Mathematical Modelling and
Simulation of Chemical Engineering Process
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Lecture 9
PDE – separation of variables (Contd.)

Hello and welcome to this class on Partial Differential Equations here we are trying to continue on the separation of variable techniques and particularly we are going to learn today about you know parabolic PDEs, related to cylindrical coordinate systems.

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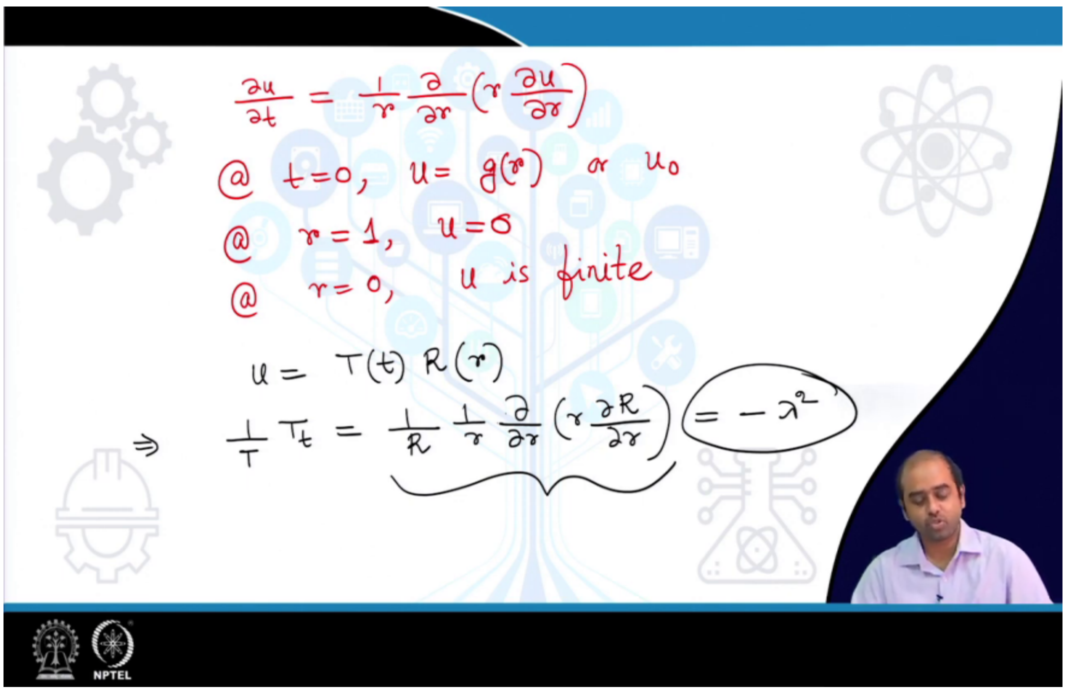

So, let us first try to understand the equation. So, this could be a scenario when we have you know, 1 dimensional transient heat conduction in a pipe, this is a likely example, where you would encountered these sorts of PDEs. So, these PDEs is in cylindrical coordinate system and this is time dependent in nature.

So, let us define the boundary conditions that at t is equal to 0, I mean this is the initial condition u is a function of is not constant or you can consider it to be constant, whatever, there could be an initial gradient of temperature also our initial distribution of temperature and at the boundary r is equal to 1, u is equal to 0. And this is sort of you know, it is it is very important to define something, I mean why I am defining as r is equal to 1, I mean, or r is equal to 0, I mean that depends on I mean, you can also consider it to be r is equal to capital R. But this C let us consider this to be a non-dimensional system, where we scaled our radius with respect to the radius scaled our radial coordinate with respect to the radius of the tube. And the other boundary condition is at r is equal to 0 this t sorry, this u is finite. Finite and it is symmetrical in nature.

So, this is a physical boundary condition and this cannot be violated, you cannot have at r is equal to 0 some undefined quantity. So, it has to be a finite value, we are not defining that whether it is constant or something, but it has to be a finite. The other symmetric condition is that you can have this you know this radial gradient of this u to be equal to 0 that is also a possibility for a symmetry boundary condition, but whatever this u has to be finite.

So, this is a necessary condition for this case. Now, from the separation, for the separation of this separation of variable technique, you know that we have to you know consider this u as the product of 2 linearly independent functions. So, in this case, this is something we are writing and this tells you that the corresponding equations are the formation of the auxiliary equations looks something like this (sorry).. del del r of r capital R. And you know that this has to be a negative quantity.

From the perspective of this temporal terms, this cannot be positive, it cannot be 0, otherwise it would be same, I mean the value of this variable will be same everywhere. So, this has to necessarily the negative quantity. There is also a possibility that if you consider that the special term, this term can also exist only, I mean this these equations can only be evaluated when we have or this auxiliary equation of the spatial part can only be evaluated when we have a minus quantity or a minus constant quantity.

