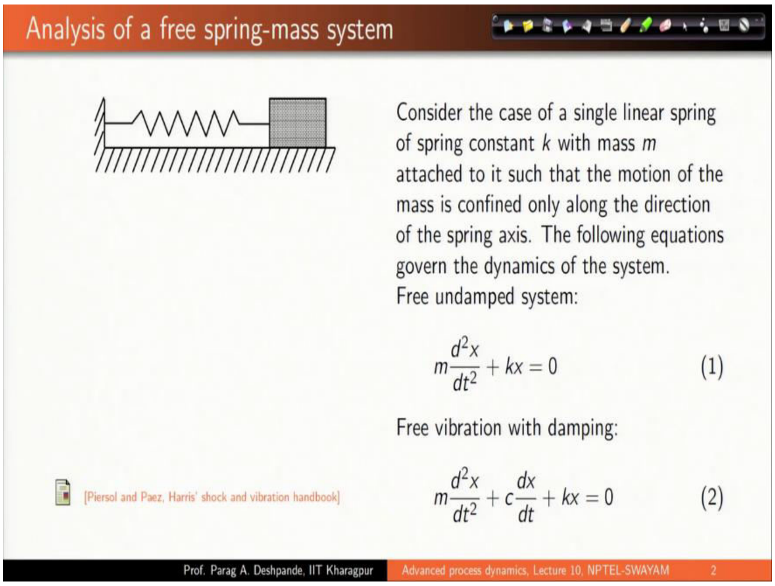
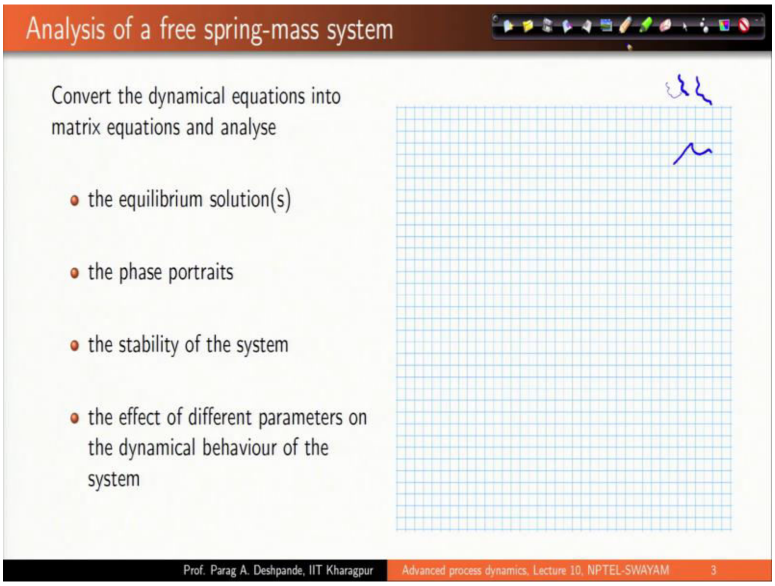
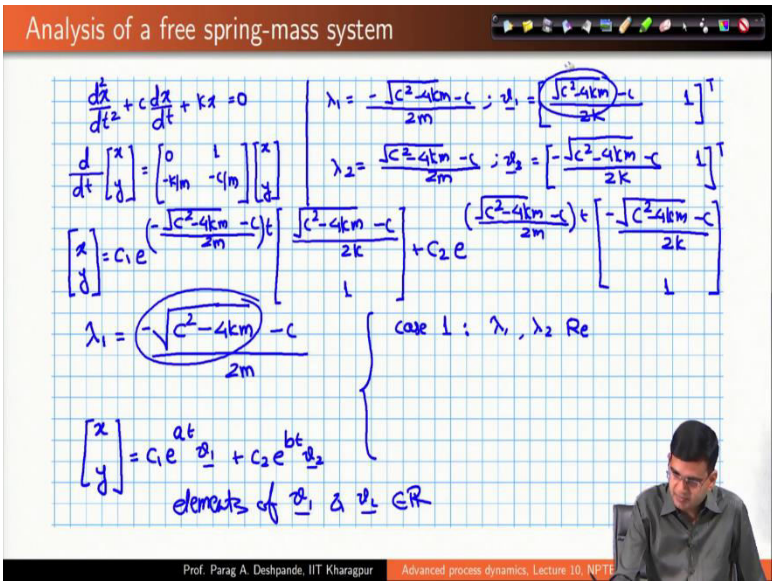
Advanced Process Dynamics
Department of Chemical Engineering
Professor Parag A. Deshpande
Indian Institute of Technology, Kharagpur
Lecture 10
Analysis of a free spring-mass system continued






