Advanced Process Dynamics

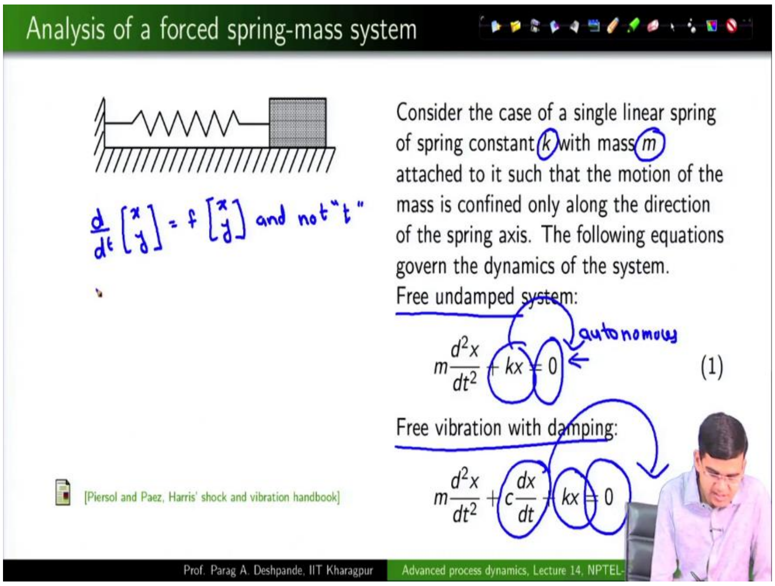
Professor Parag A. Deshpande

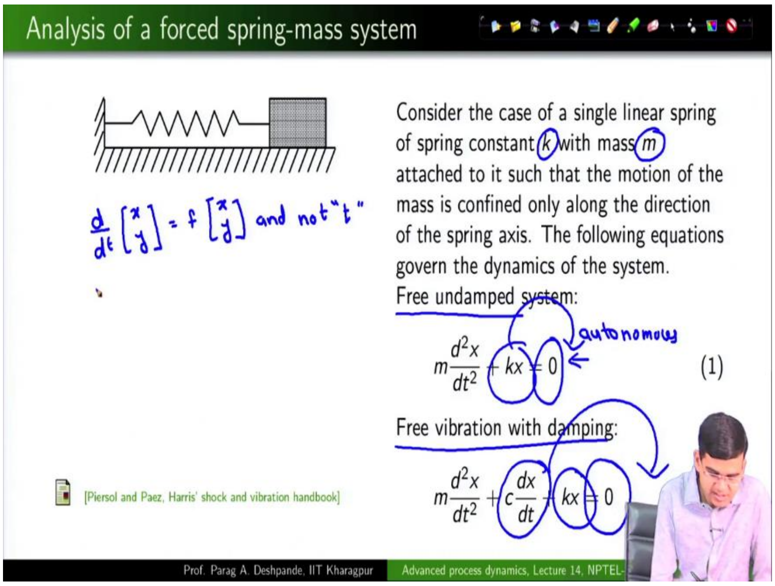
Department of Chemical Engineering

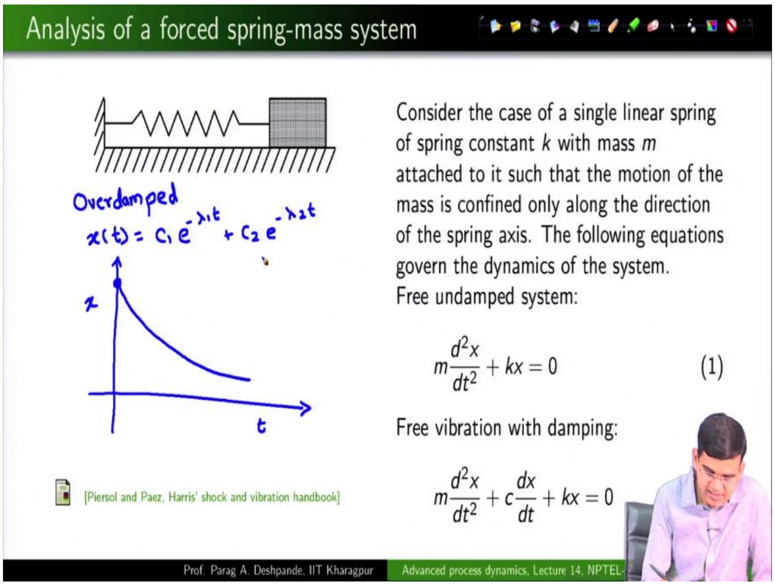
Indian Institute of Technology, Kharagpur