That is also possible to prove and that is something possibly I would leave it to you as an exercise to find out that what are the typical consequences when we consider instead of explaining for the temporary terms what happens when we try to explain the spatial terms with a 0 value or with a you know positive constant quantity.

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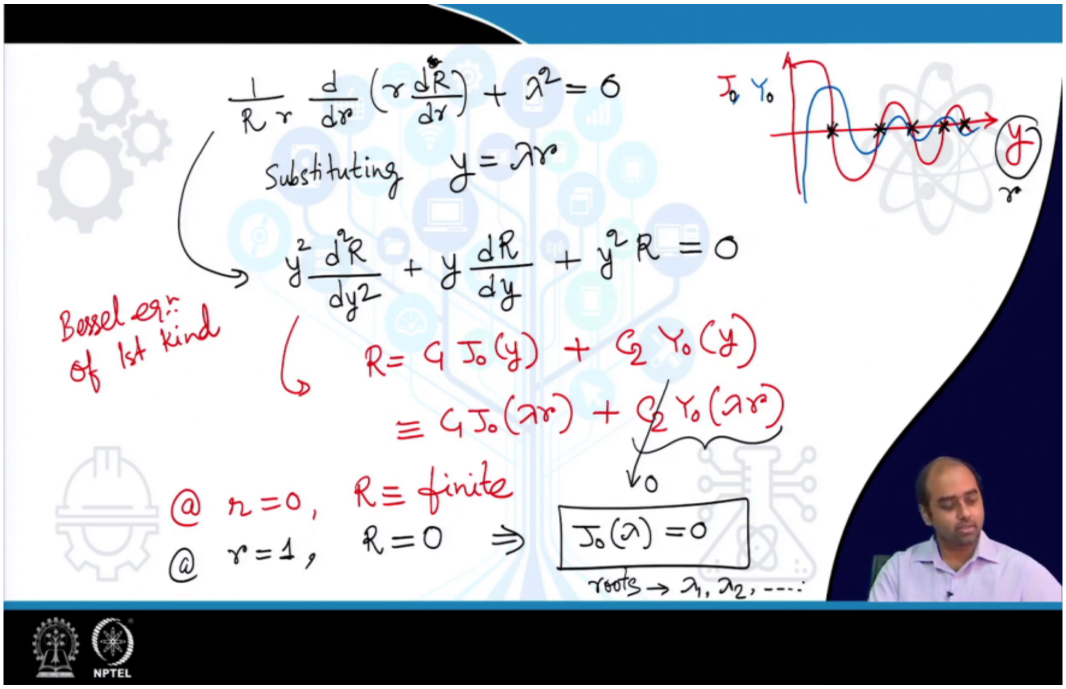

So, let us move ahead for the case when it is of course, negative, we set as the auxiliary equations of course, you know for the time part it is going to be the exponential term. So, interesting thing comes from the spatial part. So, let us work out the spatial part so, you have 1 by capital R r, d d r of r d sorry. So, the easiest way to substitute is that if you do this simple substitution that substitution thing y to be equal to lambda r, then this equation gets transformed to something like this and please try to identify what these equation looks like. So, this is Bessel equation. Bessel equation of first kind.

So, the solutions to this are in the form of j and y. So, clearly, I can say that R is equal to C1, J0 y plus C2, y0 y and of course y can be replaced by lambda r. So, I can alternatively write this to be C0, J0, C1, J0 lambda r plus C2, Y0. Now, please note that at r is equal to 0, we have this boundary condition that capital R has to be finite right and from this if you recall if you try to recall the profiles of J and Y you will see that, let us say I try to draw J and Y so, the curve corresponding to J. It starts from a finite value for you know for this dependent variable and then it looks go something like this. In the case of Y, it is unbounded like this. So, this Y can be can instead of Y, you can also represent this with respect to you know r or lambda r.

It is more or less same, because it will be only changed by a constant factor of lambda. But what is the important message that I want to pass on here sorry, these are Y0 and J0 important message is that, Y is 0 at r is equal to 0 is unbounded. But this solution tells you that at r is equal to 0 the spatial coordinate cannot be infinite or unbounded, it has to exist. So, this term corresponding to Y0 component has to go away and that means, C0 has to turn to 0, sorry C2 has to be 0 that is the only possibility that you can make this part bounded and J0 is a finite quantity from this nature of the curve, that we know.

So, at for at r is equal to this 1 or at the boundary the boundary conditions satisfy is that capital R is equal to 0 which means that J0 alpha is equal to 0 and here C1 cannot be is equal to 0 otherwise there will be no spatial distribution. So, for you know non-zero value of C1, this is the Eigen function that needs to be satisfied. So, what does this tells you, that you have to find out the solutions of you know J with respect to alpha, sorry with respect to lambda.

