Advanced Process Dynamics

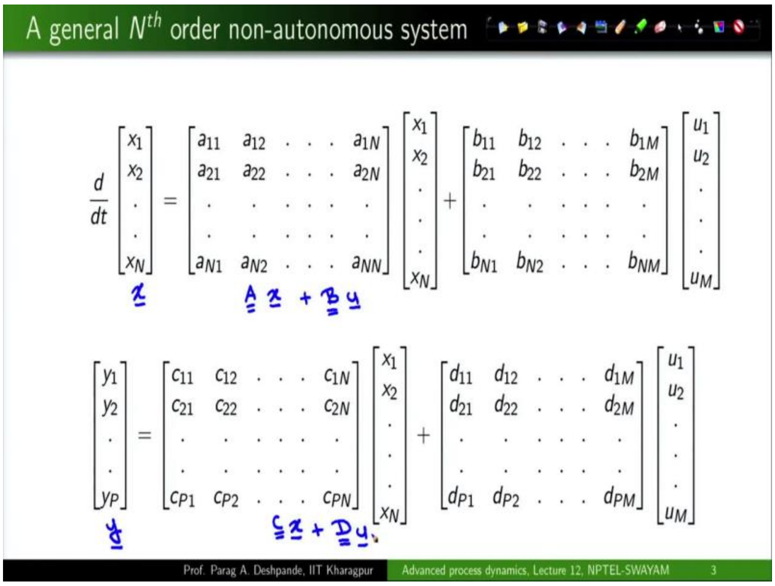
Professor Parag A. Deshpande

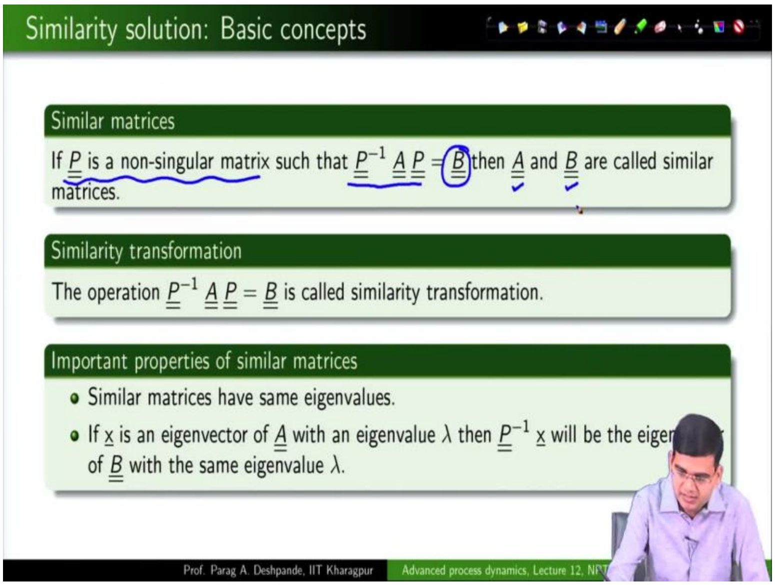
Department of Chemical Engineering

Indian Institute of Technology, Kharagpur

Lecture 12

Similarity Solution for Non-Autonomous Higher Order Dynamics





Similarity solution: Basic concepts

Similar matrices

If $\underline{\underline{P}}$ is a non-singular matrix such that $\underline{\underline{P}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$ then $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are called similar matrices.

Similarity transformation

The operation $\underline{\underline{P}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$ is called similarity transformation.

Important properties of similar matrices

- Similar matrices have same eigenvalues.
- If $\underline{x}$ is an eigenvector of $\underline{\underline{A}}$ with an eigenvalue λ then $\underline{\underline{P}}^{-1}\,\underline{x}$ will be the eigenvector of $\underline{\underline{B}}$ with the same eigenvalue λ.

Similarity solution: Basic concepts

Similar matrices

If $\underline{\underline{P}}$ is a non-singular matrix such that $\underline{\underline{P}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$ then $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are called similar matrices.

Similarity transformation

The operation $\underline{\underline{P}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$ is called similarity transformation.

