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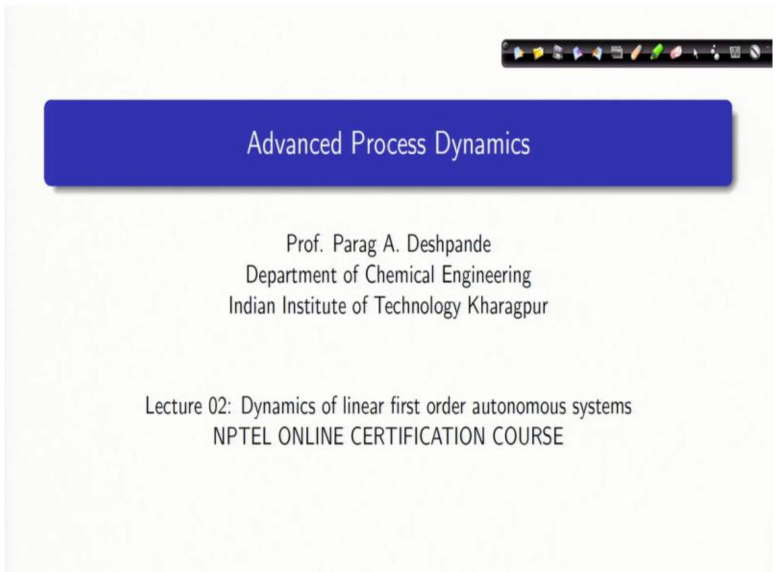
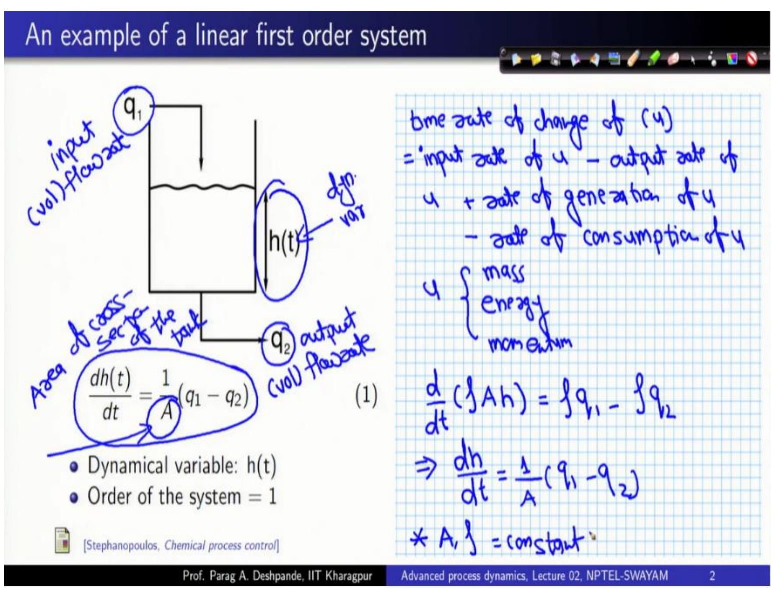
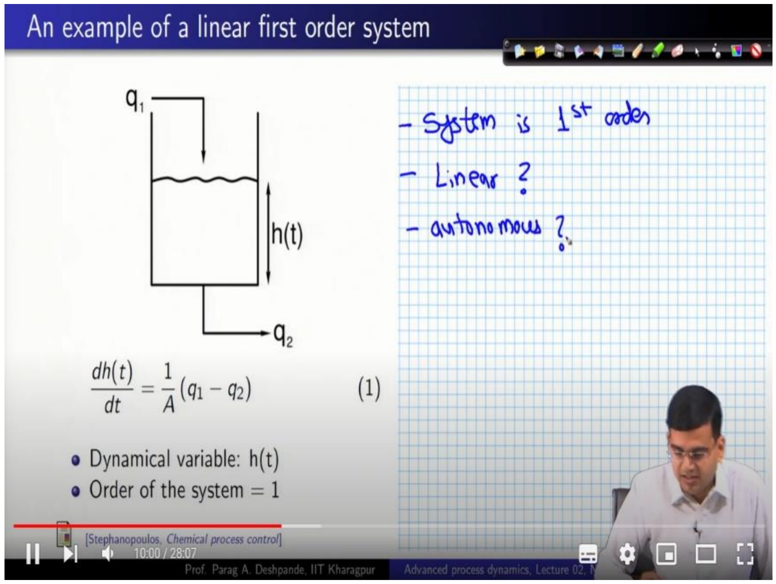
Lecture 02

