Advanced Process Dynamics

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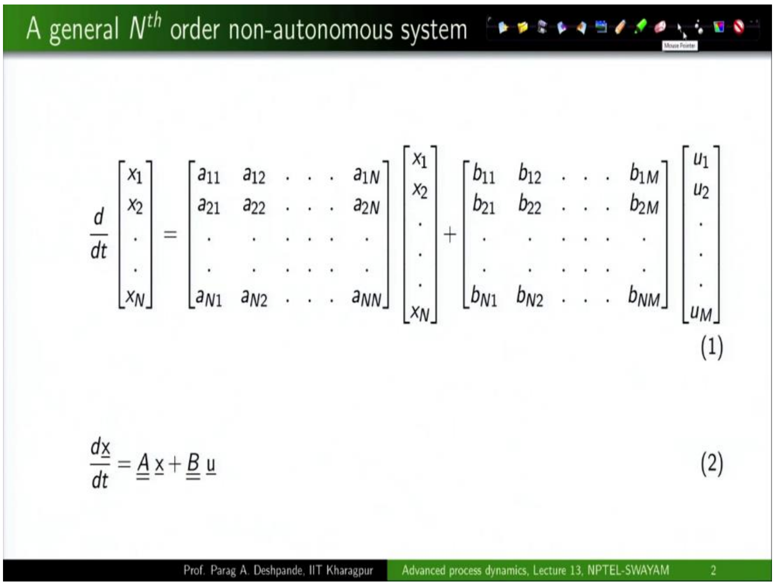
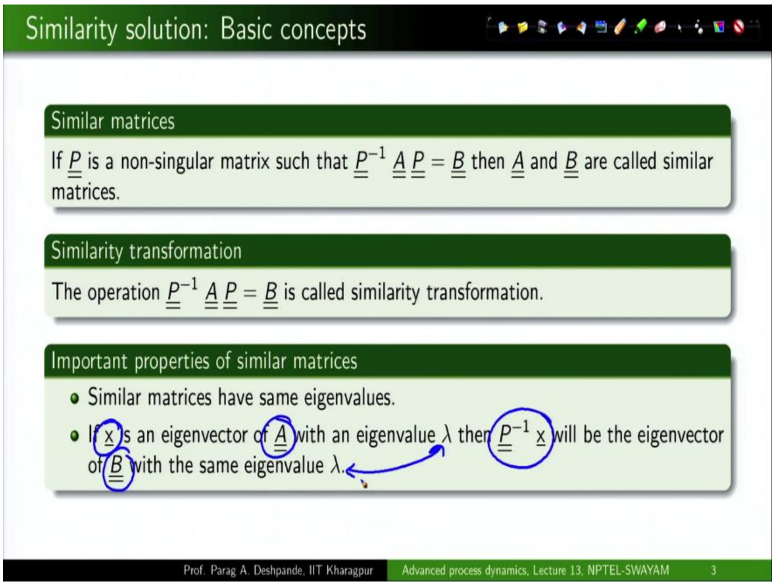
Lecture 13

Similarity solution for non-autonomous higher order dynamics continued





Similarity solution: Diagonalisation

Consider $\underline{\underline{P}}$ made from the augmentation of eigenvectors of $\underline{\underline{A}}$.

$$\begin{aligned}
\underline{\underline{A}}\,\underline{\underline{P}} &= \underline{\underline{A}}\,[\,\underline{x}_1 \mid \underline{x}_2 \mid \cdots \mid \underline{x}_N\,] \\
&= [\,\underline{\underline{A}}\,\underline{x}_1 \mid \underline{\underline{A}}\,\underline{x}_2 \mid \cdots \mid \underline{\underline{A}}\,\underline{x}_N\,] \\
&= [\,\lambda_1\,\underline{x}_1 \mid \lambda_2\,\underline{x}_2 \mid \cdots \mid \lambda_N\,\underline{x}_N\,] \\
&= \underline{\underline{P}}\,\underline{\underline{\Lambda}}
\end{aligned}$$

where,

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_N \end{bmatrix}$$

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{\Lambda}}$$

$$\underline{\underline{P}} = [\,\underline{v}_1 \mid \underline{v}_2 \mid \ldots \mid \underline{v}_N\,]$$