Lecture 14

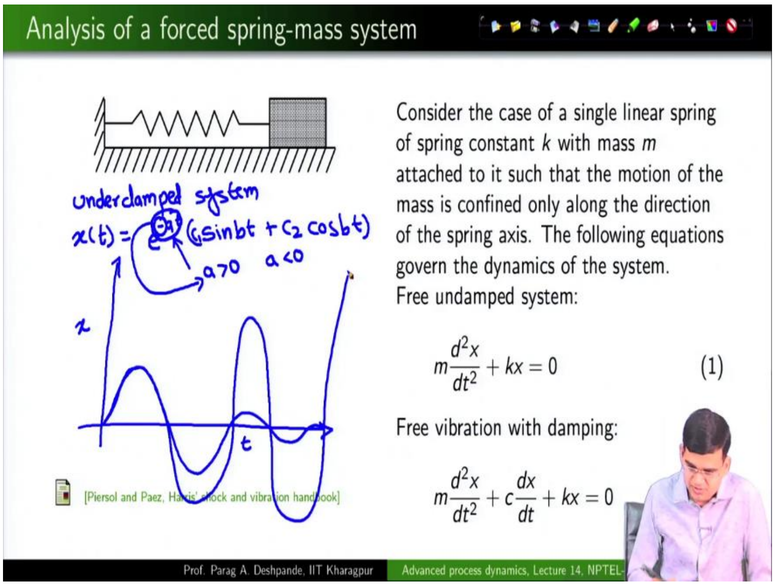
Analysis of a forced spring-mass system



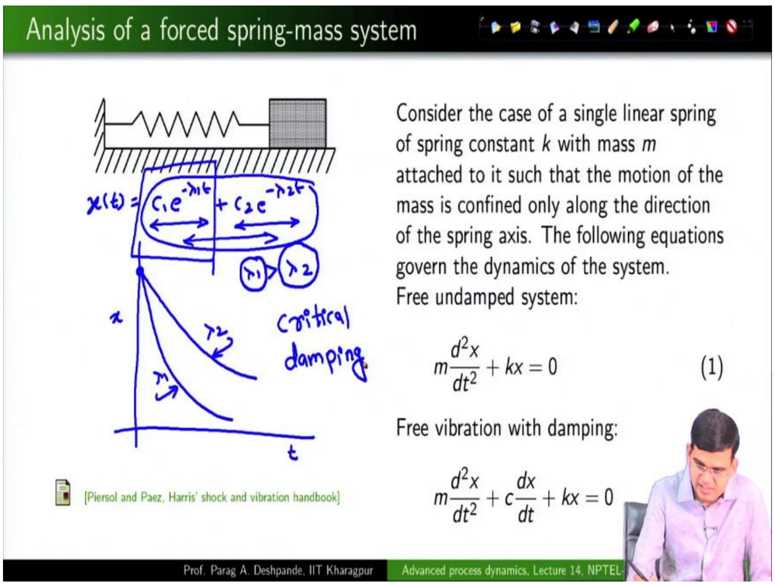




Analysis of a forced spring-mass system
Overdamped
$x(t) = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t}$
x
t
[Piersol and Paez, Harris' shock and vibration handbook]
Consider the case of a single linear spring of spring constant k with mass m attached to it such that the motion of the mass is confined only along the direction of the spring axis. The following equations govern the dynamics of the system.
Free undamped system:
$$m\frac{d^2x}{dt^2} + kx = 0 \quad (1)$$
Free vibration with damping:
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 14, NPTEL