So, for the different values of lambda which for which the function J0 is equal to 0 is something will give you the Eigen solutions and there are in finite Eigen solutions, let us term them as lambda 1, lambda 2, lambda 3, like that. This could be the infinite lambda values. Now, the easiest way is to find is to use a you know program to compute the you know J with the value of J0 for different values, I mean J0(x). You can plot and you can find out from that expression that under what is the corresponding values of this J0 this x for which you will be getting J0 to be equal to 0, that is the easiest way to find the roots of this equation.

So, you have to root find out the roots of this equation, using the easier way is to use a numerical technique or a numerical program. Inbuilt tools like Fortran, MATLAB have a direct you know library functions to calculate the Bessel solutions, I mean the Bessel functions and from there you can find out what could be the possible roots. So, the possible routes are can be designated as something like you know lambda 1, lambda 2, like this and they will be infinite such solutions.

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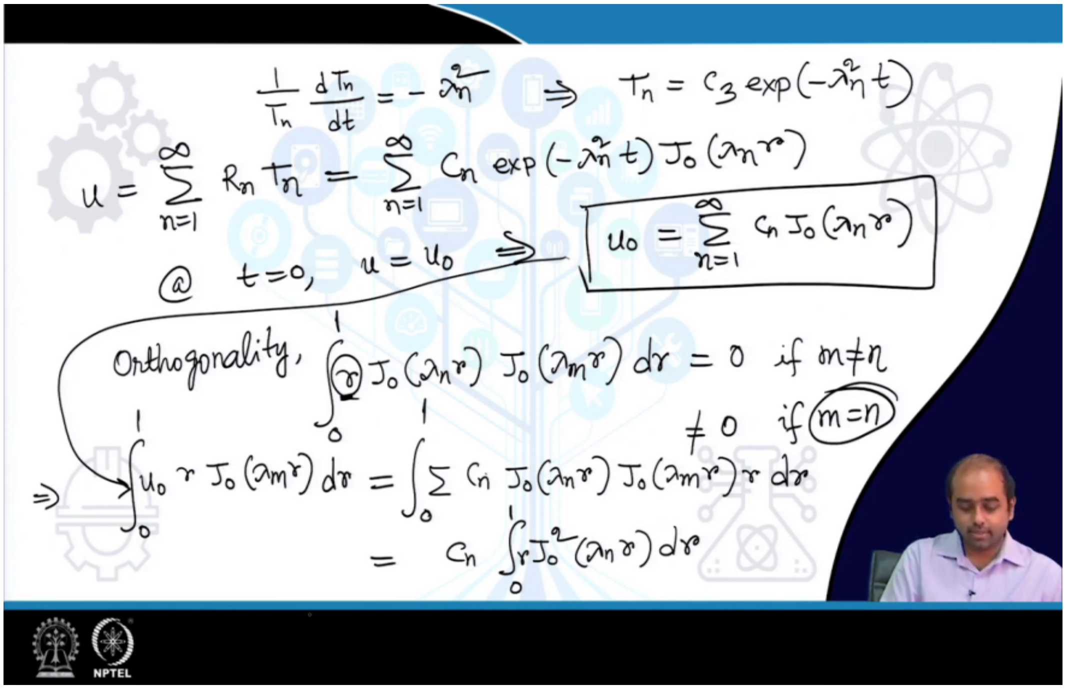

So, time part let us go back let us write that time part and let us join them together now. So, from here we can say that Tn is equal to C2 something like, sorry C3 exponential minus lambda n square because, there are n number of possible infinite solutions for lambda. So, u is equal to, the solutions of like this… where n is equal to 1 to infinity and if I join them together both the constant C1 and C3 together they will make Cn just the same way as we did it for the Cartesian coordinates, and this here it will be Bessel functions of zeroth order. We use the initial condition at t is equal to 0, u is equal to u0. So, this means I have u0…

Now, from here how we proceed? Again we bring in the criteria of the orthogonality of the Eigen functions, and in this case, I think I have told you in the couple of lectures back when we did the special function criteria, that the Bessel functions are also orthogonal in nature. So, if I So, this quantity J0 lambda n and J0 lambda m they can be multiplied together and you know their integration unless n and m are equal they will be equal to 0.

So, Bessel functions are also orthogonal in nature. So, what specifically I mean to say that is this so orthogonality of Bessel function tells you the orthogonality criteria tells you that. So, this is equal to 0 if m and n are equal sorry, not equal and this is equal to non-zero if m and n are equal. So, with this idea, we can multiply both sides to the equation and we can remove that summation term, because any n values which is not equal to m will lead to the integration equal to 0. But remember, we have to multiply this r on both sides. So, you multiply you know r J0 lambda n, on both sides for this equation. And similarly, we do the same thing on the right-hand side (sorry). So, we do the integration also 0 to 1, so, the right-hand side would simply become c and integration of J0 square of course r is there, alright. Here is how the right-hand side is going to look like.

