Advanced Process Dynamics

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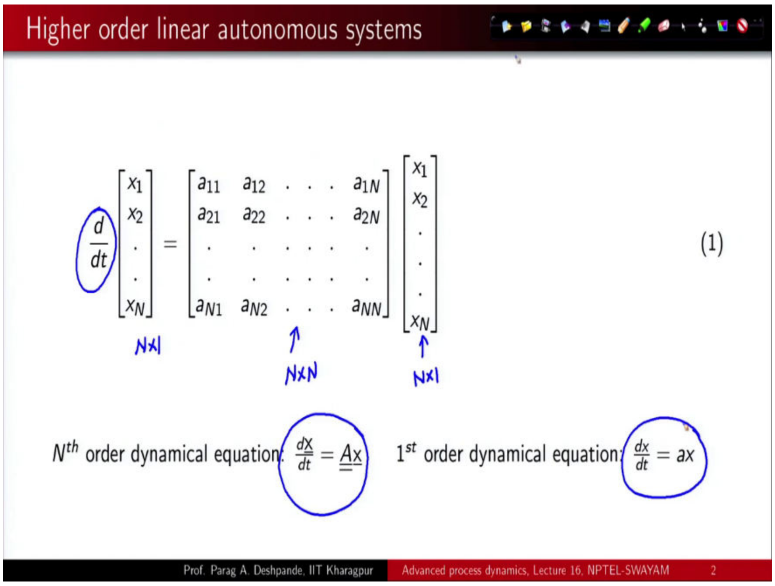
Lecture 16

Phase portraits of linear autonomous systems of order three and higher



Solution of N^{th} order linear autonomous equations

Theorem

The solutions to a linear autonomous equation of the form $\frac{dx}{dt} = \underline{\underline{A}}\underline{x}$ are given as

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

where,
λ_i's are the eigenvalues of $\underline{\underline{A}}$
$\underline{v}_i$'s are the corresponding eigenvectors
c_i's are present in the field over which the vector space of solutions is defined

Phase portraits for higher order systems

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (2)$$

$\lambda_1 = a, \lambda_2 = b, \lambda_3 = c$

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{at}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{bt}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{ct}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (1)$$

when $a, b, c \in \mathbb{R}$ | a, b

Case I : Saddle → $a > 0, b < 0$

Case II : Source → $a > 0, b > 0$

Case III : Sink → $a < 0, b < 0$

$\underline{v}_1 = [1\ 0\ 0]^T$ is a sol^n

$\underline{v}_2 = [0\ 1\ 0]^T$ is a sol^n

$\underline{v}_3 = [0\ 0\ 1]^T$ is a sol^n

[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

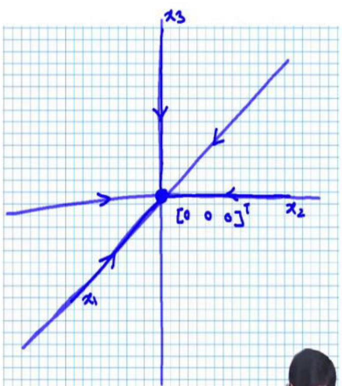
Phase portraits for higher order systems

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad (2)$$

$\lambda_1 = a, \lambda_2 = b, \lambda_3 = c$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$a < 0, b < 0, c < 0$

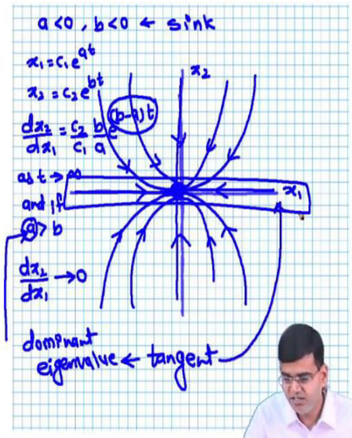
$c_1 e^{at}\underline{v}_1 + c_2 e^{bt}\underline{v}_2 + c_3 e^{ct}\ldots$