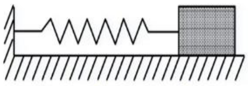


Analysis of a forced spring-mass system
Underdamped system
$x(t) = e^{at}(C_1 \sin bt + C_2 \cos bt)$
a>0 a<0
x
t
[Piersol and Paez, Harris' shock and vibration handbook]
Consider the case of a single linear spring of spring constant k with mass m attached to it such that the motion of the mass is confined only along the direction of the spring axis. The following equations govern the dynamics of the system.
Free undamped system:
$$m\frac{d^2x}{dt^2} + kx = 0 \quad (1)$$
Free vibration with damping:
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 14, NPTEL



Analysis of a forced spring-mass system
$x(t) = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t}$
$\lambda_1 > \lambda_2$
Critical damping
x
t
[Piersol and Paez, Harris' shock and vibration handbook]
Consider the case of a single linear spring of spring constant k with mass m attached to it such that the motion of the mass is confined only along the direction of the spring axis. The following equations govern the dynamics of the system.
Free undamped system:
$$m\frac{d^2x}{dt^2} + kx = 0 \quad (1)$$
Free vibration with damping:
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 14, NPTEL

Analysis of a forced spring-mass system



Consider the case of a single linear spring of spring constant k with mass m attached to it such that the motion of the mass is confined only along the direction of the spring axis. The following equations govern the dynamics of the system.

Free undamped system:

$$m\frac{d^2x}{dt^2} + kx = 0 \quad (1)$$

Free vibration with damping:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

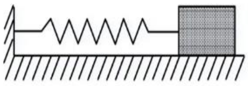
$x(t) = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}$

$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$ → λ_1, λ_2; $\lambda_1 = \lambda_2$

$x(t) = c_1 e^{-\lambda_1 t} + c_2 t e^{-\lambda t}$ ← critical damping.

[Piersol and Paez, Harris' shock and vibration handbook]

Analysis of a forced spring-mass system



non-autonomous

Consider the case of a single linear spring of spring constant k with mass m attached to it such that the motion of the mass is confined only along the direction of the spring axis. The following equations govern the dynamics of the system.

Forced vibration without damping:

$$m\frac{d^2x}{dt^2} + kx = F_0 \sin\omega t \quad (3)$$

($F_0 \sin\omega t$ = $f(t)$)

Forced vibration with damping:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \sin\omega t \quad (4)$$

($F_0 \sin\omega t$ = $f(t)$)

[Piersol and Paez, Harris' shock and vibration handbook]

Analysis of a forced spring-mass system

$m\frac{d^2x}{dt^2} + kx = 0$ ← free undamped vibration

$\frac{dx}{dt} = y$ (say)

$\frac{dy}{dt} = -\frac{k}{m}x$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad (1)$$

$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$ ~~$+ g(t)$~~

diagonalise $\underline{\underline{A}}$

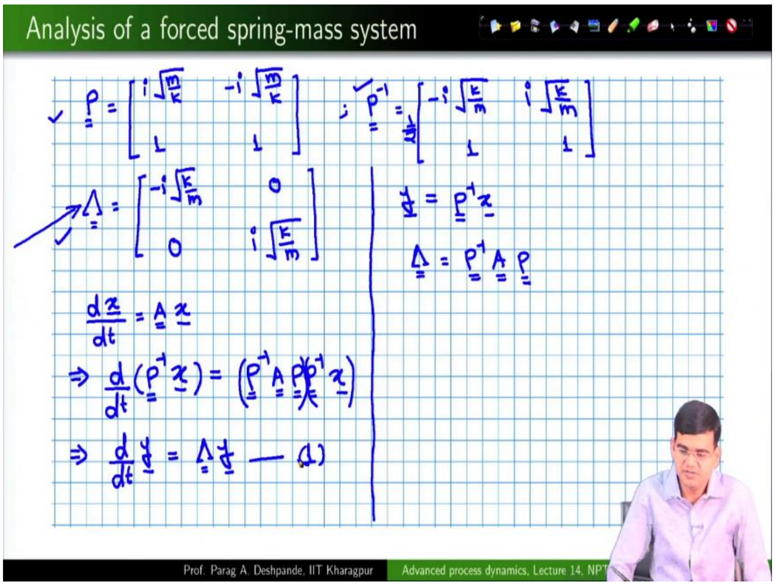
$\lambda_1 = -i\sqrt{\frac{k}{m}}$; $\underline{v}_1 = \left[i\sqrt{\frac{m}{k}} \quad 1 \right]^T$

