Advanced Process Dynamics

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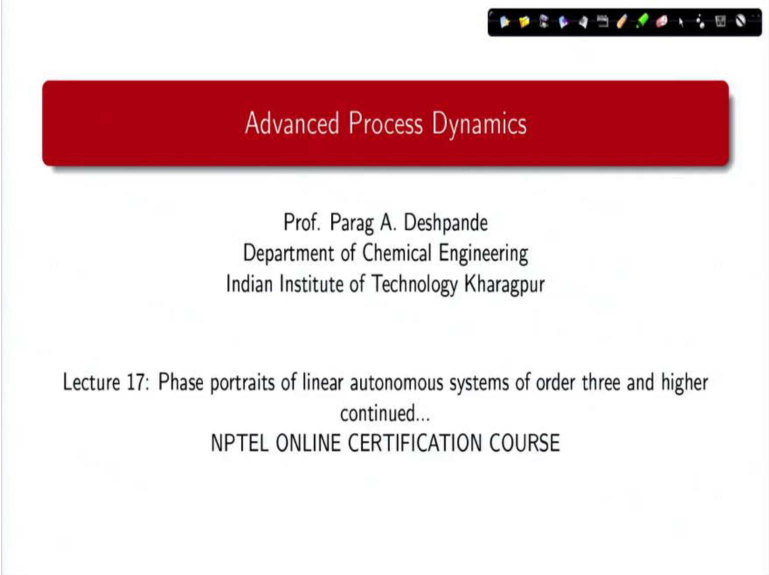
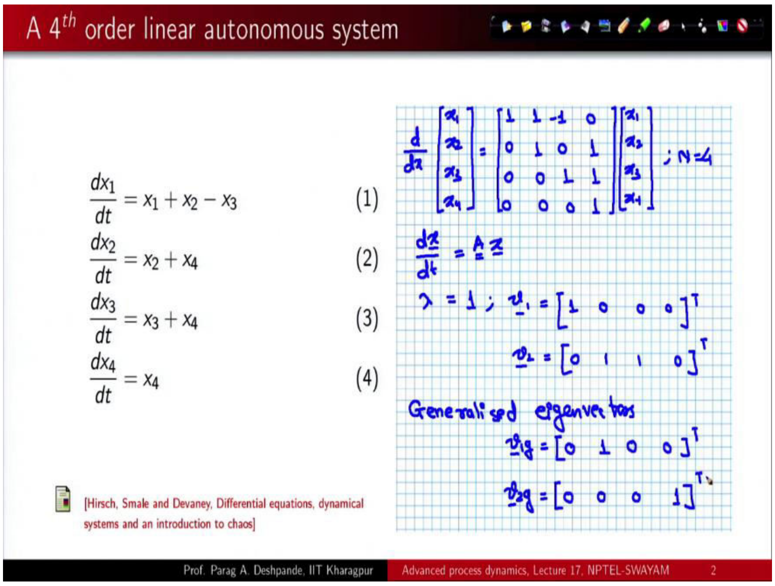
Lecture 17

Phase portraits of linear autonomous systems of order three and higher continued…





A 4^{th} order linear autonomous system

$$\frac{dx_1}{dt} = x_1 + x_2 - x_3 \qquad (1)$$
$$\frac{dx_2}{dt} = x_2 + x_4 \qquad (2)$$
$$\frac{dx_3}{dt} = x_3 + x_4 \qquad (3)$$
$$\frac{dx_4}{dt} = x_4 \qquad (4)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i \quad \text{— (a)}$$

$\underline{v}_1 \to \underline{v}_{1g}$; $\underline{v}_3 \to \underline{v}_{3g}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_1 e^{\lambda t} \underline{v}_1 + c_2 e^{\lambda t}\left(t\underline{v}_1 + \underline{v}_{1g}\right) + c_3 e^{\lambda t} \underline{v}_3 + c_4 e^{\lambda t}\left(t\underline{v}_3 + \underline{v}_{3g}\right) \quad \text{— (b)}$$

[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

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$$\frac{dx_4}{dt} = x_4 \qquad (4)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{t} \left[t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right] + c_3 e^{t} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_4 e^{t} \left[t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 t e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_4 t e^{t} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_4 e^{t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