Analysis of a free spring-mass system
$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$
$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
$\lambda_1 = \frac{-\sqrt{c^2-4km}-c}{2m}$; $v_1 = \left[\frac{\sqrt{c^2-4km}-c}{2k} \quad 1\right]^T$
$\lambda_2 = \frac{\sqrt{c^2-4km}-c}{2m}$; $v_2 = \left[-\frac{\sqrt{c^2-4km}-c}{2k} \quad 1\right]^T$
$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\left(\frac{-\sqrt{c^2-4km}-c}{2m}\right)t}\begin{bmatrix} \frac{\sqrt{c^2-4km}-c}{2k} \\ 1 \end{bmatrix} + c_2 e^{\left(\frac{\sqrt{c^2-4km}-c}{2m}\right)t}\begin{bmatrix} -\frac{\sqrt{c^2-4km}-c}{2k} \\ 1 \end{bmatrix}$
$\lambda_1 = \frac{-\sqrt{c^2-4km}-c}{2m}$
case 1 : λ_1, λ_2 Re
$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{at} v_1 + c_2 e^{bt} v_2$
elements of v_1 & $v_2 \in \mathbb{R}$
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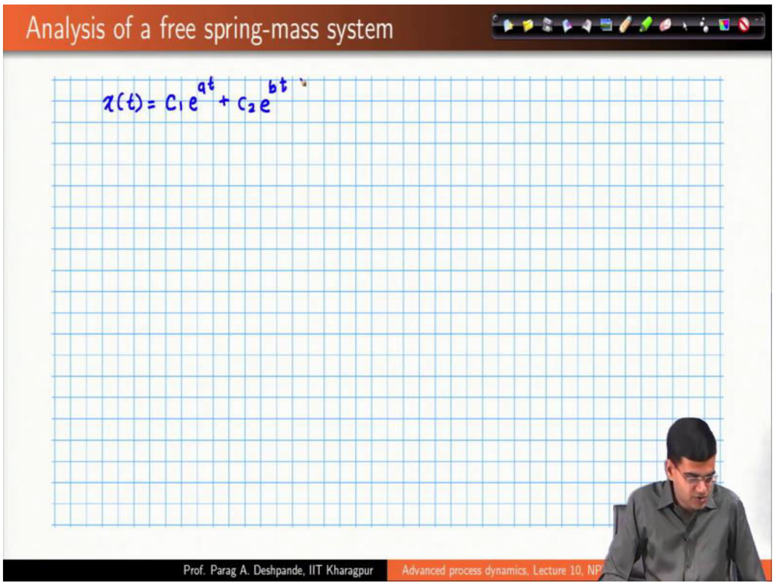
Analysis of a free spring-mass system
$x(t) = c_1 e^{at} + c_2 e^{bt}$
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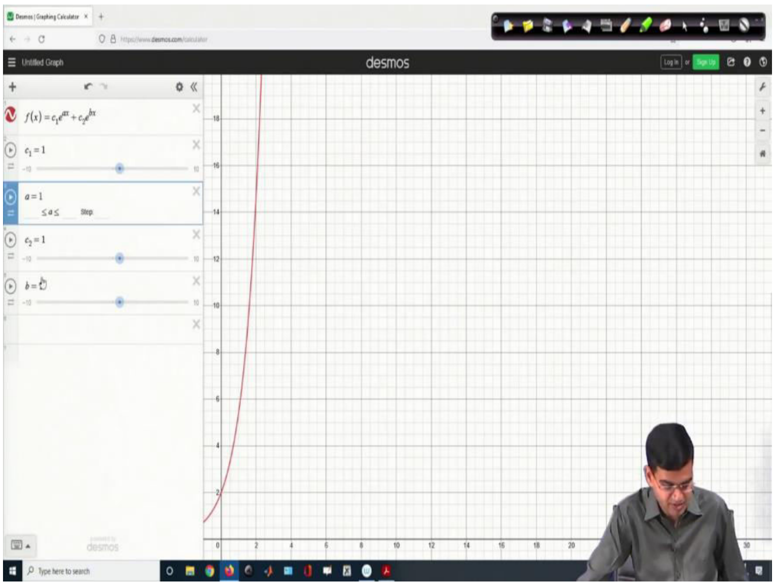
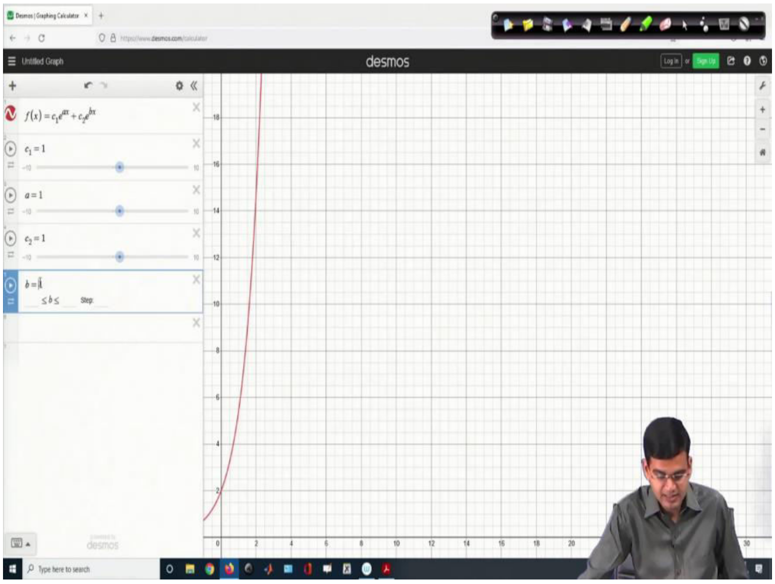
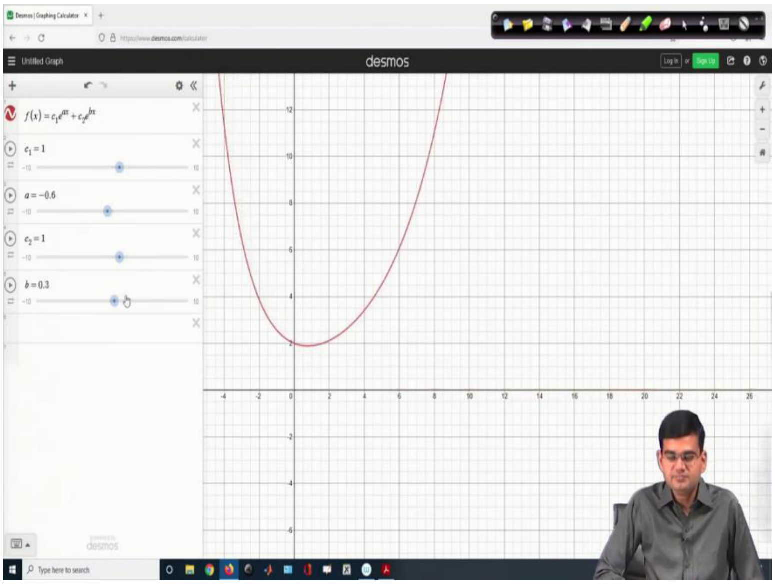
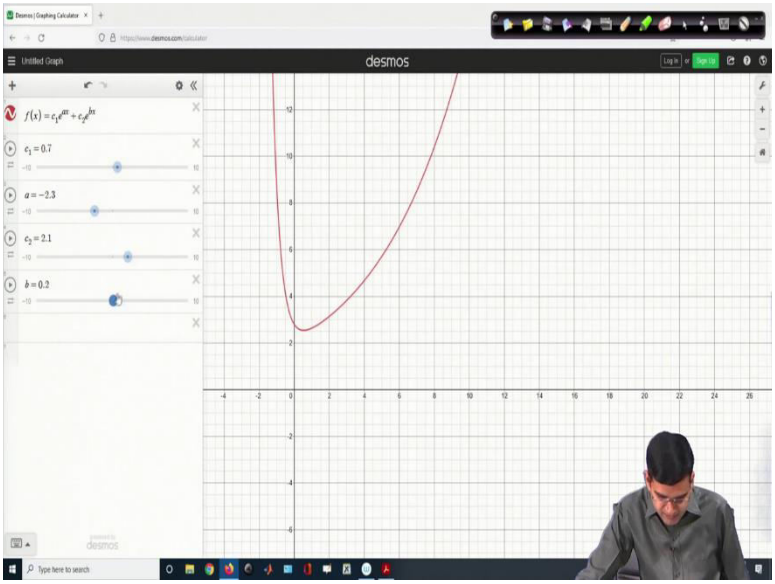
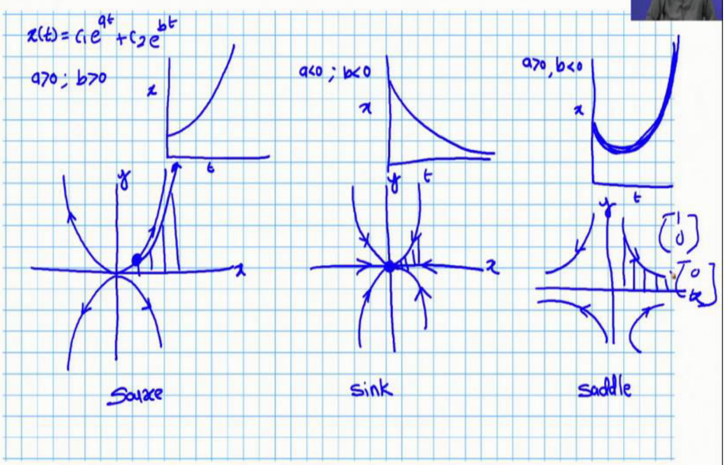