[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad (2)$$

$\lambda_1 = a,\ \lambda_2 = b,\ \lambda_3 = c$

$$\underline{v}_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \underline{v}_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\ \underline{v}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$



[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

Phase portraits for higher order systems

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad (2)$$

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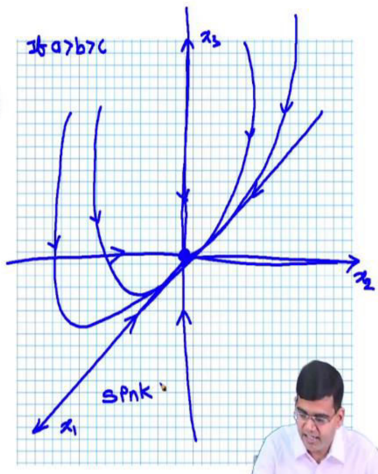


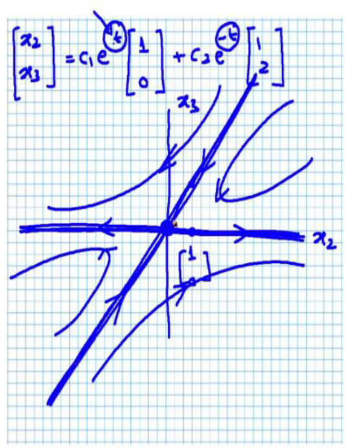
[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

Phase portraits for higher order systems

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&2&-1\\0&3&-2\\0&2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad (3)$$

$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -1$

$$\underline{v}_1=\begin{bmatrix}3\\2\\1\end{bmatrix},\ \underline{v}_2=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \underline{v}_3=\begin{bmatrix}0\\1\\2\end{bmatrix}$$



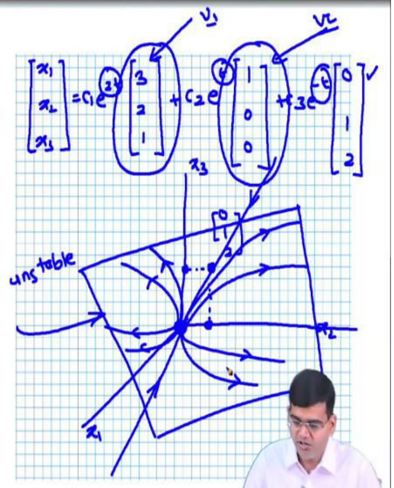
[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

Phase portraits for higher order systems

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1 & 2 & -1\\0 & 3 & -2\\0 & 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad (3)$$

$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -1$

$\underline{v}_1 = \begin{bmatrix}3\\2\\1\end{bmatrix}\ \underline{v}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}\ \underline{v}_3 = \begin{bmatrix}0\\1\\2\end{bmatrix}$



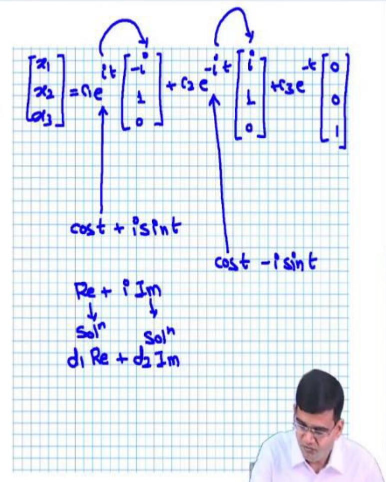
[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

Phase portraits for higher order systems

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\-1 & 0 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad (4)$$

$\lambda_1 = i,\ \lambda_2 = -i,\ \lambda_3 = -1$

$\underline{v}_1 = \begin{bmatrix}-i\\1\\0\end{bmatrix},\ \underline{v}_2 = \begin{bmatrix}i\\1\\0\end{bmatrix},\ \underline{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$