Important properties of similar matrices

- Similar matrices have same eigenvalues.
- If $\underline{x}$ is an eigenvector of $\underline{\underline{A}}$ with an eigenvalue λ then $\underline{\underline{P}}^{-1}\,\underline{x}$ will be the eigenvector of $\underline{\underline{B}}$ with the same eigenvalue λ.

Similarity solution: Basic concepts

$$\underline{\underline{B}} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} \quad -(1)$$

$$\underline{\underline{B}}\,\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{-1}\underline{\underline{A}}(\underline{\underline{P}}\,\underline{\underline{P}}^{-1}) \rightarrow \underline{\underline{I}}$$

$$\underline{\underline{B}}\,\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{-1}\underline{\underline{A}}$$

$$(\underline{\underline{B}}\,\underline{\underline{P}}^{-1})\underline{x} = (\underline{\underline{P}}^{-1}\underline{\underline{A}})\underline{x}$$

$$\underline{\underline{B}}(\underline{\underline{P}}^{-1}\underline{x}) = \underline{\underline{P}}^{-1}(\underline{\underline{A}}\,\underline{x}) \quad -(2)$$

If $\underline{x}$ is an eigenvector of $\underline{\underline{A}}$ with an eigenvalue λ

$$\underline{\underline{A}}\,\underline{x} = \lambda\underline{x}$$

$$\underline{\underline{B}}(\underline{\underline{P}}^{-1}\underline{x}) = \underline{\underline{P}}^{-1}(\lambda\underline{x})$$

$$\Rightarrow \underline{\underline{B}}(\underline{\underline{P}}^{-1}\underline{x}) = \lambda(\underline{\underline{P}}^{-1}\underline{x})$$

$$\underline{\underline{P}}^{-1}\underline{x} = \underline{y}$$

$$\Rightarrow \underline{\underline{B}}\,\underline{y} = \lambda\underline{y}$$

λ is an eigenvalue of $\underline{\underline{B}}$ with the corresponding eigenvector as $\underline{\underline{P}}^{-1}\underline{x}$

Similarity solution: Diagonalisation

Consider $\underline{\underline{P}}$ made from the augmentation of eigenvectors of $\underline{\underline{A}}$.

$$\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{A}}\,[\,\underline{x}_1 \mid \underline{x}_2 \mid \cdots \mid \underline{x}_N\,]$$
$$= [\,\underline{\underline{A}}\,\underline{x}_1 \mid \underline{\underline{A}}\,\underline{x}_2 \mid \cdots \mid \underline{\underline{A}}\,\underline{x}_N\,]$$
$$= [\,\lambda_1\,\underline{x}_1 \mid \lambda_2\,\underline{x}_2 \mid \cdots \mid \lambda_N\,\underline{x}_N\,]$$
$$= \underline{\underline{P}}\,\underline{\underline{\Lambda}}$$

where,

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_N \end{bmatrix}$$

$\underline{\underline{P}}$

$\underline{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}; \quad \underline{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$

$\underline{\underline{A}} = [\,\underline{v}_1 \mid \underline{v}_2\,] = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$\underline{\underline{P}} = [\,\underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3 \cdots \mid \underline{v}_N\,]$

Similarity solution: Diagonalisation

Consider $\underline{\underline{P}}$ made from the augmentation of eigenvectors of $\underline{\underline{A}}$.

$$\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{A}}\,[\,\underline{x}_1 \mid \underline{x}_2 \mid \cdots \mid \underline{x}_N\,]$$
$$= [\,\underline{\underline{A}}\,\underline{x}_1 \mid \underline{\underline{A}}\,\underline{x}_2 \mid \cdots \mid \underline{\underline{A}}\,\underline{x}_N\,]$$
$$= [\,\lambda_1\,\underline{x}_1 \mid \lambda_2\,\underline{x}_2 \mid \cdots \mid \lambda_N\,\underline{x}_N\,]$$
$$= \underline{\underline{P}}\,\underline{\underline{\Lambda}}$$

where,

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_N \end{bmatrix}$$

$\underline{\underline{B}} = \underline{\underline{P}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{P}}$

✓ $\underline{\underline{P}} = [\,\underline{v}_1 \mid \underline{v}_2 \cdots \mid \underline{v}_N\,]$

$\underline{v}_i \rightarrow$ eigenvectors of $\underline{\underline{A}}$

$\underline{\underline{P}}^{-1} \rightarrow \underline{\underline{P}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$

$\underline{\underline{B}} = \underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_N \end{bmatrix}$

Similarity solution: Diagonalisation

Condition for diagonalisation

An $N \times N$ matrix is diagonalisable if it has N linearly independent eigenvectors.