Untitled Graph
desmos
$f(x) = c_1e^{ax} + c_2e^{bx}$
$c_1 = 1$
$a = 1$
$c_2 = 1$


Untitled Graph
desmos
$f(x)=c_1e^{ax}+c_2e^{bx}$
$c_1=1$
$a=1$
$c_2=1$
$b=1$

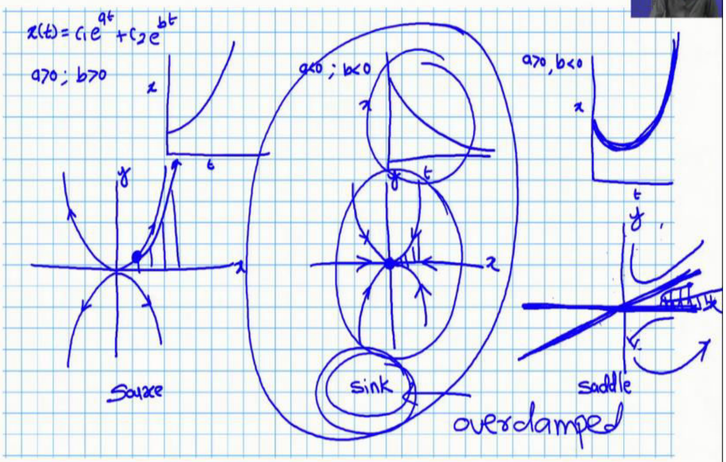

Untitled Graph
desmos
$f(x)=c_1e^{ax}+c_2e^{bx}$
$c_1=1$
$a=-0.6$
$c_2=1$
$b=0.3$