Dynamics of Linear First Order Autonomous Systems






An example of a linear first order system
q_1
$h(t)$
q_2
$$\frac{dh(t)}{dt} = \frac{1}{A}(q_1 - q_2) \quad (1)$$
Dynamical variable: h(t)
Order of the system = 1
- System is 1st order
- Linear ?
- autonomous ?
[Stephanopoulos, Chemical process control]
10:00 / 28:07
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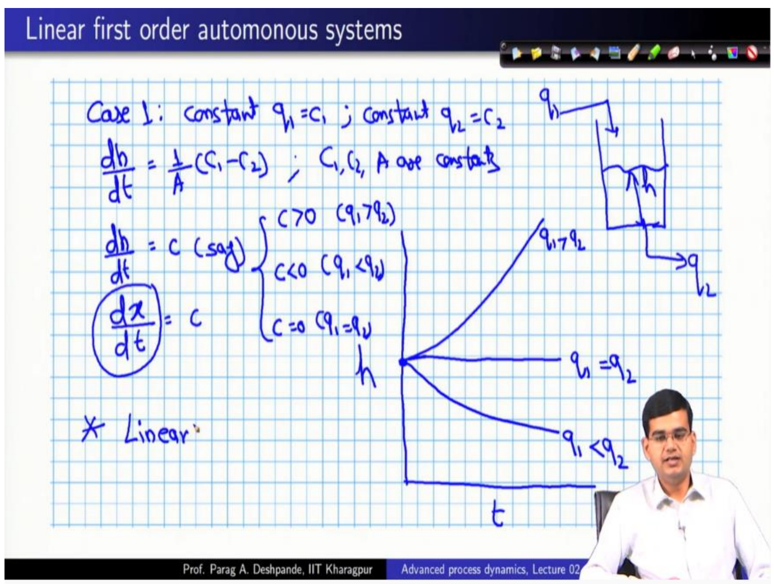
Linear first order automonous systems
Case 1: constant $q_1 = C_1$; constant $q_2 = C_2$
$\frac{dh}{dt} = \frac{1}{A}(C_1 - C_2)$; C_1, C_2, A are constants
$\frac{dh}{dt} = C$ (say)
$C > 0$ $(q_1 > q_2)$
$C < 0$ $(q_1 < q_2)$
$C = 0$ $(q_1 = q_2)$
$\frac{dx}{dt} = C$
* Linear
q_1
h
q_2
$q_1 > q_2$
$q_1 = q_2$
$q_1 < q_2$
h
t
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Linear first order automonous systems

Case 2: $q_1 = \text{constant} = C$

$q_2 = f(h)$

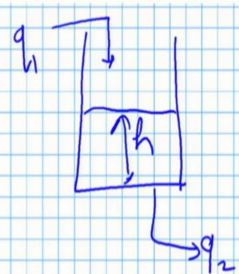
$$\frac{dh}{dt} = \frac{1}{A} q_1 - \frac{1}{A} q_2$$

$$\Rightarrow \frac{dh}{dt} = \frac{c}{A} - \frac{1}{A} f(h)$$

$$\Rightarrow \frac{dh}{dt} + \frac{1}{A} f(h) = c \text{ (say)} \quad \Big| \quad \frac{dh}{dt} = C$$

If $f(h)$ is linear, then the system is linear

Solution using IF method



Linear first order automonous systems

Case 3: ~~consta~~ $q_1 = f(t)$

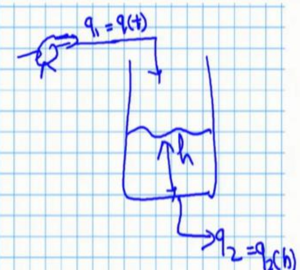
$q_2 = g(h)$

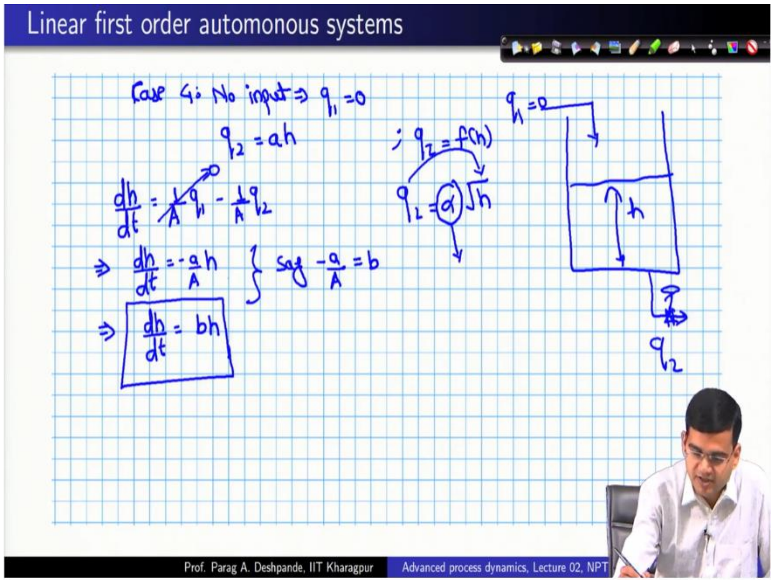
$$\frac{dh}{dt} = \frac{1}{A} f(t) - \frac{1}{A} g(h)$$