Similarity solution: Diagonalisation

Condition for diagonalisation

An $N \times N$ matrix is diagonalisable if it has N linearly independent eigenvectors.

$$\frac{dx_1}{dt} = -2x_1 - 4x_2 + 2x_3$$

$$\frac{dx_2}{dt} = -2x_1 + x_2 + 2x_3$$

$$\frac{dx_3}{dt} = 4x_1 + 2x_2 + 5x_3$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

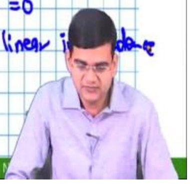
$\lambda_1 = 3$; $\underline{v}_1 = [2 \;\; 3 \;\; 1]^T$

$\lambda_2 = -5$; $\underline{v}_2 = [2 \;\; -1 \;\; 1]^T$

$\lambda_3 = 6$; $\underline{v}_3 = [1 \;\; 6 \;\; 16]^T$

The only solution to $a\underline{v}_1 + b\underline{v}_2 + c\underline{v}_3 = 0$ should be $a = b = c = 0$

– condition for linear independence



Similarity solution: Diagonalisation

Condition for diagonalisation

An $N \times N$ matrix is diagonalisable if it has N linearly independent eigenvectors.

$$a\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 6 \\ 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 2 & 1 \\ -3 & -1 & 6 \\ 1 & 1 & 16 \end{bmatrix}}_{\underline{\underline{B}}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 1 \\ -3 & -1 & 6 \\ 1 & 1 & 16 \end{bmatrix} \xrightarrow[R_3 \to 2R_3 - R_1]{R_2 \to 2R_2 + 3R_1} \begin{bmatrix} 2 & 2 & 1 \\ 0 & 4 & 15 \\ 0 & 0 & 31 \end{bmatrix} \Rightarrow a = b = c = 0$$

$2a + 2b + c = 0$ (1)

$4b + 15c = 0$ (2)

$31c = 0$ (3)

$\Rightarrow c = 0$

$\Rightarrow b = 0$

$\Rightarrow a = 0$

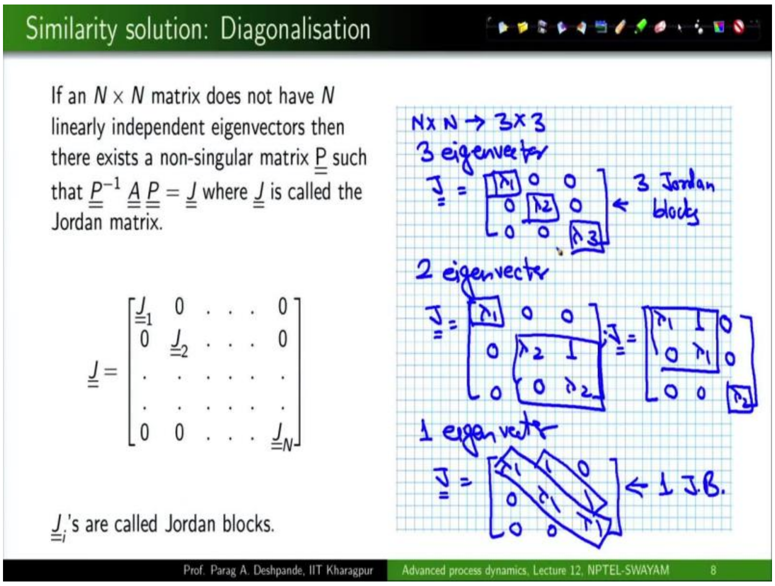
Similarity solution: Diagonalisation

If an $N \times N$ matrix does not have N linearly independent eigenvectors then there exists a non-singular matrix $\underline{\underline{P}}$ such that $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{J}}$ where $\underline{\underline{J}}$ is called the Jordan matrix.

$$\underline{\underline{J}} = \begin{bmatrix} \underline{\underline{J}}_1 & 0 & . & . & . & 0 \\ 0 & \underline{\underline{J}}_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \underline{\underline{J}}_N \end{bmatrix}$$

$\underline{\underline{J}}_i$'s are called Jordan blocks.