Untitled Graph
desmos
$f(x)=c_1e^{ax}+c_2e^{bx}$
$c_1=0.7$
$a=-2.3$
$c_2=2.1$
$b=0.2$

Analysis of a free spring-mass system



Analysis of a free spring-mass system



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Analysis of a free spring-mass system

$x(t) = c_1 e^{at} + c_2 e^{bt}$

eigenvalues are complex : $\lambda = \dfrac{\pm\sqrt{c^2-4km} - c}{2m} \rightarrow a \pm ib$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{(a+ib)t} \begin{bmatrix} c+id \\ 1 \end{bmatrix} + c_2 e^{(a-ib)t} \begin{bmatrix} c-id \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} \dfrac{\sqrt{c^2-4km} - c}{2k} & 1 \end{bmatrix}^T$$

$$x = c_1 e^{at} \underbrace{e^{ibt}}_{\cos bt + i \sin bt} (c+id) + c_2 e^{at} \underbrace{e^{-ibt}}_{\cos bt - i \sin bt} (c-id)$$

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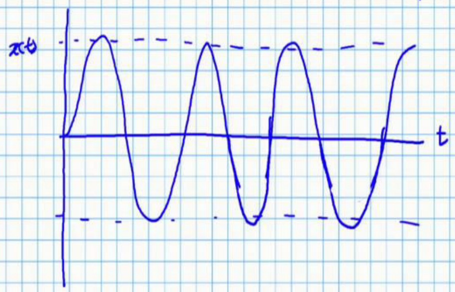
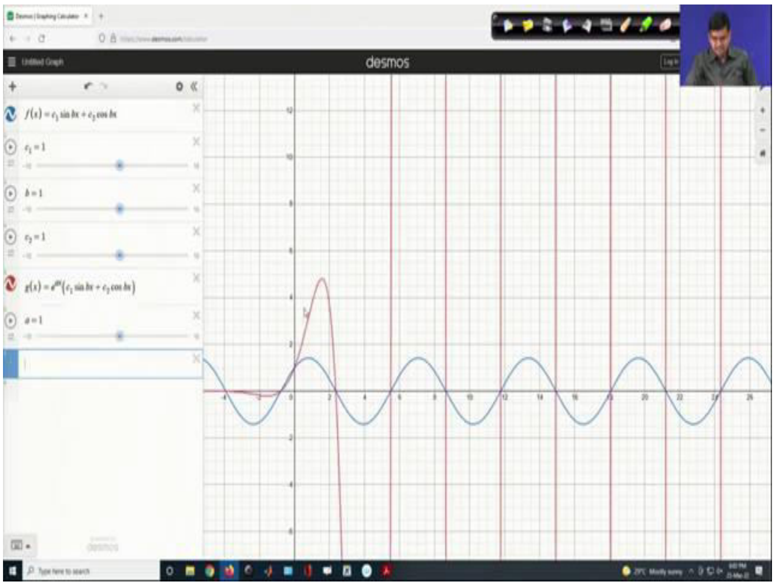
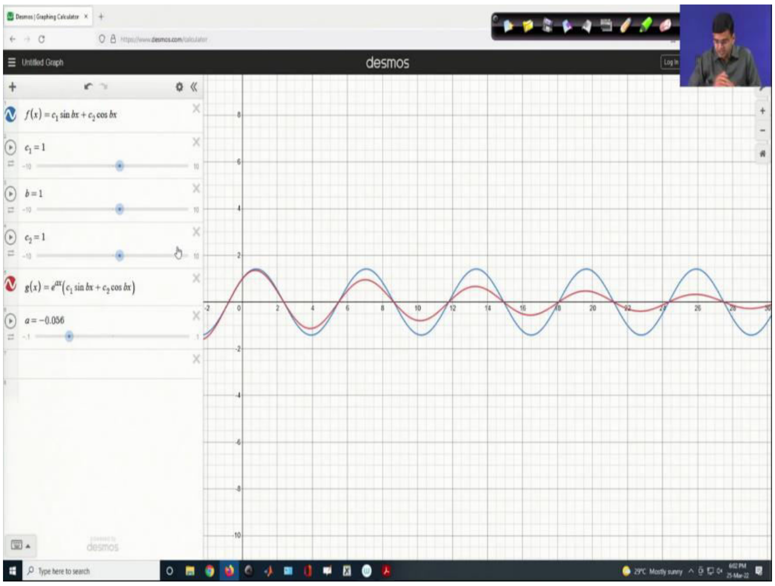
Analysis of a free spring-mass system