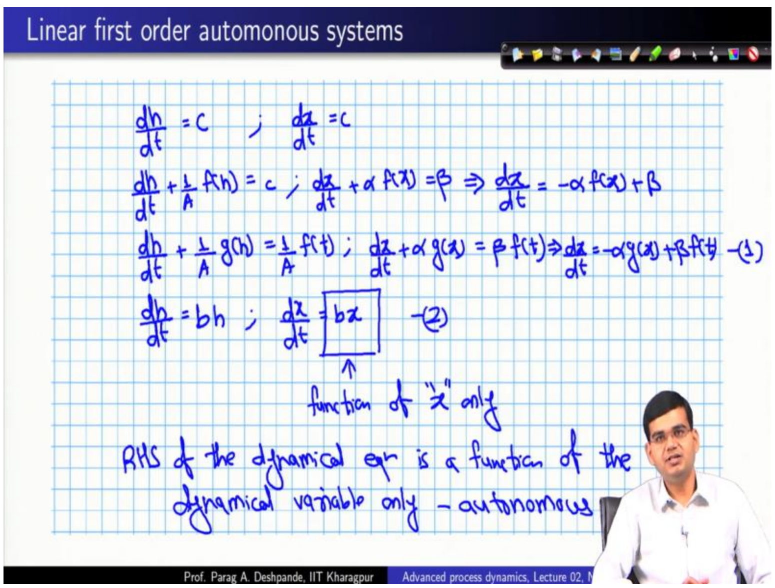
$$\Rightarrow \frac{dh}{dt} + \frac{1}{A} g(h) = \frac{1}{A} f(t)$$

If $g(h)$ is linear then the system is linear.

Solution by IF method.







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So, let us get started with analysis of dynamical systems in state space domain. Before we go into the details of examples that we would take today, and subsequently in this week, let us first formally define the meaning of state space. So, imagine that you have a thermometer and you measure the temperature of your system using thermometer. The dynamical system in this case can be very easily identified as the temperature of the system, because the temperature of the system changes with time.

So, now, the question is that how do you define the state of the system? In state space domain analysis, the state of the system is defined by the dynamical variable of the system. So, therefore, what you say is that different temperatures correspond to different states of the system. Now, in lot of dynamical models, the solution to the model equations is not unique. In fact, if the system is a linear system, then in lots of cases, the solutions belong to a vector space, which means that you have a collection of solutions.

If you model the variation of temperature in your thermometer, you would get a first order equation and the first order equation corresponding to the parameter space that you have, will have lot of solutions.

And therefore, the solutions will belong to a vector space and therefore, we call this particular analysis as state space domain analysis, where the solutions are very many and they belong to a vector space, they belong to a linear vector space in fact, and therefore, the term state space domain analysis.

So, we will take various examples, and what we will do is we will make the system gradually go from simple ones to very complex ones. The simplest ones are linear, first order autonomous equations, the most complex ones would perhaps be nonlinear, higher order non-autonomous systems. So, in today's lecture, let us take the simplest case of linear autonomous first order equations.

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You must have seen this particular example from various textbooks, which is the case of liquid level in a tank. So, there is a tank which has the provision of an input and then you can drain the liquid from this using an output. This is a toy example for most chemical engineering operations. This is a very, very simple example, but then you will see that this simple example actually gives you a lot of insight, which will subsequently be used for the analysis of further complex examples.

So, what you see here is a system which is a linear first order system. We are trying to define a linear first order autonomous system. The definition of linearity is what we saw in the previous lecture. This is a first order system, because you have only one first order ODE. So, linearity and first orderness is fine. We have not yet defined what is the meaning of autonomous system.

We will define it in today's lecture, and then subsequently see what are the characteristic features of linear autonomous first order systems. So, this is the mass balance equation for the system. So, let us first define the system and look at the system properly. You have q1 which is the input flow rate, and q2 is the output flow rate. h is the liquid level in the tank and therefore, this is your dynamical variable.