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \cdots & & \lambda_N \end{bmatrix}$$



(Refer Slide Time: 0:34)

So, we continue our discussion on non-autonomous systems, we will today take up similarity solution for solution of non-autonomous higher order dynamics. Before we go into the details, let us quickly look into the equations that we were dealing with our dynamical equation of general N^{th} order non-autonomous system was given by this vector, this matrix equation

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

where $\underline{x}$ is the dynamical vector and $\underline{u}$ is the input vector.

And then we saw that the system in fact can be multiple input multiple output type and therefore, the general the most generic form of the equations were given as this were

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

This was the dynamical equation and the corresponding equation for the output variables was given as this

$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$$

So, this all together will completely describe my system. Now, to solve a system which is of this form, what we need to do is resort to similarity transformation.

(Refer Slide Time: 2:19)

So, before we go into the details of similarity transformations, let us look into some basics. So, first of all we need to define what similar matrices are. So, the definition is here, similar matrices are the matrices which follow these conditions. So, if $\underline{\underline{P}}$ is a non-singular matrix, if $\underline{\underline{P}}$ is a non-singular matrix such that you invert the matrix $\underline{\underline{P}}$ and that is why the condition of non-singularity must be there, if you can invert the $\underline{\underline{P}}$ and get

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\ \underline{\underline{P}} = \underline{\underline{B}}$$

If this is possible, then the matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are called similar matrices.

Let me repeat quickly that if you have a matrix $\underline{\underline{A}}$ and you identify a non-singular matrix $\underline{\underline{P}}$ and do an operation $\underline{\underline{P}}^{-1}\underline{\underline{A}}\ \underline{\underline{P}} = \underline{\underline{B}}$, then the matrices A and B are called similar matrices. Now, what is this operation $\underline{\underline{P}}^{-1}\underline{\underline{A}}\ \underline{\underline{P}}$ called. The operation of $\underline{\underline{P}}^{-1}\underline{\underline{A}}\ \underline{\underline{P}} = \underline{\underline{B}}$ is called similarity transformation you have every matrix can be looked upon as a transformation it can be looked upon as an operator which operates on a vector to give you another vector. So, when I do this operation $\underline{\underline{P}}^{-1}\underline{\underline{A}}\ \underline{\underline{P}}$, I have the operator $\underline{\underline{A}}$, I also have another operator $\underline{\underline{B}}$ because both of them matrices. So, the procedure of obtaining a new operator $\underline{\underline{B}}$ by $\underline{\underline{P}}^{-1}\underline{\underline{A}}\ \underline{\underline{P}} = \underline{\underline{B}}$ is called similarity transformation.

When you do this similarity transformation, you get certain characteristics which are interesting and important. The first feature is that similar matrices have same eigen values, this is important similar matrices have same eigen values. So, matrix $\underline{\underline{A}}$ and matrix $\underline{\underline{B}}$ would have seen eigen values and how are the eigen vectors related, if $\underline{x}$ is an eigen vector of $\underline{\underline{A}}$,

So, if I know the eigen values of $\underline{\underline{A}}$, I know the eigen values of $\underline{\underline{B}}$ too because I know that similar matrices have same eigen values, but if I know the eigen vectors of $\underline{\underline{A}}$, what can I say about eigen vector of $\underline{\underline{B}}$. So, if $\underline{x}$ is an eigen vector of $\underline{\underline{A}}$ with an eigen value λ, then $\underline{\underline{P}}^{-1}\underline{x}$ would be the eigen vector of $\underline{\underline{B}}$ with same eigen value.