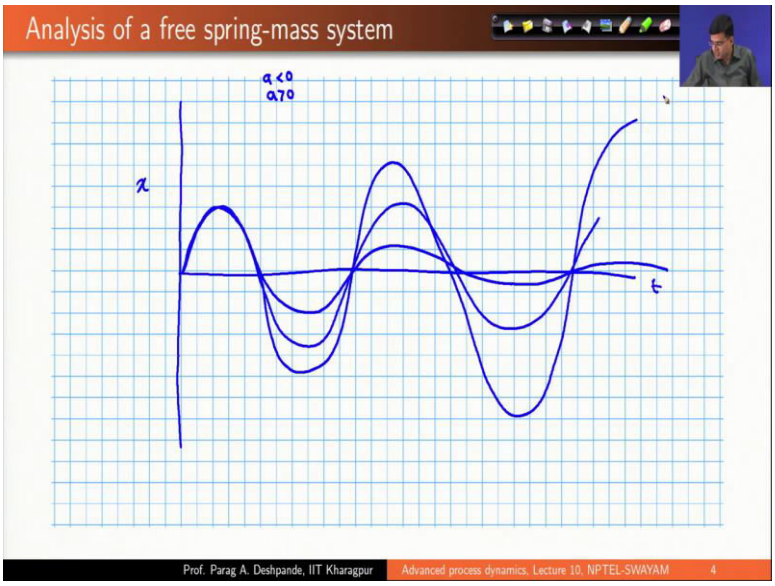
$x(t) = e^{at}(c_1 \cos bt + c_2 \sin bt) \leftarrow$ damped

$x(t) = c_1 \cos bt + c_2 \sin bt \leftarrow$ undamped









Analysis of a free spring-mass system
a<0
a>0
x
t
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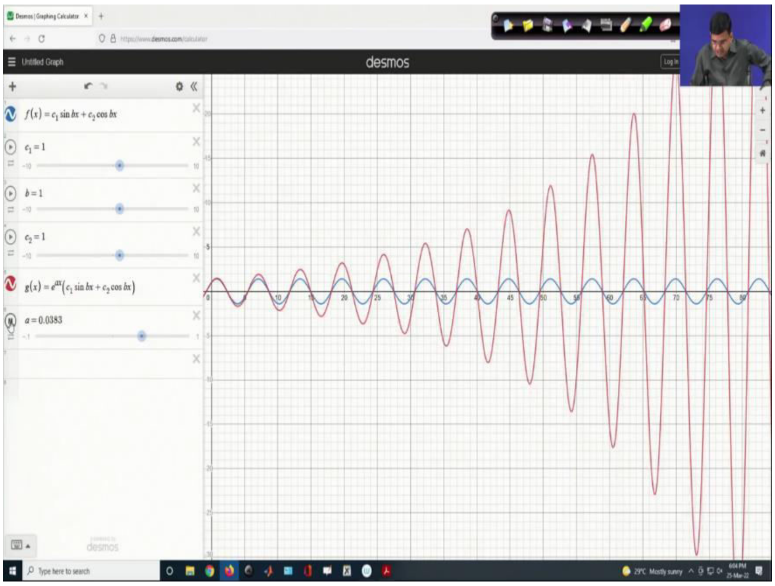
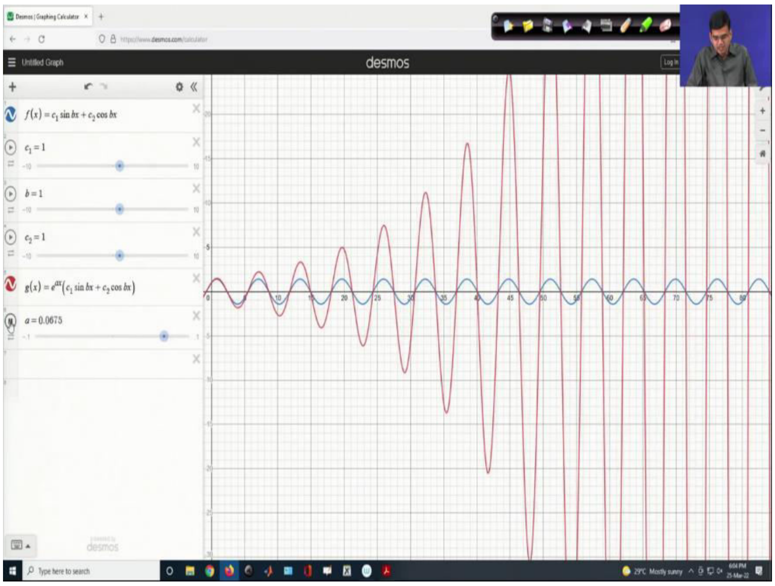
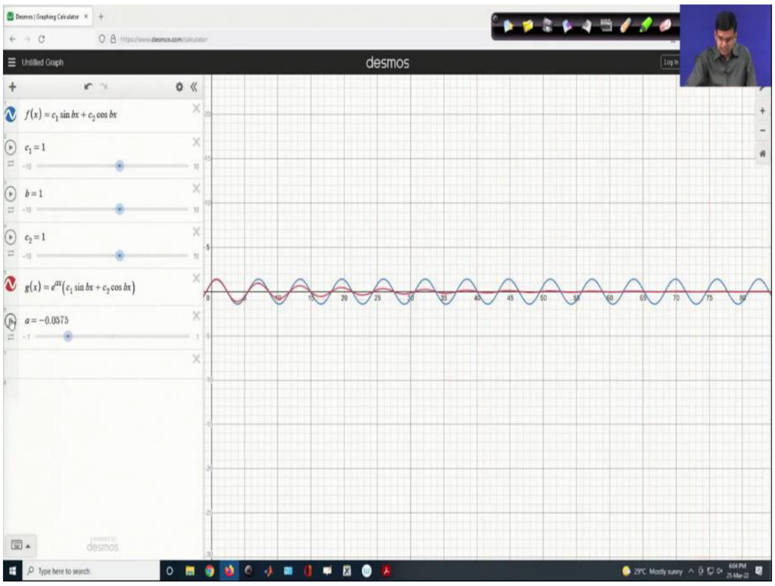
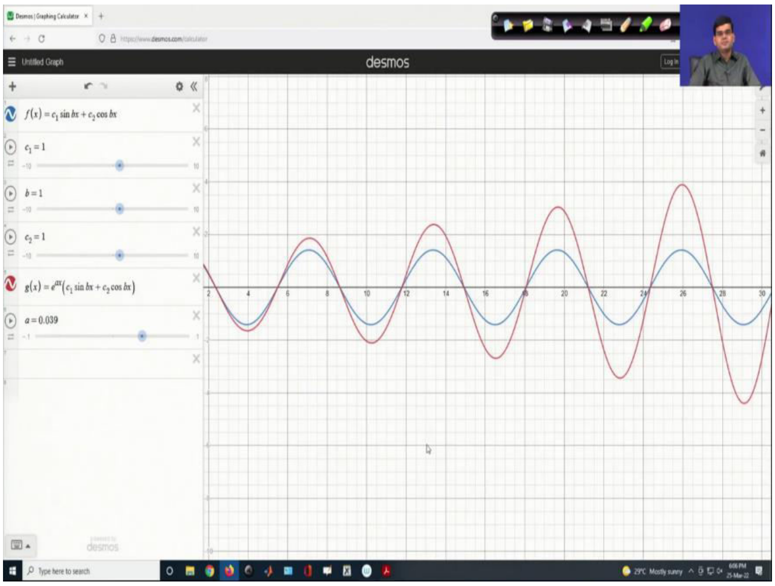