Similarity solution: Procedure

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} \leftarrow \underline{g}(t)$$

$\underline{x} \rightarrow N \times 1$

$(N\times N)(N\times 1) \rightarrow N\times 1$

$(N\times M)(M\times 1) \rightarrow N\times 1$

Similarity solution: Procedure

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{g}(t) \quad -(0)$$

$$\frac{d}{dt}(\underline{\underline{P}}^{-1}\underline{x}) = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{x} + \underline{\underline{P}}^{-1}\underline{g}(t)$$

$$\frac{d}{dt}(\underline{\underline{P}}^{-1}\underline{x}) = (\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}})(\underline{\underline{P}}^{-1}\underline{x}) + \underline{\underline{P}}^{-1}\underline{g}(t) \quad -(1)$$

If $\underline{\underline{P}}$ is made from augmented eigenvectors of $\underline{\underline{A}}$ then

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{\Lambda}} \quad -(2)$$

Let $\boxed{\underline{\underline{P}}^{-1}\underline{x} = \underline{y}}$ and $\boxed{\underline{\underline{P}}^{-1}\underline{g}(t) = \underline{b}(t)}$ $-(3)$

$$\frac{d\underline{y}}{dt} = \underline{\underline{\Lambda}}\,\underline{y} + \underline{b}(t) \quad -(4)$$

diagonal matrix

$$\frac{d\underline{y}}{dt} = \underline{\underline{A}}\,\underline{y} + \underline{b}(t)$$
$$\Rightarrow \frac{d\underline{y}}{dt} - \underline{\underline{A}}\,\underline{y} = \underline{b}(t)$$
$$\Rightarrow e^{-\underline{\underline{A}}t}\frac{d\underline{y}}{dt} - \underline{\underline{A}}\,e^{-\underline{\underline{A}}t}\underline{y} = e^{-\underline{\underline{A}}t}\,\underline{b}(t)$$
$$\Rightarrow \frac{d}{dt}\left(\underline{y}\,e^{-\underline{\underline{A}}t}\right) = e^{-\underline{\underline{A}}t}\,\underline{b}(t)$$
$$\Rightarrow d\left(\underline{y}\,e^{-\underline{\underline{A}}t}\right) = e^{-\underline{\underline{A}}t}\,\underline{b}(t)\,dt$$
$$\Rightarrow \underline{y}\,e^{-\underline{\underline{A}}t} = \int e^{-\underline{\underline{A}}t}\,\underline{b}(t)\,dt + \underline{c}$$
$$\Rightarrow \underline{y} = \underset{?}{\left(e^{-\underline{\underline{A}}t}\right)^{-1}} \int e^{-\underline{\underline{A}}t}\,b(t)\,dt + \left(e^{-\underline{\underline{A}}t}\right)^{-1}\underline{c}$$

$$\frac{dy}{dt} = \lambda y + b(t)$$
$$\Rightarrow \frac{dy}{dt} - \lambda y = b(t)$$
$$\Rightarrow e^{-\lambda t}\frac{dy}{dt} - \lambda e^{-\lambda t} y = e^{-\lambda t} b(t)$$
$$\Rightarrow \frac{d}{dt}\left(y e^{-\lambda t}\right) = e^{-\lambda t} b(t)$$
$$\Rightarrow d\left(y e^{-\lambda t}\right) = e^{-\lambda t} b(t)\,dt$$
$$\Rightarrow y e^{-\lambda t} = \int e^{-\lambda t} b(t)\,dt + c$$
$$\Rightarrow y = e^{\lambda t}\int e^{-\lambda t} b(t)\,dt + e^{\lambda t} c$$