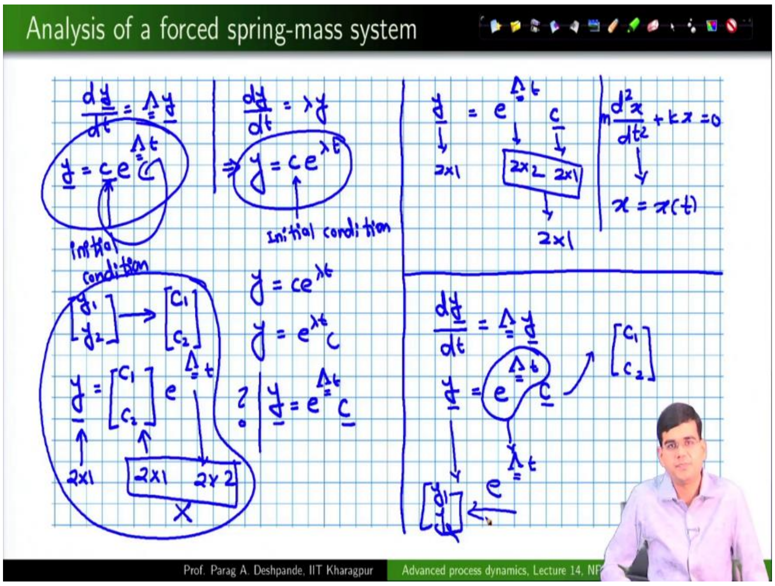
$\lambda_2 = +i\sqrt{\frac{k}{m}}$; $\underline{v}_2 = \left[-i\sqrt{\frac{m}{k}} \quad 1 \right]^T$

$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$

when $\underline{\underline{P}} = [\underline{v}_1 \mid \underline{v}_2]$

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$





So, let us continue our discussion on non-autonomous systems, we learnt about similarity transformation and how similarity transformation can help us solve non-autonomous linear problems.

(Refer Slide Time: 0:43)

We will take an example today, in fact we have come across this example in the previous week a variant of this example in fact where we have a single linear spring of, spring constant k attached to mass m and what we saw in the previous week was that this system can be modelled as an autonomous system, you can see that this equation

$$m\frac{d^2x}{dt^2} + kx = 0 \quad \ldots\ldots\ldots\ldots.. (1)$$

Equation number (1) is an autonomous system and we had two cases free undamped system and free vibration with damping.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \ldots\ldots\ldots\ldots\ldots (2)$$

The second term in equation (2) gave us the damping effect.

And both of these equations corresponded to the autonomous system because here on the right-hand side you do not have any function of t and when you transfer the other portions also to the right-hand side you will see that you can write equations of the form $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix}$, where x, y are the components of our dynamical vector, this would only be functions of x, y and not explicit functions of t.

And what we saw was that the behaviour of such autonomous systems could be very easily depicted like this that for a free undamped system you had sustained oscillations, the magnitudes remained constant so this is x, this is t and the magnitudes were or amplitudes of oscillations were constant, so this was corresponding to free undamped system, for vibration with damping we in fact had several cases.

(Refer Slide Time: 3:12)

In the first case where you had overdamped system, you had overdamped system the solutions were of the form

$$x(t) = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}$$

and a general behaviour of such a system, qualitative behaviour can be shown like this that you have x, now you have t you have x initial displacement and you go down they can be local extrema's which are possible depending upon the magnitudes of c_1, c_2 and relative magnitudes of λ_1 and λ_2. So, this is the case of overdamped system.

We had the case of underdamped system, so underdamped system where the solutions were of the form. I am not writing the exact solution but the qualitative nature of the solution the solutions are of the form

$$x(t) = e^{-at}(c_1 sinbt + c_2 cosbt)$$

This is the qualitative behaviour, qualitative in fact the dynamical solution and the behaviour is like this that you have oscillations which would reduce in magnitude and this happens when a > 0 you are already accompanied with negative sign.

So, when a > 0, this will happen and you have vibrations with damping which means that the amplitudes keep the reducing with time. Where that, you have the case where a < 0 which means that the overall coefficient becomes positive you can see an increase in amplitude with time and we also took the examples where such a case would arise.