Untitled Graph
desmos
f(x) = c1 sin bx + c2 cos bx
c1 = 1
b = 1
c2 = 1
g(x) = e^{ax}(c1 sin bx + c2 cos bx)
a = 0.0383



Untitled Graph
desmos
f(x) = c1 sin bx + c2 cos bx
c1 = 1
b = 1
c2 = 1
g(x) = e^{ax}(c1 sin bx + c2 cos bx)
a = 0.0675





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So, let us continue our discussion on spring mass system, which is second order linear system. So, we took the case of free undamped system in the previous lecture, and today we would be interested in free vibration with damping. We took the example of the cases where we do see damping in the system and the purposes or the motivation behind damping.

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So, what we are trying to understand is we are trying to analyse the system in state space domain by converting the second order equation into two first order equations and then subsequently analyse the dynamics. We did develop the phase portrait of the system for undamped system, let us see what can be done for the damped system and how does it differ from the undamped system.

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So, we have the equation for our damped system as

$$m\frac{d^2x}{dt^2} + C\frac{dx}{dt} + \ kx = 0$$

and we converted it to a set of two equations and subsequently to a matrix equation and we got the matrix equation as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

So, now this matrix is different from the matrix which we get for the free system or the undamped, or the undamped system.

So, the since the matrix is different the corresponding eigen values and eigen vectors would be different. So, for this particular case, I have already determined the eigen vectors and eigen values for you. So, I have λ_1 which is equal to $\frac{-\sqrt{C^2-4km}-C}{2m}$; the corresponding eigen value is $[\frac{\sqrt{C^2-4km}-C}{2k} \ \ 1]^T$.

And the second eigenvalue is $\frac{\sqrt{C^2-4km}-C}{2m}$; eigen vector was $[\frac{-\sqrt{C^2-4km}-C}{2k} \ \ 1]^T$.. So, clearly these eigen values are different from what we got in the previous case, but, you see the only difference between this case and the previous case is that in the previous case you do not have C.

So, what you can as well do is you can simply substitute C is equal to 0 in this case and what you would realize is that you get the same eigenvalues and eigenvectors which you got in the previous lecture. So, it looks like everything is consistent. So, in that case my solution x y would take the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1\, e^{\frac{-\sqrt{C^2-4km}-C}{2m}t} \begin{bmatrix} \frac{\sqrt{C^2-4km}-C}{2k} \\ 1 \end{bmatrix} + C_2\ e^{\frac{\sqrt{C^2-4km}-C}{2m}t} \begin{bmatrix} \frac{-\sqrt{C^2-4km}-C}{2k} \\ 1 \end{bmatrix}$$

So, this is going to be the solution and from here what we can do is we can determine x explicitly, but before we do that, we see an interesting thing here right so, let us see any one of the eigen vectors, $\lambda_1 = \frac{-\sqrt{C^2-4km}-C}{2m}$, now we at this point of time do not know anything about C or k or m the magnitudes.

So, we do not know whether $C^2 - 4km$ would be greater than 0 or less than 0 or the under root of it would be a real quantity or an imaginary quantity. So, let us take both the cases. So, case 1 you have λ_1 and λ_2 which are real λ_1 and λ_2 both of them are real.

So, what is going to happen is your solution would assume the form xy is equal to C_1 e to the power some real number. Let us say that real number is at times the eigen vector. So, let me simply write it as the eigen vector to avoid wastage of any time here, $v_1 + Ce^{bt}\, \underline{v}_2$.

So, this is going to be the functional form of the solution exact values of a C1 etc we need to be determined. Now, since I have considered this as a real quantity what appears inside the eigen vector the same quantity appears inside the eigen vector so, that also is going to be a real quantity. So, therefore, the elements of v_1 and v_2 are going to be real, because $\sqrt{C^2 - 4km}$ is the real number. So, would be minus of that number.