Problem : $e^{-\underline{\underline{A}}t}$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \cdots$$
$$e^{-\underline{\underline{A}}t} = \underline{\underline{I}} + (-\underline{\underline{A}}t) + \frac{1}{2!}(-\underline{\underline{A}}t)^2 + \cdots$$
$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 t & 0 & \cdots & 0 \\ \vdots & & & & \\ & & \cdots & & \lambda_n t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (\lambda_1 t)^2 & 0 & \cdots & \\ 0 & (\lambda_2 t)^2 & \cdots & \\ 0 & 0 & \cdots & (\lambda_n t)^2 \end{bmatrix} + \cdots$$

$$\frac{(\underline{\underline{A}}t)^2}{2!}$$
$$\underline{\underline{A}}^2 = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & & \lambda_n \end{bmatrix}^2$$
$$(\underline{\underline{A}}t)^2 = \begin{bmatrix} (\lambda_1 t)^2 & 0 & \cdots & 0 \\ 0 & (\lambda_2 t)^2 & \cdots & \\ \vdots & & & \\ 0 & \cdots & & (\lambda_n t)^2 \end{bmatrix}$$

$$e^{-\underline{\underline{A}}t} = \begin{bmatrix} \underbrace{1 - \lambda_1 t + \frac{\lambda_1^2 t^2}{2!} \cdots}_{e^{-\lambda_1 t}} & 0 & 0 & \cdots & 0 \\ 0 & 1 - \lambda_2 t + \frac{\lambda_2^2 t^2}{2!} \cdots & 0 & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & & \cdots & 1 - \lambda_n t + \frac{\lambda_n^2 t^2}{2!} \cdots \end{bmatrix}$$

$$e^{-\underline{\underline{A}}t} = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & e^{-\lambda_n t} \end{bmatrix}$$

$$\int \underline{\underline{A}}\,dt = \begin{bmatrix} \int a\,dt & \int b\,dt & \cdots \\ \vdots & & \\ \int c\,dt & \cdots \int c\,dt & \cdots \end{bmatrix}$$

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Fine, so we continue our discussion on similarity transformations and how similarity transformations can help us solve non-autonomous equations. So, our general N^{th} order non-autonomous system was given by equation of this form

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

and then we also saw that we could have similar matrices where similar matrices were defined as the matrices which have same eigen values and the eigen vectors were related by this that if $\underline{x}$ is the eigen vector of $\underline{\underline{A}}$ then for $\underline{\underline{B}}$, $\underline{\underline{P}}^{-1}\,\underline{x}$ would be the eigen vector and the eigen values for the two cases would be the same. The matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ were called similar matrices.

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Then we also saw that if the matrices if a matrix has N linearly independent eigenvectors, then the matrix is diagonalizable so all you need to do is you need to do the operation $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}$ and this would give you a $\underline{\underline{\Lambda}}$ and for this to happen what you need to do is you need to pose $\underline{\underline{P}}$ as the one which has been formed by the augmentation of the linearly independent N eigen vectors. So, now we can in fact use this concept for solution of non-autonomous equations.

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Let us see how we can do that, so we have equation general equation which is of the form

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

Let us remind ourselves quickly of the dimensions $\underline{x}$ was of the dimension Nx1, $\underline{\underline{A}}$ was of the dimension NxN and $\underline{x}$ was of the dimension Nx1 so therefore overall, the dimension would be Nx1.

Similarly, $\underline{\underline{B}}$ was of the dimension NxM and $\underline{u}$ was of the dimension Mx1. So, overall, the dimensions were Nx1 fine, so we saw that mathematical compatibility exists and therefore we can for the sake of mathematical ease replace this, this entire operation of $\underline{\underline{B}}\ \underline{u}$ will give me a vector and let me call that vector as $\underline{g}$, I will simply call that vector $\underline{g}$ and to emphasize that g is a function of time I will write this as $\underline{g}(t)$ fine.

$$\underline{\underline{B}}\ \underline{u} = \underline{g}(t)$$

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So, let me rewrite my equation as

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{g}(t) \quad - (0)$$

This is my N^{th} order non-autonomous dynamical equation which I am trying to solve, let see how I can make use of similarity transformation to solve this problem, so let me do a trick and multiply both the sides by $\underline{\underline{P}}^{-1}$.

$$\frac{d}{dt}\left(\underline{\underline{P}}^{-1}\underline{x}\right) = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{x} + \underline{\underline{P}}^{-1}\underline{g}(t)$$

I will do one more trick here I will multiply the second term by two quantities.

$$\frac{d}{dt}\left(\underline{\underline{P}}^{-1}\underline{x}\right) = \left(\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}\right)\left(\underline{\underline{P}}^{-1}\underline{x}\right) + \underline{\underline{P}}^{-1}\underline{g}(t) \quad - (1)$$

Let us see what did I do, I inserted an identity matrix in between and I know that multiplication of identity matrix does not change the system but how did I bring that identity matrix I brought that identity matrix by multiplication of $\underline{\underline{P}}^{-1}$ with $\underline{\underline{P}}$.