(Refer Slide Time: 5:44)

One also can see that in case of an overdamped system the solution is given as

$$x(t) = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}$$

So, if I see the nature of individual parts the first part and the second part then, I can draw the solutions as this t, x and if I start with same initial condition then if this is λ_1 and this is λ_2, this is corresponding to λ_1 and this is corresponding to λ_2, then we know that $\lambda_1 > \lambda_2$.

So therefore, for a second order response whenever I have both of these effects all together the fastest response comes from the eigen value which has the highest magnitude, in this particular case λ_1. So, the addition of effect of the smaller eigen value always slows down your response and which is that case when you I have just one part not the other one well this is nothing but the response of a first order system, I cannot have just the first part because my system is second order. So, I need to have two parts here as the solutions so the response of a first order system would always be faster compared to the response of the second order system.

But what is the best that I can do if I know that I have a second order system and I need to have two parts for my solution that condition is called critical damping, critical damping is the fastest response that you can get in case of second order overdamp system, now what is the genesis of critical damping?

(Refer Slide Time: 7:59)

So, we need to first see how does the original solution come. So, when you have

$$x(t) = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}$$

This basically comes from the fact that you cast your equation as a matrix equation, matrix dynamical equation and this is represented as

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$

and then what you do is you determine the eigen values so the two eigen values that you get are λ_1 >and λ_2. These are the two eigen values which you get.

Now, what if these two eigen values are equal? So, $\lambda_1 = \lambda_2$, in that case what you will do is you will write the solutions as

$$x(t) = c_1 e^{-\lambda_1 t} + c_2 t e^{-\lambda_1 t}$$

So, this will be the solution and this happens when the two eigen values are equal and when the two eigen values equal, then you get this as the solution and this is the case of critical damping, this is the fastest response that you can expect from a second order overdamped system.

In all the cases λ_i's were real. When λ_i's were imaginary, then we saw that you get sustained oscillations when you have complex real, complex eigen values then you have the sustained oscillations which are on top of which you put e^{at} type function. So, when a > 0, the amplitudes of, magnitudes of oscillations keep on increasing and when a < 0, they die down.

(Refer Slide Time: 10:16)

So, in all of these cases what you saw was that the equations of our this form and they could be converted to autonomous form, in non-autonomous systems we saw that you have an external forcing function and now we can see that we have forced vibrations, so now we use the term forced vibration, again you have forced vibration without damping so all I have done is that instead of the right-hand side being zero, I have now added one function which is a function of t in this particular example this is $F_0 \sin \omega t$.

Forced vibration without damping:

$$m\frac{d^2x}{dt^2} + kx = F_0 sin\omega t \quad \ldots\ldots\ldots\ldots.. (3)$$

Forced vibration with damping:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 sin\omega t \quad \ldots\ldots\ldots\ldots.. (4)$$

Similarly, the left-hand side of equation (4) is the same as free vibration with damping and the left-hand side, the right-hand side instead of may equated to zero, you have now made it a function of t.

So, now the system is non-autonomous, the system is no more autonomous you can in fact modulate your response based on the function which you have on the right-hand side, now I would like to do a similar analysis as we did in the previous lecture. But this is going to be an incredibly large problem in terms of solving the problem by hand and I am afraid that given the time frame that we meet here for half an hour every day it might not be feasible for me to solve the entire problem right from the first to the last step.

So, to understand how to solve this problem we will take two approaches first we will take that sub part of this problem which can be solved feasibly here and then we will analyse the problem and secondly, for this entire problem what I will do is I will jot down on and highlight the steps leaving aside the trivial mathematical steps which you, I would expect you to already know for example multiplication of matrices or doing integration of functions so these are the steps which I assume you already know very well. So, we will leave those steps, so that you can make those solutions at home but how to do the solutions, what are the steps to be followed I will highlight those steps here.

(Refer Slide Time: 13:06)

So, let us first start with something which is feasible and something which is quite feasible is to solve this equation,

$$m\frac{d^2x}{dt^2} + kx = 0$$

Now, this is an autonomous equation and we were in fact solving non-autonomous systems. So, non-autonomous systems are obtained by imposing an external function, the input function or the forcing function over autonomous systems, so therefore autonomous systems basically are the limiting cases of non-autonomous systems.