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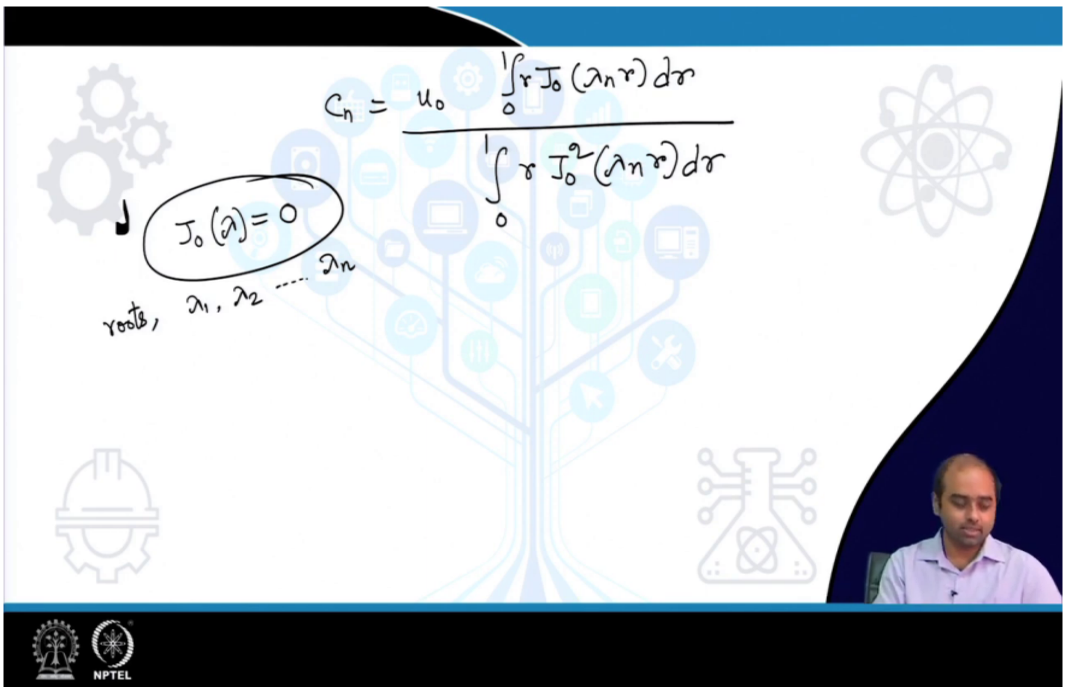

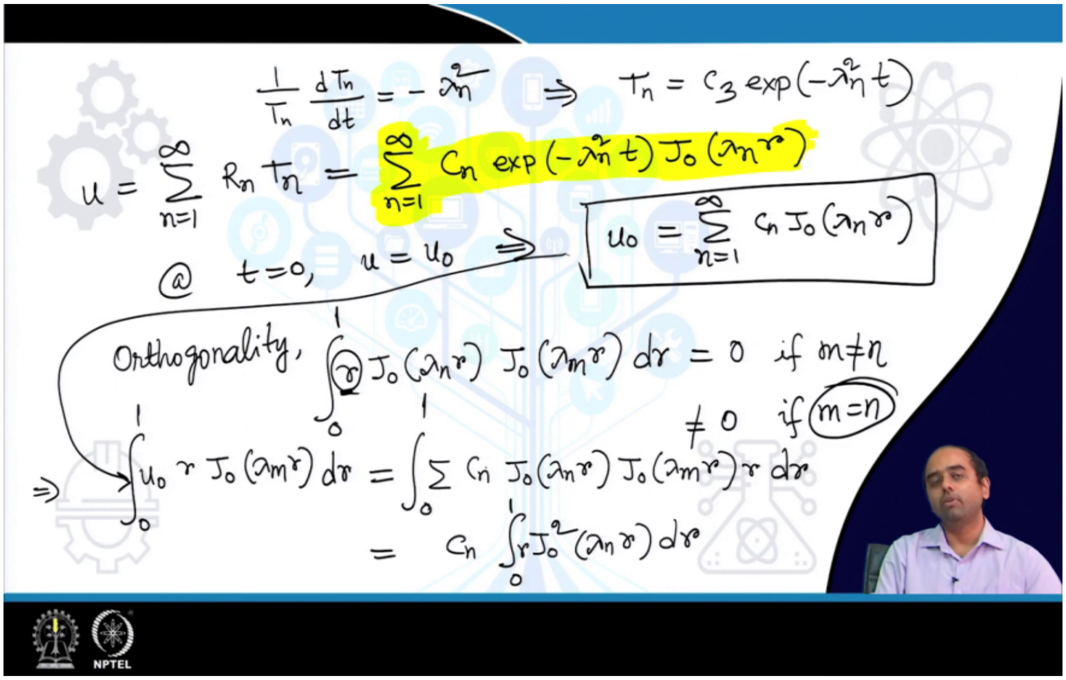

So, Cn in this case is u0, there is one integration here divided by another integration. So, you can use any of the recursive formulae for the integration of the Bessel functions. And once you know the (you know this) value of your Cn, the final solution is actually quite well defined now.

