all the analysis that I am doing is the fractional population. So, therefore, my maximum population for equation number (4), maximum population, which is the carrying capacity, carrying capacity is the maximum population which ecosystem consist. So, the maximum population will be community. So, instead of number N, I have number 1. So, let us try to solve this problem.

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So, I have

$$\frac{dy}{dt} = ay(1-y) \ldots\ldots\ldots (1)$$

and I can solve this by first doing a rearrangement can write this as

$$\frac{1}{y(1-y)}dy = adt$$

And how do I, what do I do with the left-hand side, if you remember, I need to do a partial fraction. For this particular case doing partial fraction is not very difficult, I believe this should be

$$\left(\frac{1}{y} + \frac{1}{(1-y)}\right)dy = adt \dots \dots \dots (2)$$

Now, I can integrate on both the sides determine the expression for the integration constant by putting the initial condition and get the final expression. Those previous steps I am leaving for you and what you should get is

$$y(t) = \frac{y(0)e^{at}}{1 - y(0) + y(0)e^{at}} \dots \dots \dots (3)$$

I repeat that what you will do is you will integrate equation number (2) on both the sides you will introduce the integration constant and determine the value of integration constant using initial condition.

So, initial condition here is y(0) so, y(0) that means write here explicitly y(0) is the initial condition. So, if x(0) is the initial population, then

$$y(0) = \frac{x(0)}{N}$$

and we will substitute here. And, by converting x to y, I have just normalized my population. So, there is nothing very difficult which has been done here.

It is just change of variables. So, instead of going from 0 to N, I will go from 0 to 1. So, let me have a look into equation number (3) and then try to understand the behavior of this solution. Now, what I see is that if

$$y(0) = 0$$

which means I do not have any population at all then what will happen

$$y(t) = 0$$

Does this make any sense physically?

Yes, it does. If you do not have any population, then how would the population grow. Quite trivial observation. But if see if I have

$$y(0) = 1$$

So, then

$$y(t) = \frac{e^{at}}{1 - 1 + e^{at}} = 1$$

and this is independent of time.

So, what is the physical meaning of this particular observation? This observation means that if I have a population which has reached the normal normalized population 1, in other words, if the carrying capacity is N and the population has reached the carrying capacity of N, then there would not occur any change in the population of your system.

But, what kind of state my system is in such that the population is not changing with time remember that y(t) is my dynamical variable y(t) is my dynamical variable which means it should be a function of time, but $y(0) = 0$ that results in $y(t) = 0$, which means independent of time and $y(0) = 1$ result in $y(t) = 1$, which again means, your system is independent of time.

And, when does your system become independent of time? Your system becomes independent of time when there are no gradients in your system, which means that $y(t) = 1$ and $y(t) = 0$ must be your equilibrium solutions.

So, when I look into this particular system, now, I see that fine I have got my solution as $y(t) = \frac{y(0)e^{at}}{1-y(0)+y(0)e^{at}}$. But my system has equilibrium states, my system has equilibrium solutions and not only that my system has equilibrium solutions.

There are two equilibrium solutions in my system one is y = 0 and the other one is y =1. For all the cases that we studied till now, we did not observe two equilibrium solutions we always observed one equilibrium solution. So, we have now here the case where you have two populations, which are equilibrium populations.

What is the physical meaning of these two equilibrium populations? This is something which we want to know. So, what we will do is that we will continue this analysis in the next lecture and try to understand the physical meaning of these two equilibrium populations, equilibrium solutions that we obtain and then we will try to know the intermediate dynamics of the system. Till then thank you.