[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

Phase portraits for higher order systems

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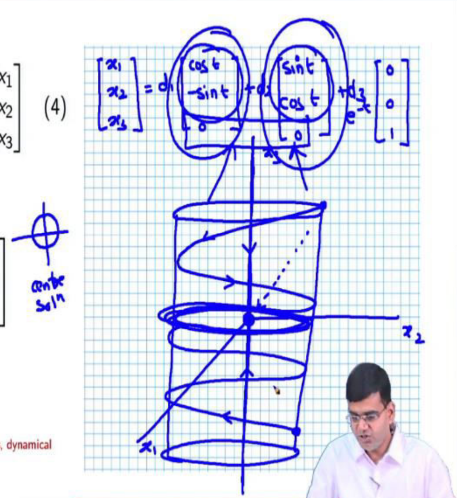

[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

Welcome back, we are studying Advanced Process Dynamics. And till the last week, we studied first and higher order systems, which were linear, we studied both autonomous as well as non-autonomous systems. So, before we move ahead, let us have a final look into some features of higher order autonomous systems.

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We previously focused on higher order systems by taking specific examples of second order systems. Second order systems are planar, which means that the first portraits can be drawn on an XY-2-dimensional plane. While they make a good case for understanding the general characteristics of higher order systems, we can actually look into higher order systems explicitly and comment more upon the general procedures which may be adopted for the analysis of higher order systems. So, in this lecture, for example, we will take some examples of third order system and we will see how third order systems can be analyzed, which are autonomous.

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So, our general autonomous system was given by a matrix equation. So, our dynamical vector was an N x 1 vector and it was applied with the time derivative. And that was equal to a matrix which was N x N. And matrix was operated on the dynamical vector again, which was N x 1. So, the N^{th} order dynamical equation was giv $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$ where $\underline{x}$ is a vector and $\underline{\underline{A}}$ is a matrix. And to compare it against the analogous first order dynamical equation, we had the first order dynamical autonomous equation as $\frac{dx}{dt} = ax$.

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We saw that for such a system, the general solution is given by this expression

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

What is important to see is that λ_i's the eigenvalues and $\underline{v}_i$ are the corresponding eigenvectors. So, this was the general solution of such an autonomous system. So, let us take some specific examples to see how we can analyze such systems.

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So, the first example that I have in front of me is given by this equation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So, the three components of my dynamical vector are x_1, x_2, x_3. And this = a diagonal matrix where the diagonal elements are a, b and c multiplied again by the vector x_1, x_2, x_3. It is not very difficult to see that the eigenvalues are simply a, b and c. Since it is a diagonal matrix, the elements on the diagonal would be the eigenvalues.

And again, since it is a diagonal matrix the corresponding eigenvectors would be simply [1 0 0], [0 1 0] and [0 0 1]. So, given that I have this equation, this matrix equation along with the eigenvalues and eigenvectors which have been given here. I can write the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 e^{a t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{bt} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{a t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In the previous analysis of planar systems, we said that when a, b and c are real in that case you had only a and b, in this case you had a, b and c and when a, b, c all three of them are real, then you have three cases. Case one, you have saddle solutions. Case two, when you have source solutions. And case three, when you have sink solutions.

So, for a planar system, when you have only a and b, you will get saddle solution when say $a > 0$, and $b < 0$, one of the eigenvalues is positive, the other eigenvalue is negative. You would get a source solution when $a > 0$, and b is also greater than 0, and you will get a sink solution when both a and b are less than 0, both of them are negative. Now, since we have a third dimension, we need to see what is the effect of c.

So, we now need to see that if I put c here, what is going to be the effect. So, before we look into the effect of a, b and c, one thing which is quite apparent from looking at the solution given by equation one here is that $v_1 = [1\ 0\ 0]^T$ is a solution, we know that this is going to be a solution. How do I know this?