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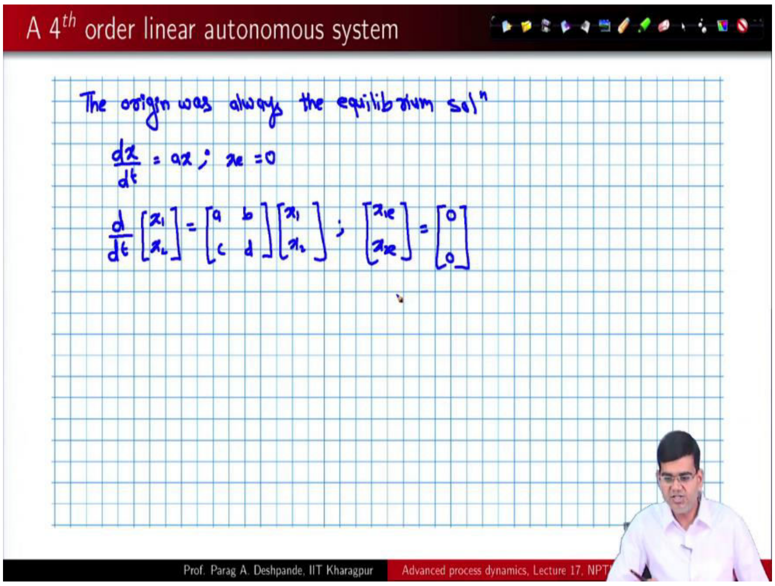
$$x_1 = c_1 e^{t} + c_2 t e^{t}$$
$$x_2 = c_2 e^{t} + c_3 e^{t} + c_4 t e^{t}$$
$$x_3 = c_3 e^{t} + c_4 t e^{t}$$
$$x_4 = c_4 e^{t}$$

[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]


A 4th order linear autonomous system
The origin was always the equilibrium sol^n
dx/dt = ax ; xe = 0
d/dt [x1 x2] = [a b; c d] [x1 x2] ; [x1e x2e] = [0 0]
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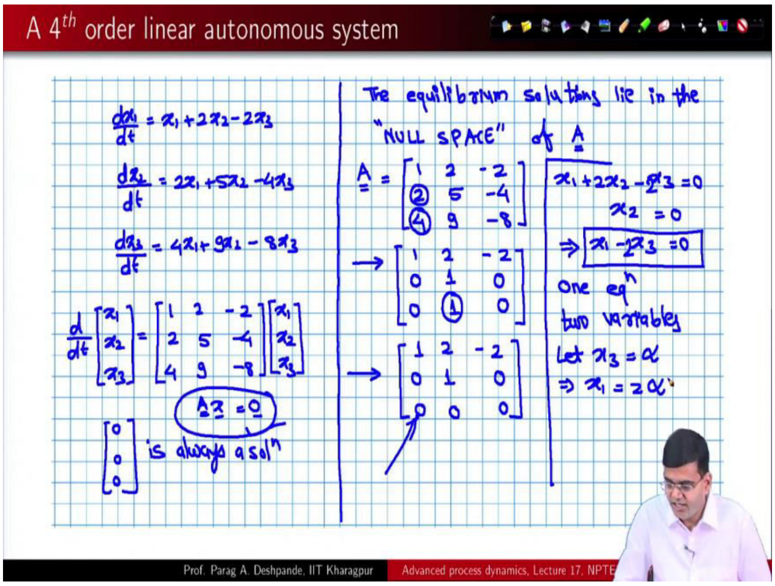
A 4th order linear autonomous system
dx1/dt = x1 + 2x2 − 2x3
dx2/dt = 2x1 + 5x2 − 4x3
dx3/dt = 4x1 + 9x2 − 8x3
d/dt [x1 x2 x3] = [1 2 −2; 2 5 −4; 4 9 −8] [x1 x2 x3]
Ax = 0
[0 0 0] is always a sol^n
The equilibrium solutions lie in the "NULL SPACE" of A
A = [1 2 −2; 2 5 −4; 4 9 −8]
→ [1 2 −2; 0 1 0; 0 1 0]
→ [1 2 −2; 0 1 0; 0 0 0]
x1 + 2x2 − 2x3 = 0
x2 = 0
⇒ x1 − 2x3 = 0
One eq^n
two variables
Let x3 = α
⇒ x1 = 2α
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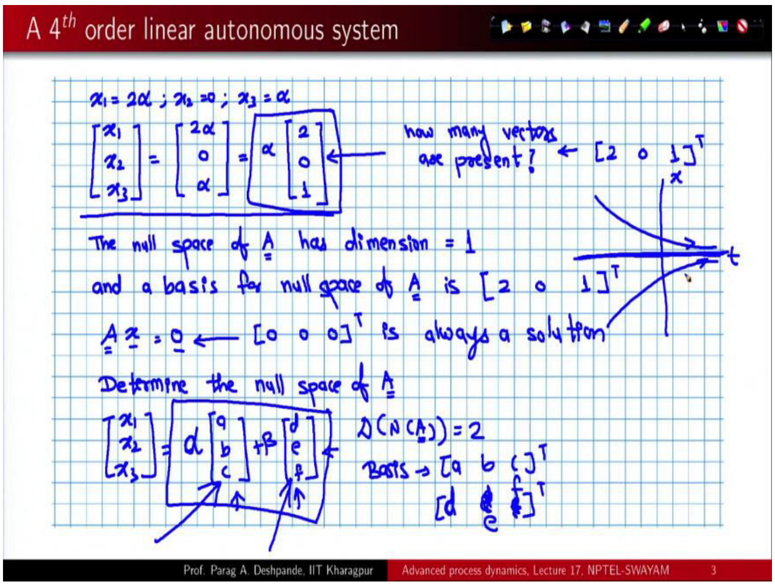
A 4th order linear autonomous system
x1 = 2α ; x2 = 0 ; x3 = α
[x1 x2 x3] = [2α 0 α] = α [2 0 1]
how many vectors are present? ← [2 0 1]^T
x
t
The null space of A has dimension = 1
and a basis for null space of A is [2 0 1]^T
Ax = 0 ← [0 0 0]^T is always a solution
Determine the null space of A
[x1 x2 x3] = α [a b c] + β [d e f]
D(N(A)) = 2
Basis → [a b c]^T
[d e f]^T
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Let us continue our discussion on phase portraits of higher order systems. In the previous lecture, we took examples from the third order systems, and we try to establish an analogy between second order and third order systems, and how we can extrapolate the observations from second order planar systems to the third dimension. Today, we will take an example from fourth order system.