So, $\underline{\underline{P}}^{-1}\underline{\underline{P}}$ is an identity matrix. This is what I did and then sorry as instead of $\underline{\underline{P}}^{-1}\underline{\underline{P}}$, I will need to do this thing $\underline{\underline{P}}\ \underline{\underline{P}}^{-1}$. So, a small correction here. I will make $\underline{\underline{P}}\ \underline{\underline{P}}^{-1}$ and then I will draw some brackets you will quickly see why did I change $\underline{\underline{P}}^{-1}\underline{\underline{P}}$ to $\underline{\underline{P}}\ \underline{\underline{P}}^{-1}$ because when I say this now, I have the equation which is in a little elegant form.

So, if $\underline{\underline{P}}$ is made from augmented eigen vectors of $\underline{\underline{A}}$ this is what we learned in the previous lecture that if $\underline{\underline{P}}$ is made from augmented eigen vectors of $\underline{\underline{A}}$, then I know that

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{\Lambda}} \text{ .... (2)}$$

Then let

$$\underline{\underline{P}}^{-1}\underline{x} = \underline{y} \quad \text{and} \quad \underline{\underline{P}}^{-1}\underline{g}(t) = \underline{b}(t) \text{ .... (3)}$$

So, now I will substitute equations (2) and (3) in equation (1) to get this

$$\frac{d\underline{y}}{dt} = \underline{\underline{\Lambda}}\,\underline{y} + \underline{b}(t) \quad - (4)$$

Let us compare it with equation number (0). So, instead of $\frac{dx}{dt}$ I have $\frac{dy}{dt}$. Instead of the matrix $\underline{\underline{A}}$, I have the matrix $\underline{\underline{\Lambda}}$ and instead of the vector $\underline{g}(t)$ which came from the input vector I have another vector $\underline{b}(t)$, I know the relationships between them so here instead of $\underline{x}$, I have $\underline{y}$ what is the relationship between $\underline{x}$ and $\underline{y}$?

Well, I have this, so if I know one of these quantities I can know the other quantity, I have $\underline{\underline{A}}$ and I have $\underline{\underline{\Lambda}}$, $\underline{\underline{A}}$ is known to me and $\underline{\underline{\Lambda}}$ is also known as a diagonal matrix and $\underline{g}$ and $\underline{b}$, what is the relationship between $\underline{g}$ and $\underline{b}$ again I know the relationship so if one is known the other one is also known.

But what is the advantage of doing these intermediate steps so as to reach equation (4) the clear difference between equation (0) and equation (4) is the appearance of this diagonal matrix which would be a key advantage during the subsequent steps, so let us see how do I solve this equation so I somehow have made up my mind that I do not want to solve equation (0) directly it is going to be a little tedious what I am going to do is I am going to convert it to equation number 4 and now I am going to solve it and I expect that there would be certain advantages.

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Let us see how to handle equation number (4). So, the equation is

$$\frac{d\underline{y}}{dt} = \underline{\underline{\Lambda}}\,\underline{y} + \underline{b}(t)$$

So, before we solve this equation let us write an analogous equation in one variable and that analogous equation would be

$$\frac{dy}{dt} = \lambda y + b(t)$$

So what have we done here, we have converted the vector $\underline{y}$ to simply the dependent variable y, we have converted the $\underline{\underline{\Lambda}}$ to simply a scalar λ and also we have converted the vector $\underline{b}$(t) as a simple function b(t).