Well, I will simply make $c_2 = 0$, I will make $c_3 = 0$, I will make $c_1 = 1$ and at initial condition $t = 0$, this is 1. So, therefore, $[x_1, x_2, x_3]$ would be [1 0 0]. And I can say that [1 0 0] is a solution. And similarly, I can write $\underline{v_2} = [0\ 1\ 0]^T$ is a solution. And finally, $\underline{v_3}$ which $= [0\ 0\ 1]^T$ is also a solution. So, when I make a phase portrait, I will consider these three facts. So, let us try to draw the phase portrait.

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Since I have x_1, x_2, x_3 in my case, so, the phase portrait now would be 3-dimensional. So, let me draw the axis and the three axes would be x_1, x_2, x_3. I saw that [1 0 0] is a solution. So, I can do one thing, I can make this as, draw this as a solution. This goes and this is a 3-dimensional plot. So, the line is going in the plane which you can see here.

So, this is one solution, this is another solution, which is [0 1 0]. And then you have a third solution, which is [0 0 1]. These are the three solutions we know for sure exist. Now, I do not know whether these solutions are stable or unstable. But I know for sure that this point [0 0 0] transpose is an equilibrium solution. How would you know that? Simply equate this equation to with 0 and you will get $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$.

Therefore, x_1, x_2, x_3 are, in fact, x_1, x_2, x_3 is vector which is the equilibrium solution. So, now what I want to know is the direction of arrows of time as t tends to infinity on this phase portrait. So, now, if $a < 0$, $b < 0$, and also $c < 0$, all these three components are negative, all these three eigenvalues are negative. So, what is going to happen I will have e^{at} c_1 $\underline{v_1}$ and what would happen to this when a is negative, this will tend to 0.

Similarly, plus c_2 e^{bt} $\underline{v_2}$ and this would tend to 0 when b is negative plus c_3 e^{ct} $\underline{v_3}$ and this would tend to 0 when c is negative. So, therefore, on all these three solutions, which are along [1 0 0], [0 1 0] and [0 0 1], I can draw the arrows like this. Now, I have c_1, c_2 and c_3 which are constant multipliers.

And therefore, these constant multipliers when they assume nonzero values would take me away from these straight-line solutions, which means then that now I will move away from the straight-line solutions and I would be somewhere in this 3-dimensional space. So, I need to know how would the curves look like, and for that, I do one thing I developed an analogy with my 2-dimensional system.

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So, for my 2-dimensional system, when $a < 0$, and $b < 0$, I want to develop the phase portrait, I know that the phase portrait is a sink solution. So, what I do is that I draw these two axes. And now, I want to draw the phase lines, these two lines are indeed the solutions and I know the directions for them, these are the directions. And I want to know the curves, which are away from these two lines.

So, what is going to happen, I will have to do an analysis of the solution when a < 0, b < 0 and equation is of the form $\frac{dx}{dt} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} bx$; then I know that $x_1 = c_2 e^{at}$ and $x_2 = c_2 e^{bt}$. This will be the solution, so I know from here that

$$\frac{dx_2}{dx_1} = \frac{C_2 b}{C_1 a} e^{(b-a)t}$$

I want to know the gradients of the lines which are on this plane. So, what do I understand from this as t tends to infinity, and if a > b, the case when the eigenvalue a >eigenvalue b, then what is going to happen $\frac{dx_2}{dx_1}$ will tend to 0 because this term will become negative and as t tends to infinity your gradient would tend to 0. So, I now need to draw the curves such that the gradient or the derivative tends to 0 as time t tends to infinity.

And on this curve, which point shows t tends to infinity, it is this equilibrium point. So, therefore, in close proximity of this point the gradient should be 0, and therefore, I can draw these phase lines. For the other case, you would see that the curves would be angled 90°, rotated by 90°.

And I can then draw several other curves and this is the general phase portrait which we solve for 2-dimensions. Now, for 3-dimensions, I can do this extension by observing that the dominant eigenvalue decides the direction which would act as the tangent. So, the dominant eigenvalue in the first case was a and direction corresponding to a was this line x_1. So, therefore, tangent for the curves would be along x1. So, I can do the same analysis.