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So, what we have in front of us is an equation, a dynamical equation with four components x_1 x_2 x_3 and x_4, the equation is given in front of you. So, the first thing that we would like to do is to convert this equation, this set of equations to a matrix equation and I can identify the dynamical vector as [x_1 x_2 x_3 x_4]. This would be multiplied by a matrix and you would get x_1 x_2 x_3 x_4.

So, what would be the elements of the matrix? This would be [1 1 -1 0], [0 1 0 1], [0 0 1 1], and [0 0 0 1]. So, these are the elements of the matrix. Now, I have the matrix equation of the form $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$. And then, we saw that by using the theorem that the solutions are given as

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

You can in fact solve the equations of any order.

So, this four these equations involving four variables might in general seem to be quite daunting, but it is not as difficult as it looks like. So, what do I need to do? The first thing which I need to do is to determine the eigenvalues. So, I have eigenvalues in front of me, in fact, for this particular example, there is only one eigenvalue.

And, what are the eigenvectors? Let me write down the eigenvectors. I have only two eigenvectors $\underline{v_1}$ is $[1\ 0\ 0\ 0]^T$. And, v2 under bar is $[0\ 1\ 1\ 0]^T$. Now, we are in trouble because, we saw that our solution went as i going from 1 to N, which means, we assumed that we have all the eigenvalues available with me which are linearly independent and they are equal to the number of variables which I have.

The current case that is not the case and therefore, the first thing which you need to do is you since N = 4 in this particular case, but you need to do is, you need to determine the rest of the eigenvectors. So, therefore, you will need to generalize eigenvectors. I assume that you have revised the method to determine the generalized eigenvectors.

I will write the generalized eigenvectors here, please make sure that you confirm that this is also what you get. So, I will write generalized eigenvector v1g because that is the eigenvector which I will get from $\underline{v}_1$. So, $\underline{v}_{1g}$ is $[0\ 1\ 0\ 0]^T$ and the generalized eigenvector, $\underline{v}_{2g}$ the eigenvector which I would get from the second eigenvector would be $[0\ 0\ 0\ 1]^T$. So, now I have four eigenvector, which are available with me. Let us see how do we proceed from here.

$$\sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

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In general, I would have written this solution as $x_1\ x_2\ x_3\ x_4$ as . I do have four eigenvectors, but now, I cannot use this formula simply like this anymore, instead for the case of repeated eigenvalues and when you do not have sufficient number of eigen vectors, this is what you will need to do.

So, remember that I have eigenvector $\underline{v}_1$ from $\underline{v}_1$ I got $\underline{v}_{1g}$. I had $\underline{v}_2$ from $\underline{v}_2$ I got $\underline{v}_{2g}$. So, the way I modify equation a for getting the solution under this case is I do this $x_1\ x_2\ x_3\ x_4$, this would be equal to $c_1\ e^{\lambda 1t}$. In fact in this case, we have only one λ. So, let us not worry about the index of $\lambda e^{\lambda t}$. Otherwise, if you do not have you will have to multiply it with corresponding eigenvalue.