You can just see that we have already the final equation and only thing that was left was to evaluate out what could be the value of our Cn. So, with this you can easily understand or realize the importance of the Bessel functions and its application in the solution to the (you know) cylindrical type partial differential equations.

So, this is the final solution and only thing that was unknown was the value of Cn and you have seen using the principality of orthogonality, I mean orthogonal nature of the Bessel functions, you can evaluate out what should be the value of here, this constant Cn as a function of lambda n. So, this involves two steps one is calculating out the (you know, this) lambda values from the solution to that equation that already I think we have noted down, let us also write it down once again here.

So, J0 lambda should be equal to 0. So, this is the equation or the solution and the roots to this equation will give you the different values of your lambda. So, the roots are lambda 1, lambda 2, to any values of lambda n. So, far we have been discussing on this parabolic type equations both in the Cartesian and the cylindrical coordinates.

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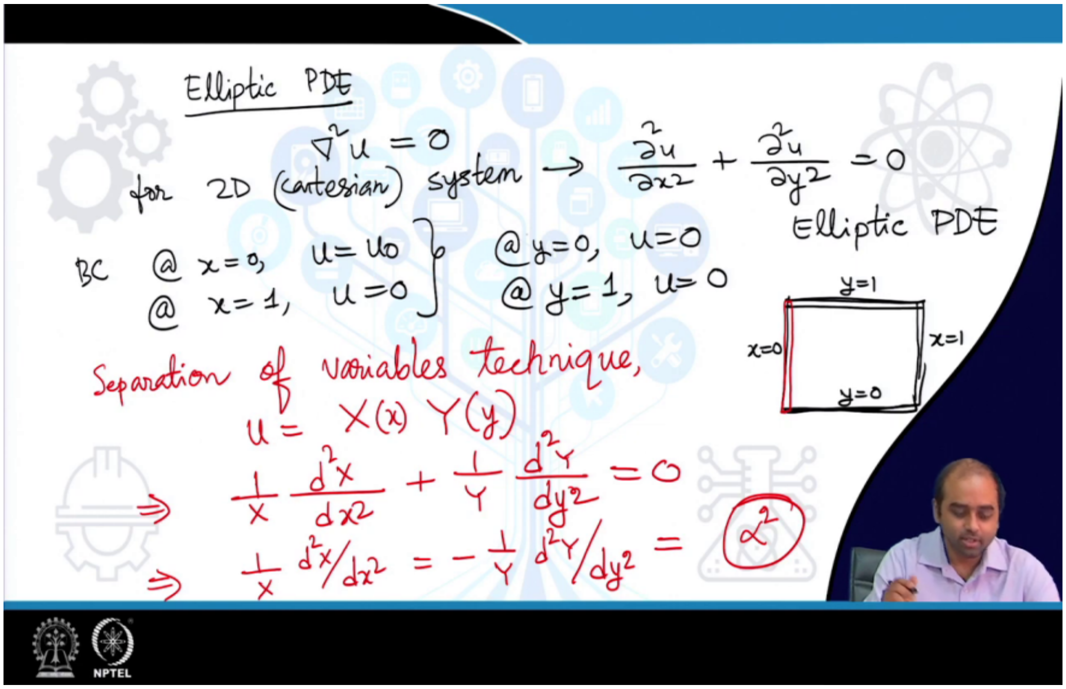

Now, let us look into a scenario when we have elliptic PDEs. Like so, these elliptic PDEs is something that you encounter in you know multi-dimensional steady state problems particularly on the scenario when the Biot number of the system is very low, the you know this you can end up with the Laplace equation after heat transfer sorry, Biot number is larger when the heat the thermal conductivity is low then you are supposed to get a distributed parameter system.

So, multi-dimensional steady state problem, so write something like this Laplace equation were for 2-dimension system, 2 dimension let us say Cartesian system you will be getting something like this. And this is an elliptic PDE. So, it is second order in x and second order in y. So, you need to have two boundary conditions in x and two boundary condition in y, let us try to define them.

So, boundary condition at x is equal to 0, u is equal to u0, at x is equal to 1 you have let us say u equal to 0. So, we are keeping all them all of them to be the Dirichlet types and again at y is equal to 0 let us say you have all these, sorry y is equal to 1 is equal to 0. So, you can think of a box. So, this situation is very similar to the case like this, let us say these are the four box so, this one is like this bottom line is x is equal to 0, top line is sorry, bottom line is this left hand side is x is equal to 0 right hand side is x is equal to 1, top is y equals to 1 and this is y is equal to 0.