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Now, for my 3-dimensional case, I have x_2, this is x_1, this is x_3, this entire line is a solution, this entire line is a solution, this entire line is a solution and the solutions tend to [0 0 0] as time t tends to infinity. And now, whichever curve I draw, which are away from this axis should be such that they should be tangential to the dominant eigenvalue.

So, if a >b >c, all of them being negative, if this is the case, then I should have the curves which are tangential to this axis, my x_1 axis and the direction of the arrows would be like this and then I can draw several of them. And whichever curve I draw, I should draw in such a manner that they should come tangentially to the x_1 axis.

So, this is a sink solution and what would happen to a source solution, well, exactly opposite of this particular case. To understand the nature of saddle solution, let us take another example.

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So, now, the example that we have in front of us is given here, where the three values which you have for the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = -1$. If imagine that I had only the later two x_2 and x_3, which means that I had the system which was confined only to this much then what would have happened? I would have written the planar solution as simply this $[x_2\ x_3]^T =$ c1 e^t, which means simply e^t times $[0\ 0]^T$, well [0 0] cannot be an eigenvector. So, this is not an ideal example.

So, let us take the 3-dimensional case, because for this particular example, we have [0 0] here and [0 0] can never be an eigenvector. But in case, instead of [0 0] imagine that you had some other quantity, for the sake of understanding we change it to say $[1\ 0]^T + c_2\ e^{-t}\ [1\ 2]^T$, then what would the phase portrait look like, the phase portrait would look like this, this would be x_2, this would be x_3.

So, I now, along x_2 direction have [1 0], so, this is [1 0]. So, this means this entire line is a solution and then I have [1 2], which means 1 and 2. So, this entire line is a solution. And now, I need to decide upon the stability. Here, you have plus 1, so, this means this axis is unstable, so I would draw curves like this, this is minus 1, so, I will draw curves like this and beyond these two, if I draw a curve like this, then you will have to draw the arrow direction of arrow like this, in this the direction of arrow would be this.

There would be a curve like this where the direction of arrow would be this, and here, you would again have a curve where the direction of arrow would be this. So, this is a 2-dimensional phase portrait and this is the equilibrium solution, not very difficult to see how we get a saddle solution.

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But now, when you introduce a third dimension, what is going to happen, I can write my $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as $c_2\ e^{2t} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + c_2\ e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3\ e^{-t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. And now, I need to develop the 3-dimensional phase portrait. So, let me first simply draw the axes this is x_1, this is x_2, this is x_3. The eigenvalue along the x_1 direction is positive + 2, along x_2 this is positive + 1, and along x_3, this is negative -1.

So, one thing which I know for sure is that x_3 is a stable solution axis. So, where is x_3? I have [0 1 2], [0 1 2]. So, if this point is [0 1 2], then this entire curve which passes through two points [0 0 0] and [0 1 2] is a solution and it is going to be a stable solution, because the eigenvalue here is negative. So, I can quite easily draw the direction of axes like this. Now, I have two solutions given by the direction of first eigenvector and the direction of second eigenvector, and along both of these the eigenvalues are positive.

So, in the first case, when I consider only the third eigenvector [0 1 2] with eigenvalue minus 1, there was no effect of first two eigenvalues, the eigenvalue +2 did not have any effect, the eigenvalue plus 1 also did not have any effect. So, I got an entire curve which was stable. Directed towards the equilibrium solution [0 0 0]. So, directed inwards.

Now, if I identify a set of points, where I do not have the influence of the third eigenvalue at all, which means that I have influence of only the eigenvalue which is equal to 2 and only the influence of eigenvalue which is equal to 1 then for that particular subspace I will have only unstability, I will not have any stable solution at all. Now, for two points, I can identify a unique straight line, for three points in 3-dimensions, I can identify a unique plane.