You see here that the model equation is given in terms of h(t),

$$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$$

A is the area of cross section of the tank. What is the genesis of this model equation? It is not very difficult to see. We can define a general balance equation or conservation equation as time rate of change of -- I will write the quantity u, I will define what u is,

time rate of change of u = input rate of u - output rate of u + rate of generation of u - rate of consumption of u

And what is u? u is the quantity which is conserved. So, you can be mass, u can be energy, u can be momentum. So, the time rate of change of mass for example, in the system will be given as the rate of influx of mass minus the rate of out flux of mass, if the system is reactive, then for reactants you will have negative rate of consumption, for products you will have positive rate of production.

And similarly, you can write the same equations for energy as well as momentum. In this case, in this particular case, we are concerned with the system which is flow system which does not have any reaction going on and therefore, you have simply the time rate of change of u which means the time rate of change of mass. So, this will be equal to

$$\frac{d}{dt}(\rho A h) = \rho q_1 - \rho q_2$$

q_1 and q_2 are input and output volumetric flow rates. There is no generation of mass or no consumption of mass because the liquid is simply coming into the vessel and it is leaving the vessel, it is not a reactor.

So, therefore, we get this as simply

$$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$$

and we should know that the underlying assumption is that A and ρ are constants, under this assumption the model equation stands correct. So, now, now that we know or understand how does this equation come, let us try to see the behavior or the dynamic behavior of the system.

At this point of time, all we can say is that the system, that the system is first order. All we can say is that the system is first order, linear or not, we do not know and autonomous or not, we have not even defined what is the meaning of autonomous. We will subsequently come to that concept. So, let us first see what are the different cases which can arise.

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So, case one, let us take the example of case one, case one. You have constant q1, let us say that,

$$q_1 = c_1 \ ; \ q_2 = c_2$$

And let us remember that our system is like this q_1, q_2 and this is the height. So, my model equation is

$$\frac{dh}{dt} = \frac{1}{A}(c_1 - c_2)$$

Now, c_1, c_2 and A are constants. So, therefore, I can write

$$\frac{dh}{dt} = c$$

Let us say it is a constant c. So, now, what does this tell me? This is an equation of the form

$$\frac{dx}{dt} = c$$

So, now depending upon the value of c your sign of $\frac{dx}{dt}$ or the derivative will change. So, I can draw here the variation of h with time. So, c and c > 0 which means the case when $q_1 > q_2$. The input flow rate is larger than the output flow rate. Then what is going to happen? If you have certain initial level of the liquid in the tank, the derivative is going to be positive. So, you can expect that the liquid level will arise with time. So, this is the case when q1 is greater than q2, input is greater than the output.

Similarly, c < 0, which means $q_1 < q_2$, well, you are draining more liquid and you are not giving enough input to the system. So, what is going to happen? The liquid level is going to reduce with time and then you can have a balance c = 0 which means $q_1 = q_2$ your liquid level will remain constant. So, this is $q_1 < q_2$ and $q_1 = q_2$. So, depending upon the input flow rates and the output flow rates your system -- the level of the liquid will either increase or decrease or remain constant with time, this was a very simple system and it is not very difficult to find out that the system is linear, that the system is in fact linear.

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Then, what we can do is, we can do something like this case two. In case two, what we do is this q_1 is a constant. And let us say that the constant is c

$$q_1 = \mathrm{c}$$

So, the input flow rate is a constant. And the output flow rate is a function of the height or the liquid level.

$$q_2 = f(h)$$

let me quickly draw this q_1, q_2, h. So, you send the feed at a constant rate. And it is not very difficult to realize from physics that the output will be a function of the liquid level inside the tank.

So, q_2 is a function of h. So, my model equation was

$$\frac{dh}{dt} = \frac{1}{A}q_1 - \frac{1}{A}q_2$$

which means this is equal to

$$\frac{dh}{dt} = \frac{c}{A} - \frac{1}{\mathrm{A}}f(h)$$

and this can be rearranged as

$$\frac{dh}{dt} + \frac{1}{\mathrm{A}}f(h) = c_1$$

So, now, you can immediately see the difference between the behavior in the previous case, your equation was, in the previous case, the equation was

$$\frac{dh}{dt} = c$$

now, your equation is

$$\frac{dh}{dt} + \frac{1}{\mathrm{A}}f(h) = c_1$$

and if f(h) is linear, if f(x) is linear, then the system is linear, then the system is linear right.

And how will you solve typically such an equation? You will solve it by solution using integrating factor method. Not very difficult to see that this kind of equations can be solved very simply by integrating factor method provided you know the functional form of f(h) and values of A and c_1.