So, on all the on all the sides, we say that the temperature is equal to 0. Something like this and only on this left-hand side there is a temperature which is non-zero, so perhaps that wall is heated and the rest of the wall are maintained at a different temperature there is thermal conduction or heat conduction from this left-hand side wall. So, this is the physical interpretation of this system of the equation. So, now, this is homogeneous conditions and homogeneous equation and all the boundary condition is also homogeneous.

So, we straightaway apply the separation of this variable technique, we define let us say u is equal to capital X as a function of x and a capital Y as a function of y. So, auxiliary equations look something like this. So, if I bring them on two sides of the equation, then I will have one part which is like exclusively a function of x and then another part which will be exclusively a function of y and these two would be equal only on the case when this is a constant.

So, what to choose shall this constant it as defined this constant as alpha square, so, shall this constant b positive or negative that is that is very important to realize. So, we will see that what happens when the constant is you know positive or what happens when the constant is negative? So, I we will do in this class today when the constant is positive and I will leave it to you to figure out what happens when the constant is negative. So, instead of plus alpha squared we will have minus alpha squared.

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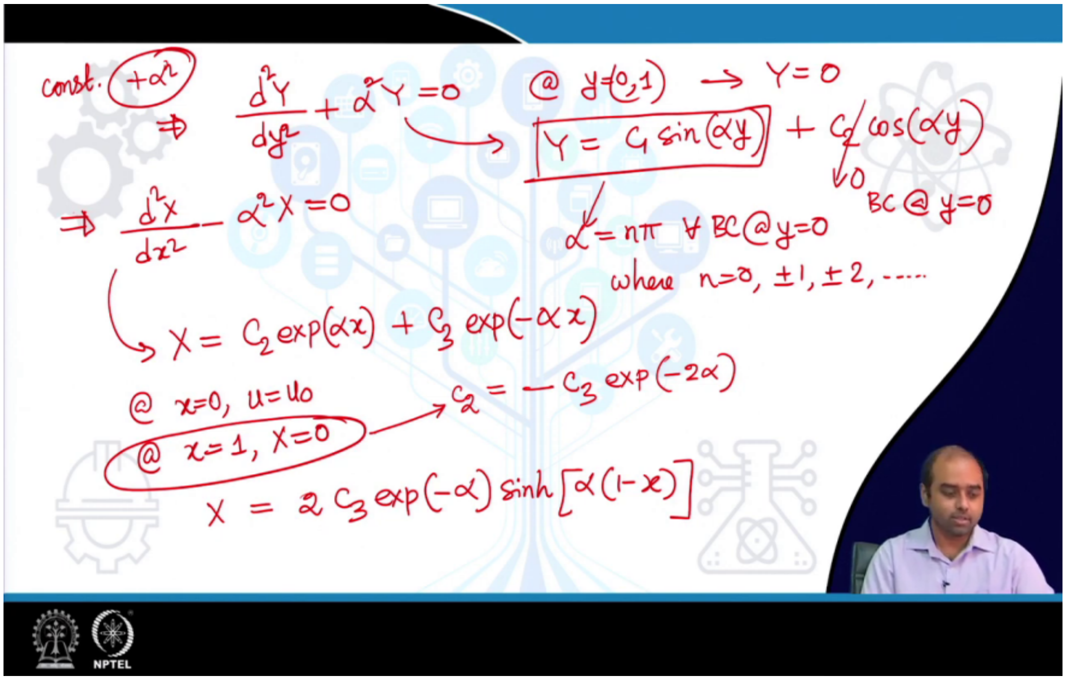

So, let us consider that the constant is positive, so, it is plus alpha square and try to set up the auxiliary equations. So, we say it is plus alpha square. So, the constant so, the constant is plus alpha square. So, what do we get d2Y/dy square plus lambda square Y is equal to 0. This is one scenario we get and another scenario another auxiliary ODE these dX/ dx square minus alpha square X is equal to 0.

So, in this case at y is equal to 0 and y is equal to 1 in both the cases we have capital Y to be equal to 0. So, what does this mean the solution to this equation capital Y is equal to C1 sin lambda y plus C2 cos, sorry alpha y and clearly you can understand that C2 cannot exist because that y is equal to 0, the solution has to be 0. So, this part sin part is 0, but the cos part will be equal to 1 so, C2 has to be 0.

So, this is the part when we consider that the boundary condition at y is equal to 0, and from here you can also find out that I mean you can also tell that alpha is equal to nothing but n pi from the from the boundary condition based on the boundary condition, based on the boundary condition at y is equal to 0, so, this n would be equal to the values of n equals 0, plus minus 1, plus minus 2, like that.

So, now, coming to this part that what happens in this case here, in this case the solutions are given in terms of you can write this in terms of exponential functions or you can write them in terms of hyperbolic sin and hyperbolic cosine functions also both are interchangeable, but it is only slightly you know in a different expression you will be getting.