So, therefore, I can identify a unique plane passing through the vector $\underline{v}_1$ the first eigenvector and the second vector $\underline{v}_2$ the second eigenvector and the equilibrium solution [0 0 0] which is always there. So, passing through these three points, I would identify a unique plane. So, let us imagine that the plane looks something like this, this is the plane. So, I have a plane which passes through $\underline{v}_1$, $\underline{v}_2$ and [0 0 0].

So, what is possible now to be done is that I can say that any point on this plane will have the effect only of λ_1 and λ_2, and there will not be any effect of λ_3 which is negative, which means that this entire plane is the unstable plane, every solution which would lie here would be an unstable solution. So, if now I can find my system only to this particular plane, then I can draw curves, which all of which would pass through [0 0 0] the way I did previously, and what would be the direction of time these are unstable, so you would move away from it.

So, this is the case of a saddle solution, where you can identify a subspace corresponding to stable solutions and a subspace corresponding to unstable solutions. If you confine yourself only to the stable subspace, your system will always be stable, if you confine yourself exclusively to the unstable subspace your solutions would be unstable and anything beyond this would have a saddle characteristics, which means, along one particular direction, you would have stability or along some other direction you would have unstability.

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Now, final example that we take is given here were the eigenvalues now are i and -i which means they are imaginary, purely imaginary and the third eigenvalue is negative. So, I can quickly write the solutions as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 e^{it} \begin{bmatrix} -i \\ 0 \\ 0 \end{bmatrix} + C_2 e^{-it} \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now, I need to analyze the system. Have you done this analysis before? Yes, we did.

What was the procedure that we adopted for a planar system? I would convert e^{it} to cos t + i sin t. I would convert e^{-it} to cos t - i sin t. I would take it in, multiply and change this entire exponential with negative with imaginary index to a real part plus i times the imaginary part and what you would get is that the real part is a solution and the imaginary part is also a solution, and therefore, you can write this as d_1 times the real part where d_1 is arbitrary multiplier plus d_2 times the imaginary part.

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So, when you do this what you would get is what I have already jotted down, you should get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = d_1 \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} \sin t \\ \cos t \\ 0 \end{bmatrix} + d_3 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

It is a little tricky to analyze the first two terms and by the way d_1, d_2, d_3 are arbitrary multipliers the way we used to have c_1, c_2, c_3. So, if I want to draw this in 3-dimensions, one thing which I can definitely do is this that the third axis along x_3 is the one which is accompanied by negative eigenvalue and this is simply [0 0 1].

So, the axis itself and the eigenvalue is negative, so, therefore, I would have the arrows like this, this is the stable solution. The problem is how to analyze the first two axes, well, you will realize that these are the parametric equations of a circle, individually if you take this if you take this, they are the parametric equations of circle which means that if I had only a 2-

dimensional system then what would have happened. For a 2-dimensional system, I know that if you have purely imaginary system that then I have a center solution.

In this case also, and you see around the third dimension you have 0 and 0, which means you have planar solutions which have been probably shifted away from the plane. So, how can I draw a center solution such that I have these equations of the circle but still have a tendency to come towards [0 0 0]. So, for that imagine that I have a point which satisfies the solution then this point should keep on encircling this axis but at the same time it also has a tendency to come towards this.

So, solution would be to draw a cylinder, so you would have a solution which would be on the cylinder and the system has a tendency to come towards [0 0 0]. So, therefore, I would spiral around here and then the moment I will keep spiraling here in circles, similarly if I am here I would spiral around and the moment I reach here I will keep on going in circles.

So, if you are away from the x_1 x_2 plane where the value of x_3 is non-zero, you have a tendency to come towards [0 0], so you will keep spiraling till you reach the x1 x2 plane and the moment you reach [x_1 x_2]plane you simply have the 2-dimensional effect and you will have the center solution. So, this is what we saw the 3-dimesional system, so general features which can be extracted for the phrase portraits of a system which is of third order. We will continue this discussion for higher order systems and take up an example from a system which is of the order four. Thank you.