(Refer Slide Time: 5:47)

So, let us see how does this work? So, I have with me

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}} \quad \ldots\ldots\ldots\ldots(1)$$

This is a condition for similarity. Now, what I do is I do an operation where I post multiply both sides by $\underline{\underline{P}}^{-1}$. So, what is going to happen

$$\underline{\underline{B}}\,\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,(\underline{\underline{P}}\,\underline{\underline{P}}^{-1})$$

Find a post multiplied $\underline{\underline{P}}^{-1}$ on both sides now, I know

$$\underline{\underline{P}}^{-1}\,\underline{\underline{P}} = I$$

An identity matrix multiplied by any other compatible matrix will give the matrix back. So, therefore, I can write

$$\underline{\underline{B}}\,\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{-1}\underline{\underline{A}}$$

Now, $\underline{\underline{B}}$ is a matrix. $\underline{\underline{P}}^{-1}$ is a matrix. $\underline{\underline{A}}$ also is a matrix and since their multiplication is compatible, they all together will make an operator individually. So, let me operate this on the same vector which means, I can write here

$$(\underline{\underline{B}}\,\underline{\underline{P}}^{-1})\,\underline{x} = (\underline{\underline{P}}^{-1}\underline{\underline{A}})\,\underline{x}$$

Where, $\underline{x}$ is any vector. I can do some rearrangements in the bracket I can write this as

$$\underline{\underline{B}}\,(\underline{\underline{P}}^{-1}\,\underline{x}) = \underline{\underline{P}}^{-1}(\underline{\underline{A}}\,\underline{x}) \ldots\ldots\ldots\ (2)$$

So, now if $\underline{x}$ is an eigen vector of $\underline{\underline{A}}$ with an eigen value λ.

So, when $\underline{x}$ is an eigen vector with an eigen value λ, then I can write

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$$

That is the definition of eigen value and eigen vector. So, therefore, I can write $\underline{\underline{B}}$ so, I can write equation (2) as

$$\underline{\underline{B}}\,(\underline{\underline{P}}^{-1}\,\underline{x}) = \underline{\underline{P}}^{-1}(\lambda\,\underline{x})$$

and λ is simply a scalar which means, I can write

$$\underline{\underline{B}}\,(\underline{\underline{P}}^{-1}\,\underline{x}) = \lambda\,(\underline{\underline{P}}^{-1}\underline{x})$$

also on the right-hand side, but what do I see here? What I see here is that I have $\underline{\underline{P}}^{-1}\underline{x}$ on both sides and $\underline{\underline{P}}^{-1}$ is a matrix $\underline{x}$ is a vector. So, let me imagine that this is equal to a vector which can be written as $\underline{y}$.

$$\underline{\underline{P}}^{-1}\underline{x} = \underline{y}$$

So, I can write

$$\underline{\underline{B}}\,\underline{y} = \lambda\,\underline{y}$$

and this is nothing but the fact that λ is an eigen value of $\underline{\underline{B}}$ with the corresponding eigen vector as $\underline{y}$, and this is what we wanted to prove that λ is an eigen value of $\underline{\underline{B}}$, λ is an eigen value of $\underline{\underline{A}}$, so, $\underline{\underline{A}}$ and $\underline{\underline{B}}$ had same eigen values and they satisfy $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$ and when $\underline{x}$ was the eigen vector of $\underline{\underline{A}}$, then $\underline{\underline{P}}^{-1}x$ became the eigen vector of $\underline{\underline{B}}$ with that proved both of them.

(Refer Slide Time: 11:47)

Now, if this is the case, if this is the case then how what can we how can we make use of this So, what we can do is that we can make a matrix $\underline{\underline{P}}$ which comes from the augmentation of eigen vectors of $\underline{\underline{A}}$. What is the meaning of augmentation? So, if vector $\underline{v}_1$ is given by

$$\underline{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$$

and vector $\underline{v}_2$ is given by

$$\underline{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

then a vector matrix $\underline{\underline{A}}$ which is formed by augmentation of $\underline{v}_1$ and $\underline{v}_2$ would be the matrix which goes like this

$$\underline{\underline{A}} = [\,\underline{v}_1 \mid \underline{v}_2\,] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

You use the vectors as the columns of the matrix and if that is the case, then for our particular example, what we have done we have made $\underline{\underline{P}}$ by augmentation of the eigen vectors of $\underline{\underline{A}}$ which means that you will do

$$\underline{\underline{P}} = [\, \underline{v}_1 \mid \underline{v}_2 \mid \ldots\ldots\ldots \underline{v}_N]$$

for NxN matrix, NxN system.