So, how would we solve the equation on the right-hand side, we can solve using the method of integrating factor and for that let me do one rearrangement let me write it as

$$\frac{dy}{dt} - \lambda y = b(t)$$

So, if this is the equation then the integrating factor can be identified very easily as

$$IF = \mathrm{e}^{-\lambda t}$$

So, I will multiply $\mathrm{e}^{-\lambda t}$ on both the sides

$$\mathrm{e}^{-\lambda t}\frac{dy}{dt} - \lambda \mathrm{e}^{-\lambda t} y = \mathrm{e}^{-\lambda t} b(t)$$

from where I can write the left-hand side as

$$\frac{d}{dt}(\mathrm{e}^{-\lambda t} y) = \mathrm{e}^{-\lambda t} b(t)$$

So, this makes it

$$d\left(\mathrm{e}^{-\lambda t} y\right) = \mathrm{e}^{-\lambda t} b(t)\, dt$$

and then I can integrate it on both the sides for which I will get

$$y\mathrm{e}^{-\lambda t} = \int \mathrm{e}^{-\lambda t} b(t)\, dt + c$$

So, I can write the final expression for y as

$$y = \mathrm{e}^{\lambda t} \int \mathrm{e}^{-\lambda t} b(t)\, dt + \mathrm{e}^{\lambda t} c$$

So, when I have a simple first order linear ODE in one variable given by $\frac{dy}{dt} = \lambda y + b(t)$, then I can adapt a simple integration, integrating factor method to get the final expression for y. Now can I do analogous steps for the solutions for the equation given on the left-hand side. So, let us see let me write this as

$$\frac{d\underline{y}}{dt} - \underline{\underline{A}}\,\underline{y} = \underline{b}(t)$$

So, you can see a correspondence between the steps which are given on the right-hand side and the steps which are given on the left-hand side.

So, if somehow the steps which are given on the right-hand side could be followed for the system which is given on the left-hand side, then let me repeat, if somehow, I can do whatever I did for the right-hand side also to the left-hand side, then my integrating factor would be for the left-hand side.

$$e^{-\underline{\underline{A}}t}\frac{d\underline{y}}{dt} - \underline{\underline{A}}\,e^{-\underline{\underline{A}}t}\,\underline{y} = e^{-\underline{\underline{A}}t}\,\underline{b}(t)$$

In this particular step I have taken the exponential of a matrix multiplied by t that may sound to be a little strange but will come to that point a little later at this point of time let us try to focus on establishing a correspondence between the two sides and an appreciable expression so I can write this as

$$\frac{d}{dt}\left(\underline{y}\,e^{-\underline{\underline{A}}t}\right) = e^{-\underline{\underline{A}}t}\,\underline{b}(t)$$

From where I can write

$$d\left(\underline{y}\,e^{-\underline{\underline{A}}t}\right) = e^{-\underline{\underline{A}}t}\,\underline{b}(t)\,dt$$

Finally,

$$\underline{y}\,e^{-\underline{\underline{A}}t} = \int e^{-\underline{\underline{A}}t}\,\underline{b}(t)\,dt + \underline{c}$$

I wrote a constant you will need to see whether I can really write a constant here or something else what kind of a quantity would that be.

At this point of time, I would write here a constant vector why, well see on the right-hand side would come from the initial conditions but on the left-hand side you have a vector so you would have a constant initial condition vector and therefore you would have, you will have to write $\underline{c}$

and then finally what you will do is the way you took $e^{\lambda t}$ on the right-hand side, I will write here

$$\underline{y} = \left(e^{-\underline{\underline{A}}t}\right)^{-1} \int e^{-\underline{\underline{A}}t}\, b(t)\, dt + \left(e^{-\underline{\underline{A}}t}\right)^{-1} \underline{c}$$

So, now why did I write

$$\left(e^{-\underline{\underline{A}}t}\right)^{-1}$$

rather than simply, you know putting it in the denominator for that you must know what kind of quantity is this, in fact all of these operations which I have now shown for the left-hand side to give you the explicit expression for the vector $\underline{y}$ are not very difficult except that I will now need to raise the exponential with the matrix times multiplier times t and if I can get hold of this particular quantity, we can actually find out the entire solution. So, let us then look into how we can actually get the exponential of the matrix.