Now, in this case you have the effect of the boundary condition that at x is equal to 0, sorry at x is equal to let us revisit the boundary conditions once again, at x is equal to 0 you have capital x to be equal to 1, and you have the other boundary condition at… So, these are the two boundary conditions, we have for the x dimension.

Now, let us use this condition first, that x is equal to 1 capital X should be equal to I mean essentially u is equal to 0 and capital X is equal to 0. So, what does this tell you? that this simply tells you I mean these are the two conditions we have that capital X is equal to 0 at x is equal to 1 so, if I use this condition here x is equal to 1, capital X is equal to 0 this gives me a ratio, I mean relation between the two constants. So, this means C2 is equal to minus C3 exponential minus 2 alpha.

So, clearly you can see that I can represent one of the (I mean) I can remove one of the constants I can again get an explicit expression of X or something like in terms of C3 maybe so, I can write X as equal to 2 C3 exponential minus alpha sinh alpha 1 minus x. So, again this C3 is something that needs to be found out. So, this is a likely scenario that I am getting I hope all of you are getting it this. Based on this boundary condition I will get C2 as minus C3 into exponential minus 2 alpha and from there, I substituted into the main equation and I get my X component or the X function or X part of the function of this solution.

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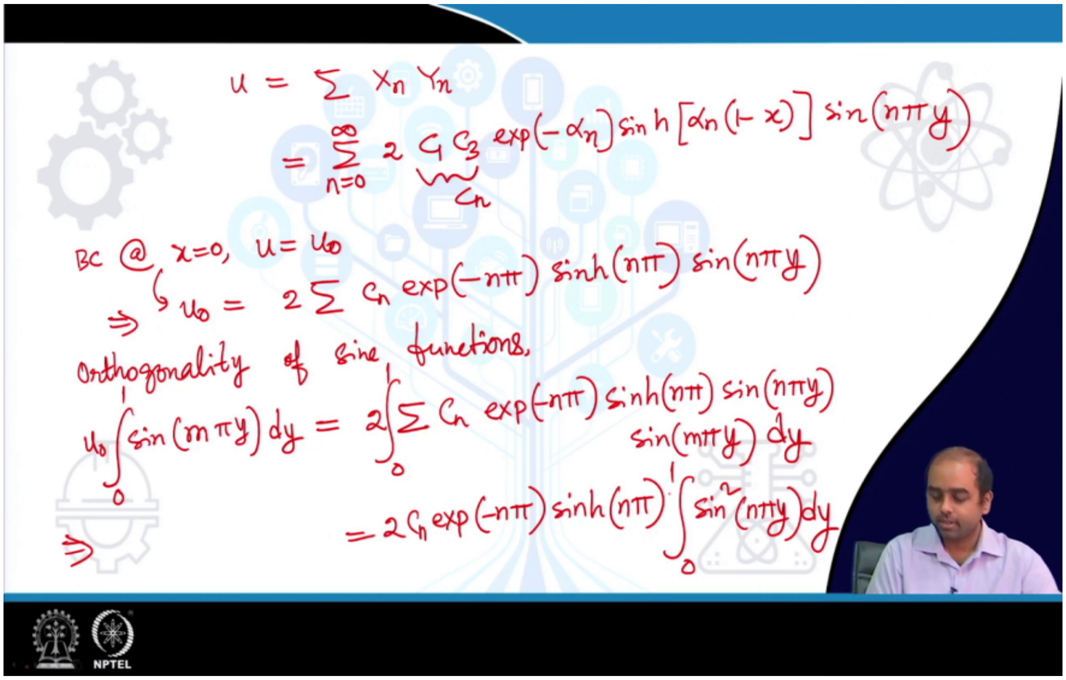

So, now again my u is a summation of the you know this X part and the Y part solutions. So, I have 2 C1, C3 exponential minus instead of writing alpha let us write them as alpha n because alpha is equal to n pi. And let us club them together and write them as Cn where n is equal to from 0 to infinity. Now, it is time to use the other boundary condition you know that, I mean by now, you have already guessed it.

So, the other boundary condition is the remaining boundary condition. So, far we have utilized three boundary condition is that x is equal to 0 we have u is equal 0 to u0. So, this simply means, I have u0 on the left-hand side to summation of Cn, exponential minus n pi substituted lambda into n pi sinh n pi sin n pi y, this is an x is equal to 0.

And again, we use the orthogonality, orthogonality of the sign or the trigonometric functions and from there you can easily so, you multiply again on the both the sides with sin functions m pi y dy, integrate it out, same thing goes to the right-hand side and you know that the summations will go away for the case when m is not equal to n. So, this summation term will go away for the case when we have m is equal to n and if you work out so, this right-hand sin will be, let me write it out clearly then for you, so, it is sin square n pi y dy.