This would be multiplied by the eigenvector one, the original eigenvector, plus c_2 the eigenvalue remains the same. So, I will write $e^{\lambda t}$. Now, the second term will be such that you would have t the independent variable multiplied by the original eigenvector, remember plus the generalized eigenvector v_{1g}.

Then going for the effect of second eigenvector and second generalized eigenvector I can write $c_3\ e^{\lambda t}\ \underline{v}_2$ under bar simply the effect of independent eigenvector which you originally had plus $c_4\ e^{\lambda t}$, t the original eigenvector linearly independent eigenvector $\underline{v}_2 + \underline{v}_{2g}$. This is the formula. Formula one for the linearly independent eigenvector, formula second given by b is when you make use of generalized eigenvectors.

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So, let us write down the solution, what would the solution be? $[x_1\ x_2\ x_3\ x_4]$ this is equal to c_1 the eigenvalue is simply 1. So, e to the power t times the first eigenvector, first original nearly independent eigenvector. So, that eigenvector was [1 0 0 0] + c_2 e to the power t multiplied by t times the first eigenvector which means [1 0 0 0] plus the first generalized eigenvector and the first generalized eigenvector that I had was [0 1 1 0].

Then plus c_3 e^t this would be equal to the second eigenvector, the second eigenvector was 0, no so, we think made a mistake in writing the generalized eigenvector, this was not the generalized eigenvector, the first generalized eigenvector was 0 1 0 0. Now, I have second eigenvector which is [0 1 1 0] + c_2 e to the power t t multiplied by the second eigenvector, this is [0 1 1 0] plus simply the second generalized eigenvector, this was [0 0 0 1].

So, I can simplify this

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = C_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_2 e^t \left(t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) + C_3 e^t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + C_4 e \left(t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

. So, this is the expression for x_1 x_2 x_3 x_4. So, what will you will do, you will solve, you will simplify this expression and write down the individual values for x_1 x_2 x_3 x_4. In fact, I have the final simplified expression for you.

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So, please do this simplification and see if you get the same answer,

$x_1 = c_1 e^t + c_2 t e^t$,

$x_2 = c_2 e^t + c_3 e^t + c_4 t e^t$

$x_1 = c_3 e^t + c_4 t e^t$

$x_1 = c_4 e^t$

So, what we saw was that this seemingly difficult set of simultaneous equations or set of equations governing the dynamics of fourth order systems. Can in fact be solved easily using matrix method.

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Now, in all of our analysis, what we did was that we always had the case from the beginning where you had the equilibrium solutions as [0 0] or [0 0 0] or in this previous case [0 0 0 0], which means that the origin was always the equilibrium solution. When we took the example of a first order system $\frac{dx}{dt} = ax$, x equilibrium was 0.

When we took the examples like this

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The equilibrium solution x_1 equilibrium x_2 equilibrium were [0 0]. We saw this to be the case for third order systems as well, which means for autonomous systems [0 0 0 0] is always an equilibrium solution. In certain cases, we found that the equilibrium solution was stable, in certain other cases we found that they were unstable.

But now, the question is, is it always true that you will have only an only [0 0 0 0] as the equilibrium solution, can you have a system where the origin is not only the origin is the solution, but you have other solutions also which are equilibrium solutions.

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So, let us take an example and see if this is the case. So, we have a dynamical third order system which is given as

$$\frac{dx_1}{dt} = x_1 + 2x_2 - 2x_3$$
$$\frac{dx_2}{dt} = 2x_1 + 5x_2 - 4x_3$$
$$\frac{dx_1}{dt} = 4x_1 + 9x_2 - 8x_3$$

. Now, to analyze this system what I will do is I will convert this to a matrix equation.

So, I have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and I need to solve for Ax = 0 to obtain an equilibrium solution. And [0 0 0] is always a solution. okay. So, [0 0 0] is always the one of the equilibrium solutions, but I want to know can I have any other solutions as well.

And the answer to that comes from this fact that the equilibrium solutions lie in the null space of A. What is the meaning of null space of A? All these vectors which would satisfy $\underline{\underline{A}}\underline{x} = 0$, then they lie in the null space. So, linear vector space whose elements are the solutions of $\underline{\underline{A}}\underline{x} = 0$ is the null space of $\underline{\underline{A}}$. So, how do I determine the null space of A? So, I write A as $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix}$

And I now do elementary row operations to row reduce the matrix A. So, the first step would be to make these elements 2 and 4 0, so, I have 1 2 minus 2. So, 2 times R2 minus 1 time R1 is 0, sorry, one time R2 minus 2 times R1, 0 5 minus 4 is 1, minus 4 minus 4 0. Similarly, R3 minus 4 times R1, 0, this becomes 1, this becomes 0.