(Refer Slide Time: 13:35)

So, when you have $\underline{\underline{P}}$ remember our usual definition of a condition for similarity was quite simply this that $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$, this was quite simply the condition for similarity and if this is the case then the eigen values of $\underline{\underline{B}}$ and eigen values of $\underline{\underline{A}}$ would be same. Now, you have a specific matrix $\underline{\underline{P}}$ which is made by augmentation of the eigen vectors. So, v_i are the eigen vectors of $\underline{\underline{A}}$ and in such a case when you do this operation, you make the matrix $\underline{\underline{P}}$, you determine its inverse and you do the operation $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$, but what would that matrix $\underline{\underline{B}}$ be, it would be a diagonal matrix it could be a matrix with a specific feature.

So, since it is a diagonal matrix, we denote it by

$$\underline{\underline{\Lambda}} = \underline{\underline{B}} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \ldots \ldots 0 \\ 0 & \lambda_2 & 0 & 0 \ldots \ldots 0 \\ 0 & 0 & \lambda_3 & 0 \ldots \ldots 0 \\ & & . & \\ & & . & \\ 0 & 0 & 0 & 0 \ldots \ldots \lambda_N \end{bmatrix}$$

So, let me repeat the condition for similarity would hold true as long as you can identify any matrix $\underline{\underline{P}}$ and do this operation $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$ and since you need $\underline{\underline{P}}^{-1}$, the matrix should be non-singular and if you obtain a matrix $\underline{\underline{B}}$, then the eigen values of $\underline{\underline{B}}$ and eigen values of $\underline{\underline{A}}$ would be same, but if you make $\underline{\underline{P}}$ from the augmentation of the eigen vectors of $\underline{\underline{A}}$ then $\underline{\underline{B}}$ would be a diagonal matrix and the eigen values would appear along the diagonals and since diagonal all have the eigen values and rest other elements are zero, we can quite simply see that the eigen values of $\underline{\underline{B}}$ which is the which is nothing but matrix $\underline{\underline{\Lambda}}$ would be same as the eigen values of $\underline{\underline{A}}$.

(Refer Slide Time: 16:43)

Now, what we do is we make use mean we observe affect, the observation is that if NxN matrix is diagonalizable if it has N linearly independent eigen vectors. So, now, the question would be that can I always do this that I take a matrix $\underline{\underline{A}}$ and by diagonalization, I mean obtaining a matrix such that the eigen values appear on the diagonals and all the off-diagonal elements are zero, is it always possible? So, the answer is that it is always possible as long as the eigen vectors are linearly independent. So, we come across this term linearly independent eigen vectors, let us see what is the meaning of linearly independent eigen vectors using one example.

So, in the previous one of the previous lectures, we took this example of a dynamical system given like this

$$\frac{dx_1}{dt} = -2x_1 - 4x_2 + 2x_3$$

$$\frac{dx_2}{dt} = -2x_1 + x_2 + 2x_3$$

$$\frac{dx_3}{dt} = 4x_1 + 2x_2 + 5x_3$$

Now this particular example is as an autonomous system. But, we are trying to understand the meaning of linear independence.

So, this example serves well, so, I can convert this system of equations to a matrix equation like this

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

I wrote the eigen values and eigen vectors as this

$$\lambda_1 = 3 \; ; \underline{v}_1 = [2 \;\; 3 \;\; 1]^T$$

$$\lambda_2 = -5; \underline{v}_2 = [2 \;\; -1 \;\; 1]^T$$

$$\lambda_3 = 6; \underline{v}_3 = [1 \;\; 6 \;\; 16]^T$$

So, for this particular example, we got these 3 eigen vector. We want to know whether these 3 eigen vectors are linearly independent or not. So, if the 3 eigen vectors are linearly independent then the only solution to

$$a\underline{v}_1 + b\,\underline{v}_2 + c\,\underline{v}_3 = 0$$

should be

$$a = b = c = o$$

This is the condition for linear independence.

So, let me write this condition for linear independence, this is the condition for linear independence which means I take a multiplication constant

$$a\underline{v}_1 + b\,\underline{v}_2 + c\,\underline{v}_3 = 0$$

Then the only solution should be a = b = c = 0. Now, let us see if that is the case in the present case.