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So, if that be the case, I can simply write the final form of x(t) as $C_1e^{at} + C_2e^{bt}$. So, my dynamical equation, my dynamical system will follow this equation and what is the nature of this system. So, let us plot it.

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So, I have $f(x) = C_1e^{ax} + C_2e^{bx}$. I have two exponential functions here. And as we saw in case of first order dynamics, if you have an exponential function with $a > 0$, your system is going to blow up. Now, instead of one exponential function, you have two exponential functions.

So, if both of them are going to be greater than 1, greater than 0, which is currently the case, a is equal to 1, b is equal to 1, then the system is going to blow up to infinity, looks consistent. Can we make both of them negative? When you make both of them negative, the system converges to 0. So, you have a stable system.

So, for the case where both of the, both a and b are positive, the system tends to infinity. For the case where *a* and *b* are both negative, the system tends to 0. Now, I have one interesting case where one of them is negative and the other one is positive. What is going to happen? So, let me make one of them positive. And the system goes to infinity. Can I modulate this?

Or can I have any way so that the system does not diverge to infinity, but I can do one thing, I can adjust the weightages here.

And does not matter what I do, does not matter what I do, I always see here, see I am changing the weightages. Here, I am changing the weightages here. And in all cases, what I see is that the system diverges to infinity. And for certain cases, so I will keep a as less than 1 and b as greater than 1. And I will make b very small, but still greater than 1, what I see is that the system now has a minima, it starts going down and then it goes up. So, what do I learn from all of these observations?

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Well, I have the equation of the form x(t), which is equal to $C_1e^{at} + C_2e^{bt}$.So, when $a > 0$, and b > 0, what is going to happen? You are going to see the system go to infinity. X(t), when $a <$ 0 and b < 0, what you are going to get is x(t), the system will converge to 0.

And the third case when say $a > 0$, and b < 0, then depending upon the weightages you may have an initial minima in your system or you may not have but ultimately as time t tends to infinity, you will always see that the system goes to infinity. Then let us ask ourselves whether we saw this in the phase portraits which we developed. Well, $a > 0$, $b > 0$, which means the eigen values are both positive, we saw that the system has source solution.

Which means, if I start at one value, the value of the variable is always going to increase, always. In sink solution, what I saw was that, the value always tends to come to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. although this is x and y, if you trace just x, so x is increasing here, x is decreasing here and this is the arrow of time, arrow gives direction of time. So, these two are consistent. What about the third one, third case is interesting. We saw so, this is the source solution, this is the sink solution.

And this is we know that for $a > 0$, $b > 0$; it should be a saddle solution. But, if I go with the conventional saddle, which we drew in the previous lecture, which should look like this then what you would find is that the value of x is monotonically decreasing, it is never increasing.

So, again this non-monotonic behaviour, which you can see here on the top is not something which you see here at the bottom. So, looks like the phase portrait and the dynamical behaviour do not match, but that is not correct. That is not correct, because, now, you have the eigen vectors which are not oriented to (1, 0) and (0, 1). See for this case, which we considered in the previous lecture the eigen vectors were oriented along (1, 0) and (0, 1) this is no more the case.

So, in the current case what is going to happen is that you will have the eigen vectors or the straight-line solutions, which would look something like this. So, let me first get rid of this one and have eigen vectors for example, would look like this.

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So, you have xy xy and if the eigen vector is this and this so, this is one eigen vector, this is another eigenvector. So, how will I draw a phase portrait or a phase line if this is the eigen vector. Well, I can draw something like this, the phase lines would be like this. So, now if I start from here the value of x will decrease it will go through minima and then it will start increasing again you can see you the value of x comes down come goes through minima and goes again.

So, again you have the saddle solution which saddle phase portrait which is in correspondence with the xt diagram which you developed here. And in case of spring mass system, what you have is that your values of m, k and c would be such that you would confirm to the central behaviour, sink behaviour.

Physically your values would be such that you start with a perturbation in your system and you come down and settle to your equilibrium value, this happens as the third case which is the sink case and if that is the case you call it as over damped system, the example of door closer or the clutch pedal of car which I gave are actually the cases where you have overdamped system. The system follows this behaviour or this phase portrait but then you may as well have a third case.

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So, let us look into this, you have a case where, so the first case was that x(t), was equal to $C_1e^{at} + C_2e^{bt}$ and we looked into different values of a and b their magnitudes and you found solutions. Now, the other case is when the eigen values are complex. So, our eigen value in this case was λ is equal to what I got was $\frac{-\sqrt{C^2-4km}-C}{2m}$.

So, the first case we said that there would be two values and these if you solve for λ_1 and λ_2, say you get *a* and *b*. Now, in second case, you get two complex values and let us see those complex values are $a \pm ib$ the two values that you get a plus or minus ib. So, now your solution would be $\begin{bmatrix} x \\ y \end{bmatrix}$ = C_1 $e^{(a + ib)t}$ times the eigen vector, the eigen vector itself would be of the form $\begin{bmatrix} (C + id) \\ 1 \end{bmatrix}$ 1 + C_2 $e^{(a - ib)t}$, $\begin{bmatrix} (C - id) \\ 1 \end{bmatrix}$ in case you have confusion of why I am writing C + id and C - id, the eigen vector which we wrote was like this $[\frac{-\sqrt{C^2-4km}-C}{2k}\ 1]^T$.