So, the method which we developed for solution of non-autonomous systems are in fact applicable for solution of autonomous systems as well, so let us see if we can solve this equation

$$m\frac{d^2x}{dt^2} + kx = 0$$

which is the case of free undamped vibration using similarity transformation method, so the first thing which I would do is I would convert this equation to a matrix equation, so I will say that let dx by dt is equal to y, I will say that

$$\frac{dx}{dt} = y$$

So, I can write

$$\frac{dy}{dt} = -\frac{k}{m}x$$

from where I can write

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \ldots\ldots\ldots\ldots (1)$$

So, this is my dynamical equation and now I would like to solve this equation by the method of similarity solution, so what is the first step for determining the solution by similarity solution well we identify this system of this equation as

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$

For a non-autonomous system, you would have here say g(t) which currently you are not considering so therefore it does not matter whether you have g(t) are there or not the solution method remains the same when you are solving for non-autonomous systems you will have to consider g(t).

So, now the question is that if I have the equation of this form what do I do, I diagonalize $\underline{\underline{A}}$, how do I diagonalize $\underline{\underline{A}}$? I determine the eigen values and eigen vectors of $\underline{\underline{A}}$. So, I have determined the eigen values and eigen vectors of $\underline{\underline{A}}$ for you that in front of me. So,

$$\lambda_1 = -i\sqrt{\frac{k}{m}} \;;\; \underline{v}_1 = \begin{bmatrix} i\sqrt{m/k} & 1 \end{bmatrix}^T$$

$$\lambda_2 = +i\sqrt{\frac{k}{m}} \;;\; \underline{v}_2 = \begin{bmatrix} i\sqrt{m/k} & 1 \end{bmatrix}^T$$

So, my eigen values and eigen vectors are now in front of me and using this I will determine the matrix $\underline{\underline{P}}$, so let us remind ourselves of the matrix $\underline{\underline{P}}$, matrix so for similarity solutions we needed to have a condition that

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$$

and $\underline{\underline{A}}$ and $\underline{\underline{B}}$ had same eigen values and when $\underline{\underline{P}}$ is made from the augmentation of eigen vectors

When,

$$\underline{\underline{P}} = [\underline{v}_1 \mid \underline{v}_2]$$

Then,

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

So, these two things are in front of me.

(Refer Slide Time: 19:00)

And therefore, now what I can do is I can write $\underline{\underline{P}}$ here $\underline{\underline{P}}$. So, $\underline{\underline{P}}$ would be the matrix which is made by the augmentation of eigen vectors. So,

So, if this is $\underline{\underline{P}}$

$$\underline{\underline{P}} = \begin{bmatrix} i\sqrt{\frac{m}{k}} & -i\sqrt{\frac{m}{k}} \\ 1 & 1 \end{bmatrix}$$

My $\underline{\underline{P}}^{-1}$ can be written like this. I am writing the $\underline{\underline{P}}^{-1}$ directly. So, this is

$$\underline{\underline{P}}^{-1} = \frac{1}{2}\begin{bmatrix} -i\sqrt{\frac{k}{m}} & i\sqrt{\frac{k}{m}} \\ 1 & 1 \end{bmatrix}$$

I encourage you to go back and find out the procedure for finding the inverse, I am sure you know this, in case you have forgotten please revise.

So, if $\underline{\underline{P}}$ and $\underline{\underline{P}}^{-1}$ are in front of you then, I can write this matrix $\underline{\underline{\Lambda}}$ as the diagonal matrix which will have the eigen values along the diagonals. So,

$$\underline{\underline{\Lambda}} = \begin{bmatrix} -i\sqrt{\frac{k}{m}} & 0 \\ 0 & i\sqrt{\frac{k}{m}} \end{bmatrix}$$

So, I have $\underline{\underline{P}}$ with me, I have my diagonal matrix $\underline{\underline{\Lambda}}$ with me, so this would put me in a situation to solve for my equation.