(Refer Slide Time: 22:05)

So, let me write down the equation as

$$a\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 6 \\ 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

I need to solve for a, b, c when I need to solve for a, b, c I can do this by first converting this into a matrix equation which is of this form

$$\begin{bmatrix} 2 & 2 & 1 \\ -3 & -1 & 6 \\ 1 & 1 & 16 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, let me refer to this matrix as $\underline{\underline{B}}$ and I will do elementary row operations on $\underline{\underline{B}}$ so as to obtain an upper triangular matrix so, I will start with this

$$\begin{bmatrix} 2 & 2 & 1 \\ -3 & -1 & 6 \\ 1 & 1 & 16 \end{bmatrix}$$

The first operation that I will do is, let me keep the first row as it is. So, I will make

$$R_2 \to 2R_2 + 3R_1$$

$$R_3 \to 2R_3 - R_1$$

so, this becomes

$$\begin{bmatrix} 2 & 2 & 1 \\ 0 & 4 & 15 \\ 0 & 0 & 31 \end{bmatrix}$$

So, in fact, I have converted my matrix to a triangular matrix and then I can convert this back into set of equations as this

$$2a + 2b + c = 0 \;\ldots\ldots.\; (1)$$

$$4b + 15c = 0 \;\ldots\ldots..\; (2)$$

$$31c = 0 \;\ldots\ldots..\; (3)$$

which means

$$c = 0$$

when I substitute it in equation (2), then I get

$$b = 0$$

and when I substitute c = 0 and b = 0 and equation (1), I get

$$a = 0$$

which means

$$a = b = c = 0$$

a is equal to b is equal to c is equal to 0. This means that I have a system in which there are 3 linearly independent eigen vectors.

(Refer Slide Time: 25:44)

Now, finally, in case where you do not have linearly independent eigen vectors, what would you do in that case, you do not get a diagonal matrix. So, a diagonal matrix was of this form

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \ldots \ldots 0 \\ 0 & \lambda_2 & 0 & 0 \ldots \ldots 0 \\ 0 & 0 & \lambda_3 & 0 \ldots \ldots 0 \\ & & . & \\ & & . & \\ 0 & 0 & 0 & 0 \ldots \ldots . \lambda_N \end{bmatrix}$$

This is whatever happened when you had all the independent linearly independent eigen vectors. So, imagine that you do not have linearly independent eigen vectors in which case what you need to do you will need to determine the generalized eigen vectors.

I have encouraged you to determine to look for the procedure in case you have not yet come across it or you have forgotten the procedure to determine generalized eigen vectors when you determine generalized eigen vectors and you do this operation that if NxN matrix does not have N linearly independent eigen vectors, then there exists a non-singular matrix such that now, $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}$ will not be a diagonal matrix, it would be a Jordan matrix, it would be a Jordan matrix.

(Refer Slide Time: 27:02)

So, for the final concept for today, what we will do is we will look if NxN is in our cases 3x3 then if you have 3 eigen vectors which are linearly independent, 3 eigen vectors, then what would happen, your system is diagonalization. So, your Jordan matrix would simply become

$$\underline{\underline{J}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

and it has 3 Jordan blocks. We will see very quickly why the it has 3 Jordan blocks. When there are 2 eigen vectors which are linearly independent. So,

$$\underline{\underline{J}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix} ; \underline{\underline{J}} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

when you have 1 eigen vector only, then your Jordan block would be simply

$$\underline{\underline{J}} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

So, the super diagonal elements would become zero. So, then what is the meaning of Jordan block you would have the diagonal elements and then you have a super diagonal where elements would be 1.

So, there is there is 1 Jordan block, here you have 2 Jordan blocks, 1 Jordan block is this one, this one another Jordan block is this one in this one Jordan block is this one, the other one or the Jordan block is this one. And in the first case you have first Jordan block, you have second Jordan block and you have third Jordan block. So, the final form of your matrix would be a little different. So, we will build upon these concepts and take up an example for solution of nonlinear autonomous systems in the next lecture, followed by physical examples. Thank you.