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So, let us see how do you get this so the problem is, problem is getting

$$\left(e^{-\underline{\underline{A}}t}\right)^{-1}$$

This is the problem so can I use, can I exponentiate a matrix this is a simple problem, so I have the definition of

$$e^{x} = 1 + x + \frac{x^2}{2!} + \cdots$$

So, if this be the case then what I can do is I can do this

$$e^{-\underline{\underline{A}}t} = \underline{\underline{I}} + (-\underline{\underline{A}}t) + \frac{1}{2!}(-\underline{\underline{A}}t)^2 + \cdots$$

Instead of 1 since you have the operations with a matrix this will become an identity, corresponding identity matrix plus x, x is simply the which you see here so this will become

$$e^{-\underline{\underline{\Delta}}t} = \underline{\underline{I}} + (-\underline{\underline{\Delta}}t) + \frac{1}{2!}(-\underline{\underline{\Delta}}t)^2 + \cdots$$

So, now the question is what would be the final functional form of this? So, this will become

$$e^{-\underline{\underline{\Delta}}t} = \underline{\underline{I}} + (-\underline{\underline{\Delta}}t) + \frac{1}{2!}(-\underline{\underline{\Delta}}t)^2 + \cdots$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 t & 0 & \cdots & 0 \\ \vdots & & & & \\ \cdots & \cdots & \cdots & \cdots & \lambda_N t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (\lambda_1 t)^2 & 0 & \cdots & \cdots \\ 0 & (\lambda_2 t)^2 & \cdots & \cdots \\ 0 & 0 & \cdots & (\lambda_N t)^2 \end{bmatrix} + \cdots$$

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So, finally what you will get?

$$e^{-\underline{\underline{\Delta}}t} = \begin{bmatrix} \overbrace{1 - \lambda_1 t + \frac{\lambda_1^2 t^2}{2!} \cdots}^{e^{-\lambda_1 t}} & 0 & 0 & \cdots & 0 \\ 0 & 1 - \lambda_2 t + \frac{\lambda_2^2 t^2}{2!} \cdots & 0 & 0 & \cdots 0 \\ \vdots & & & & \\ 0 & 0 & & \cdots & 1 - \lambda_N t + \frac{\lambda_N^2 t^2}{2!} \cdots \end{bmatrix}$$

Each term in that component of the matrix in fact is the expansion of exponential. So, therefore

$$e^{-\underline{\underline{\Delta}}t} = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & e^{-\lambda_N t} \end{bmatrix}$$

And then, what you see is that the only problem which you had in otherwise solving compatible equations and doing integral. By the way

$$\int \underline{\underline{A}}\, dt = \begin{bmatrix} \int a\, dt & \int b\, dt & \cdots \\ \vdots & & \\ \int c\, dt & \cdots \int c\, dt & \cdots \end{bmatrix}$$

So, the integration is never the problem, integration of the matrix is simply the integration of all its individual elements, the problem was the exponentiation and what we saw is that you can in fact exponentiate it using the method which we just now developed. The method was nothing but using a series expansion of the exponential function.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

and therefore, what we did was we wrote this in an analogous manner that

$$e^{-\underline{\underline{A}}t} = \underline{\underline{I}} + (-\underline{\underline{A}}t) + \frac{1}{2!}(-\underline{\underline{A}}t)^2 + \cdots$$

What came to our advantage was that by doing diagonalization the matrix multiplication and taking the power of, taking N^{th} power of a matrix became very easy and then since the individual, since the exponentiation is in fact a process of taking infinite, summation of infinite number of terms you again had to do a summation of individual components and then what did you see was that since that matrix was diagonal all the off-diagonal elements were zero. You were left with only the diagonal elements and the diagonal elements themselves were nothing but the series expansion of $e^{-\lambda t}$.

And therefore, we came up with a method so that you could raise a matrix to the power of e. The subsequent steps are not very difficult as you did in case of single variable you can do simple manipulation so in case of matrices as well the only trick was to do exponentiation, so now what we will do is we will in fact use these concepts to study one of the physical systems in the previous lecture we saw that we had in the previous week in fact we saw the dynamics of a spring where you had undamped and damped systems but they were autonomous systems. Now, how to make the systems non-autonomous in case of a spring mass system and how would the dynamics vary if you have a non-autonomous system. These are the concepts that we will take in the next two lectures, thank you.