So, what will happen next so what I can do is I can write my original equation

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$

This was my original equation, so if you remember the trick that I would do is this I would do

$$\frac{d}{dt}\left(\underline{\underline{P}}^{-1}\underline{x}\right) = \left(\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}\right)\left(\underline{\underline{P}}^{-1}\underline{x}\right)$$

So, now what I can do is this, I can write this as

$$\frac{d}{dt}\underline{y} = \underline{\underline{\Lambda}}\,\underline{y} \quad \text{—— (1)}$$

and let me remind ourselves that our definition of various quantities are like this that

$$\underline{y} = \underline{\underline{P}}^{-1}\underline{x}$$

This is what we saw in our previous lecture as well and

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{\Lambda}}$$

In fact, you have $\underline{\underline{P}}$ in front of you, you have $\underline{\underline{P}}^{-1}$ in front of you, you know $\underline{\underline{A}}$. Please make sure that you do this multiplication $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}$ and ensure that you get the same answer $\underline{\underline{\Lambda}}$ here where the eigen values lie along the diagonal elements.

(Refer Slide Time: 23:57)

So, now I need to solve this equation. So, I need to solve equation (1). So, let me solve equation (1) by doing analogous steps. So, my equation is

$\frac{d}{dt}\underline{y} = \underline{\underline{\Lambda}}\,\underline{y}$ and my analogous equation is $\frac{dy}{dt} = \lambda y$

The equation on the left-hand side involves matrices and vectors, the equation on the right-hand side does not involve such quantities.

So, I know how to solve equation on the right-hand side, the solution is given simply as

$$y = ce^{\lambda t}$$

'c' we know that it has should be y(0). But let us not worry about that. So, similarly I should write

$$\underline{y} = c\,e^{\underline{\underline{\Lambda}} t}$$

what is the problem here?

Have I written this solution correctly? I now know that there should not be a great problem determining the quantity $e^{\lambda t}$, the thing which I might miss if I am doing the solution like this is that I know that this quantity signifies the initial condition, initial conditions, condition in terms of y and then you can do a transformation to, I mean from the initial conditions in terms of x you can determine the initial condition in terms of y but this is not the, so here when you look at the equation or the solution on the left-hand side in an analogous manner this should come as an initial condition.

Now, the problem is that your $\underline{y}$ is a vector which means you will have the components say

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and you will have the initial condition corresponding to this which means that the initial condition will also be a vector right so I should have here $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ as the initial condition vector so one thing is clear that I cannot simply use c, I will have to use $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

So, now is it to write this

$$\underline{y} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\underline{\underline{A}}t}$$

Is it okay to write this? Well, you need to check the compatibility of the operations there has to be under bar, $\underline{y}$ is a 2x1 system. $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a 2x1 vector and $e^{\underline{\underline{A}}t}$ will be 2x2 which means I cannot multiply these two quantities they are not compatible to be multiplied so I need to do some adjustments.

c_1 and c_2 are simply scalars. So, like on this solution whether you write $y = ce^{\lambda t}$ or you write $y = e^{\lambda t}c$ Both are the same quantities. It does not matter here since we have matrices the way to handle this situation is to write

$$\underline{y} = e^{\underline{\underline{A}}t}\underline{c}$$

So, let us see $\underline{y}$ is 2x1 is equal to $e^{\underline{\underline{A}}t}$ which is 2x2 multiplied by the initial condition vector which is $\underline{c}$ not simply c, so I will change this to $\underline{c}$ in the first instance because c simply does not make any sense and then I will have to take it towards this side so this is 2x1 which altogether will give me 2x1 solution. So, therefore to conclude what you can, what you will have to do is write

$$\frac{d}{dt}\underline{y} = \underline{\underline{\Lambda}}\,\underline{y}$$

as the model equation and the solution as

$$\underline{y} = e^{\underline{\underline{\Lambda}}t}\,\underline{c}$$

And you should know that this c̲ itself consists of two components, c_1, c_2. Now, the only thing which is left is to determine the quantity

$$e^{\underline{\underline{\Lambda}}t}$$

and if you determine that using the method which we determined in the previous lecture then what you can do is, you can multiply that with the constant initial vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and that will give you $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ but again $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is not what you are looking for, your system of equations, your original equation was

$$m\frac{d^2x}{dt^2} + kx = 0$$

So, what you want is something in, not something in terms of various other quantities but actually x as a function of time, so what you will need to do is you will need to transform y back to x, so what we will do is we will take up the solution from this point onwards and continue this solution in the next lecture, till then good bye.