Now, I need to make this element 0 and how do I do that, simply I keep the first row as it is, I keep the second row as it is and R3 becomes R3 minus R1, this is [0 0 0]. So, now, can I do any operation which would further row reduce my matrix A, the answer is no I cannot do any operation which would further row reduce A.

So, now, I would convert my system of equations, my matrix back to a system of equations. So, that would be $x_1 + 2x_2 - x_3 = 0$;

this is my first equation following the row reduced matrix A. And I have from second row x_2 is equal to 0 which means x_1 minus x_3 is equal to 0. I have wrote this equation wrong, so, it has to be 2, and therefore, x_1 minus $2x_3$ is equal to 0. I miss the coefficient there.

Now, I have one equation x_1 minus $2x_3$ is equal to 0. So, I have one equation and I have two variables, which means now I can arbitrarily set one of the variables and that would fix the second variable. So, I say let x_3 is equal to some arbitrary constant α. So, this will make x_1 is equal to 2 α.

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So, when I do this I have now $x_1 = 2\alpha$, x_2 was 0 and x_3 was α. Is this what I got from the previous analysis? Yes, in the previous slide this is what you got. So, how can I write now, x_1 x_2 x_3, $[x_1\ x_2\ x_3] = [2\alpha\ 0\ \alpha]$ and I can further write this as $\alpha[2\ 0\ 1]$. So, what are some comments which I can write about this entire system? What I can write is that the null space of $\underline{\underline{A}}$ has dimension, which is equal to 1.

How do I make this conclusion? The way I made this conclusion is I look into this and as I ask this question, how many vectors are present? And in fact, there is only one vector which is present, the vector v $[2\ 0\ 1]^T$. So, when you have only one vector which is present in this final expression, then you say that the dimension of the null space of A is 1. And a basis for the null space of A is the vector which is present.

So, the basis would be $[2\ 0\ 1]^T$. So, what do I understand by all of this? This simply means that if I have to solve for $\underline{\underline{A}}\underline{x} = 0$. Vector x is equal to 0 vector then I know that [0 0] for the present 3-dimensional case $[0\ 0\ 0]^T$ is always a solution, I know that. I want to know whether some other solutions also exist. So, what do I do, I determine the null space of A by the procedure which are highlighted.

Let me very quickly recall the procedure, I would write the matrix A, I would row reduce it, once I get a completely row reduced form, I would convert it back to the set of equations, I will analyze how many equations are there, I would analyze how many variables are there. So, I would do a degree of freedom analysis, which would tell me how many variables can be set arbitrarily.

And on that basis the number of variables which can be set arbitrarily would be given some constant values and the other vectors which cannot be set freely on the basis of the equations, which you finally got would be expressed in terms of the variables, which you had set at your will. So, for the current case, what you got was this.

So, once you write this in the present case, for example, alpha times the vector [2 0 1], then you say that the null space has a dimension of 1 which means that if you know one vector, non [0 0 0], then every single multiplier would be solutions of your equations, which would give $\underline{\underline{A}}\underline{x} = 0$. Let me repeat, if you know one vector, which solves $\underline{\underline{A}}\underline{x} = 0$, which is still not [0 0 0] vector, then all the multipliers of this vector would be the solutions of $\underline{\underline{A}}\underline{x} = 0$, this is what you got from here.

Now, imagine a case where you get where you do the degree of freedom analysis as 2, and imagine a case where you get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \beta \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

So, in this case, the dimensions of the null space of A would be 2 because there are two vectors. And the basis would be $[a\ b\ c]^T$ and $[c\ d\ e]^T$.

And the meaning of this is that [a b c] is going to be a solution of Ax = 0, [a b c] is going to be a solution [d e f] sorry, [d e f] is going to be a solution. And since these two make a linear vector space and [a b c] and [d e f] are the basis then every linear combination of these two vectors would also satisfy Ax = B.

So, therefore, it is not necessary that you have only one equilibrium solution which is [0 0 0] and so on, you can have infinitely many equilibrium solutions, you can have more than one equilibrium solution. And now, whether that equilibrium solution is a stable solution or an unstable solution is something which needs to be determined.

Previously, we determined whether the solutions are stable or unstable based upon having a look at the phase portrait. For example, when we drew for a first order system, the phase portrait which looked like this, this being x and t and this being x, then we said that this equilibrium solution 0 is a stable solution because all the solutions are converging here. This was qualitative by looking at the phase portrait.

In the subsequent lectures, we would take upon quantitative measures to determine how to establish whether the equilibrium solutions are stable or unstable. Thank you.