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The next case is when you have case three, when you have constant -- no, in this case, you have input q_1 which is a function of time and output q_2, output q_2 is some other function of h.

$$q_1 = f(t)\ ;\ q_2 = g(h)$$

So, let us say that I have put a pump here. So, q_1 I have a method to make it a function of time and then I have q_2, here is h and q_2 is a function of the level of liquid.

In this case, what is going to happen my model equation will become

$$\frac{dh}{dt} = \frac{1}{A}f(t) - \frac{1}{A}g(h)$$

which can be rearranged as

$$\frac{dh}{dt} + \frac{1}{A}g(h) = \frac{1}{A}f(t)$$

this is yet another variation of the same very, very, very simple liquid level problem.

And here you can see that if g(h) is linear then the system is linear. You can see that the condition of linearity is not required for f(t), f(t) does not need to be a linear function for this system of -- for this particular equation to be linear g(h) has to be a linear equation and you can solve again solution by integrating factor method. If you know the explicit forms of g(h) and f(t) you can solve this equation.

So, we saw that seemingly very simple system with equation as simple as $\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$ can have very, very different solutions very, very different behaviors and as was very correctly pointed out by Henri Poincare, very simple ordinary differential equations or very simple dynamical systems can have very complex or different behaviors.

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So, now we finally take one case where we have no case for no input, which means q_1 is equal to zero. So, in my scheme you have q_1 and this is equal to zero this is h and this is q_2 and imagine that I have supplied the system, I have fitted the system with a valve at q_2 such that q_2 is ah.

$$q_1 = 0 \ ; \ q_2 = ah$$

In the previous case with we said that q_2 is a function of h and f (h) can be anything, depending upon the valve that you use, in fact, for the case of gravity driven flow, this would vary as q_2 as $\alpha\sqrt{h}$. So, the dependencies and goes as square root where α will depend upon the valve character -- upon the characteristics of the pipe, the discharge coefficient, the area of cross section of the pipe and so on. But the dependence of h goes as $\sqrt{h}$.

Now, if you have a valve which is designed such that

$$q_2 = ah$$

which can be referred to as the valve coefficient times h. So, in this case what will happen to the model equation, you have

$$\frac{dh}{dt} = \frac{1}{A}q_1 - \frac{1}{A}q_2$$

from where we get

$$\frac{dh}{dt} = -\frac{a}{A}h$$

So, let us say, let us say

$$-\frac{a}{A} = b$$

So, what we get here is

$$\frac{dh}{dt} = bh$$

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Now, this is a very, very specific form for the equation, where you see in the previous -- in this case, so, let us take all the cases one by one, you had

$$\frac{dh}{dt} = c$$

which is of the form

$$\frac{dx}{dt} = c$$

In case one we saw this. Then what we saw was that

$$\frac{dh}{dt} + \frac{1}{A}f(h) = c$$

which is of the form

$$\frac{dx}{dt} + \alpha f(x) = \beta$$

We are redefining different constants. It is not very difficult to retrace those constants in terms of the original variables.

Then, we had the form

$$\frac{dh}{dt} + \frac{1}{A}g(h) = \frac{1}{A}f(t)$$

which is of the form

$$\frac{dx}{dt} + \alpha g(x) = \beta f(t)$$

And finally, I have

$$\frac{dh}{dt} = bh$$

which means the equation is of the form

$$\frac{dx}{dt} = bx \quad \ldots\ldots\ldots\ldots.. (2)$$

correct? So, what do I see here?

In fact, I can do further rearrangement where I can see

$$\frac{dx}{dt} = -\alpha g(x) + \beta f(t) \quad \ldots\ldots\ldots\ldots.. (1)$$

So, when I compare equation one with equation two, what I get is that the RHS here is a function of x only, which means that the RHS of the dynamical equation is a function of the dynamical variable only and not a function of time, and such systems are called autonomous systems.

So, what is a first order linear autonomous system? A first a linear first order autonomous system would be a system in which the dynamical equation would be a single first order ODE the corresponding ODE would be linear and autonomous means that the right-hand side when you rearrange the equation as $\frac{dx}{dt}$ is equal to some function of the variables and parameters which appear in the equation, should be only a function of the dynamical variable and not time.

Remember that the constants and parameters which appear in the equation are not considered the definition. For example, you can have area of cross section on the right-hand side for the liquid level problem, you can have valve coefficient on the right-hand side for the liquid level problem, but what you are not allowed is that you cannot have the time on the right-hand side of the equation. And such systems are called autonomous systems. We will take into the -- we will look into the details of the dynamical behavior of autonomous linear first order systems in the subsequent lectures. Thank you.