So, you can see that you in fact will have if $\sqrt{C^2 - 4km}$ is imaginary, then your eigen vectors will also have the elements which are complex numbers. And therefore, if you have the equation of this form, then you can write x as what C_1 $e^{at}e^{ibt}$ (C + id) + $C_2e^{at}e^{-ibt}$(C – id).

And then what would be your next step, your next step would be converting e^{ibt} to cos bt +i sin bt and you would convert e^{-ibt} to cos bt - i sin bt. Then you will do again the rearrangements convert the expression to the form real plus i times imaginary and the real numbers, the real part and the imaginary part themselves would be the functions of would be the solution of your equation.

So, therefore I leave this, these steps as an exercise for you and write that what is going to happen is that e^{at} will come out here and this exercise would actually be identical to what you got previously because you are going to convert into the power ibt to cosines and sines, its exactly the same method.

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So, what you would get is this, x(t) = e^{at} C_1 cos bt + C_2 sin bt. Again, I am just writing the functional form because the absolute values are not important here. What you will rearrange your entire equation is what you will do is, you will rearrange your entire equation into a form which is similar to this expression. And now, we will analyse the dynamics of the system.

So, how does the dynamics differ from the previous one? So, this is for the damped one, equation 2 and previous case was x(t) was simply C_1 cos bt + C_2 sin bt, this was undamped and we know that undamped looks like this, so this is how undamped look like I am trying to draw them. It did not happen, but I am trying to draw them with the same magnitude, same amplitude, so the amplitude would be the same.

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So, let us see if we can draw this, so my function is f(x) = C_1 sin bx + C_2 cos bx. And my second function g(x) is what? The same thing but this time multiplied by e^{ax}. Let us see the red curve looks very interesting is the damped one. So, to make the case properly, let us first make the value negative and let us keep the well amplitudes in a small range from minus 0.1 to 0.1.

So, that the systems do not become very, very blown up. So, what you see here is that the undamped system, sustained oscillation does not change the magnitude, but here does not change the amplitude. But for the for the red function, what do you see is that the amplitude goes on reducing. This means that it has a damping effect, but it has a damping effect. When does it have a damping effect? When a is negative is very important here.

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So, let us go back and draw this. So, we have x and t, sustained oscillations of same magnitude and now I have a less than 0, what is going to happen? In this case, I am going to reduce my amplitudes subsequently, such that further, as time t tends to infinity the amplitude will become 0. Now, you can expect exactly the opposite case here, when a is greater than 0, since a appears in the x in the power of the exponential, what is going to happen, the amplitudes will increase with time and so on, and the system would blow up.

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Let us see if we indeed see this. So, what I am doing now is I am making a positive and you will see here the amplitudes keep on increasing with time. So, let me quickly animate this for a less than 0 the amplitudes die out for a greater than 0 the amplitudes keep on increasing and amplitude established to infinity. So, this is an interesting behaviour and for the over damped system for an under damped system.

So, overdamped system was the previous one, this is called under damped system, where you have damping, but the damping is not complete, you do have damping the system is getting damp, but the system is called under damped for over damping, the system will asymptotically reach to the 0 value in a continuous manner, depending upon $e^{at} + C_1 e^{at} + C_2 e^{bt}$ depending on *a* and b, the system may have some number of extrema but in general, the system will assume exponentially die down.

So, you call that as over damping. This particular case is the case of under damping and when we say damping, you are assuming that a is less than 0. So, this is the case which you can see here. You have damping, but the same equation with to a greater than 0 can in fact show a divergent behaviour. And this behaviour in fact, is what you also see in various examples, two very famous examples are there, one example is the example of aerodynamic fluttering.

So, you have you have an aircraft, where the wings are pretty large and when the aircraft flies, that happens small motion of vibration in the wings now, under certain condition this fluttering can increase. So, the small amplitude of vibration of the wings may keep on increasing and that is a dangerous situation, because this would result into the mechanical failure of the wing, I would encourage you to look into fluttering of wings of aircraft, and this is the situation where the same dynamical equations applicable and your system is susceptible to failure simply because a and that condition.

So, a resulting into, resulting from the properties of aerodynamic, aeroelasticity and the properties of air and that condition will result into a greater than 0. Similarly, there is a very very famous incidence, it is called Tacoma bridge failure, Tacoma bridge is in was a suspension bridge in Washington DC. It failed I think 7-8 decades back, may be even longer back and it is was a suspension bridge over a water body and since it was a water body obviously there was wind flowing through.

So, due to small fluctuations, vibrations caused due to winds, the vibrations in the bridge was were setup and the vibrations increasingly, in increase magnitude resulted into the failure of the bridge. Both of these examples which I am giving are there on YouTube due to probable proprietary nature of the videos I am not showing them here, I encourage you to search for flutter and Tacoma bridge failure on YouTube and you will find both of this videos and this will show you that how the same equation which results, which governs the damping of the door closure or the damping of car pedals will they do result in mechanical failure of the aircraft wings as well as the failure of huge suspension bridge.

So, this is what we have to learn about the autonomous systems, now from next week onwards, we will take a little more complex topic of non-autonomous system, thank you.