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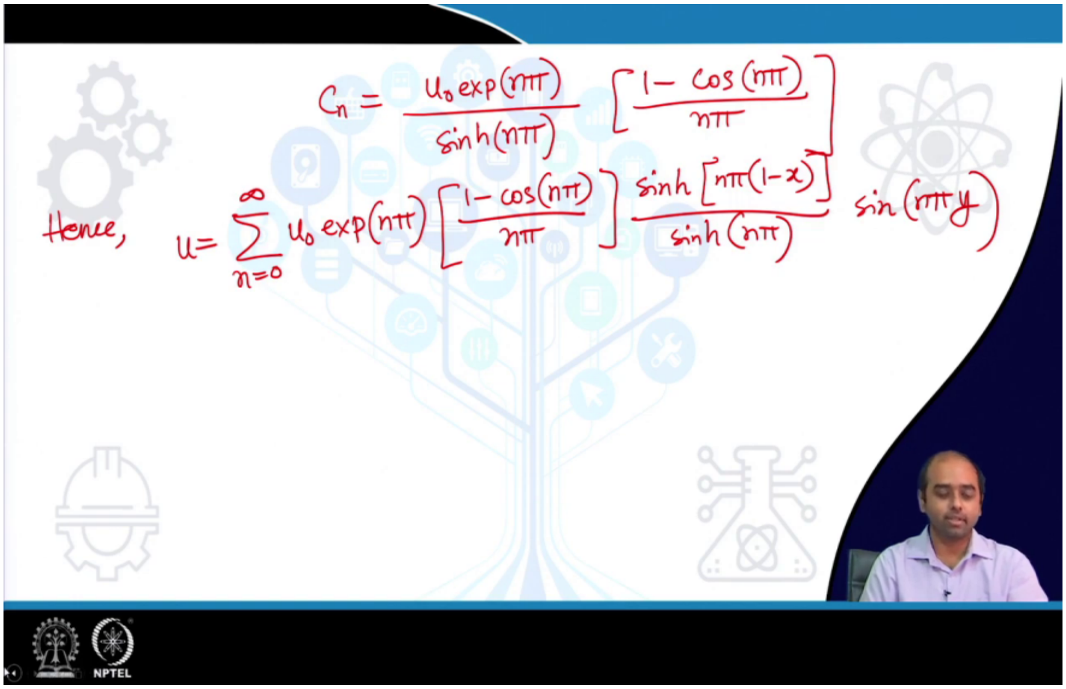

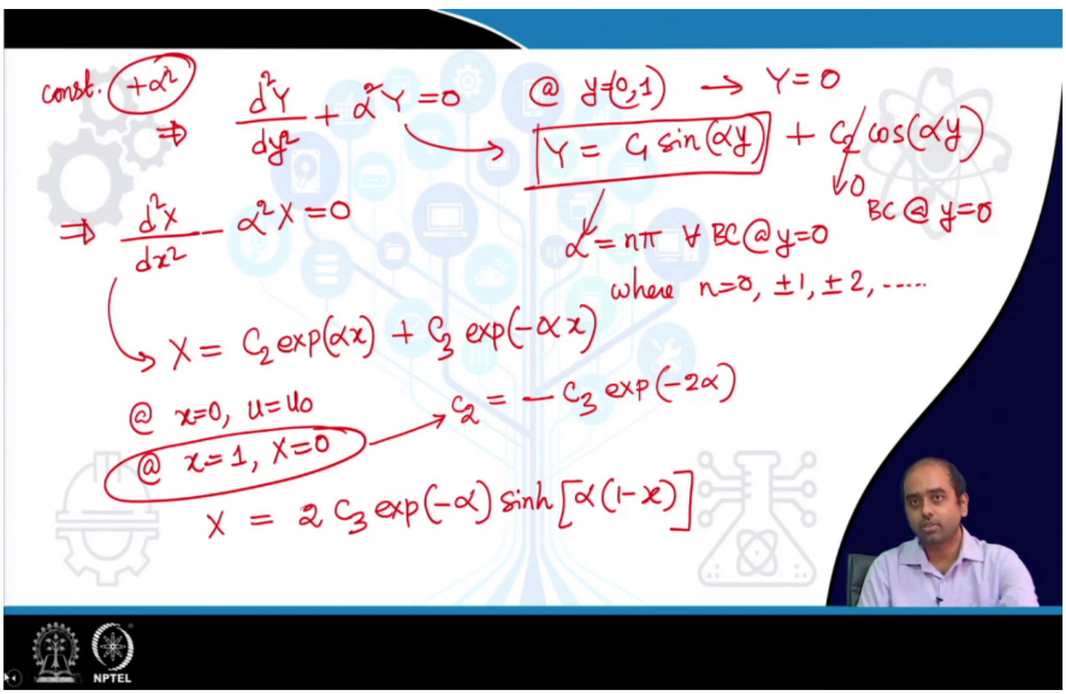

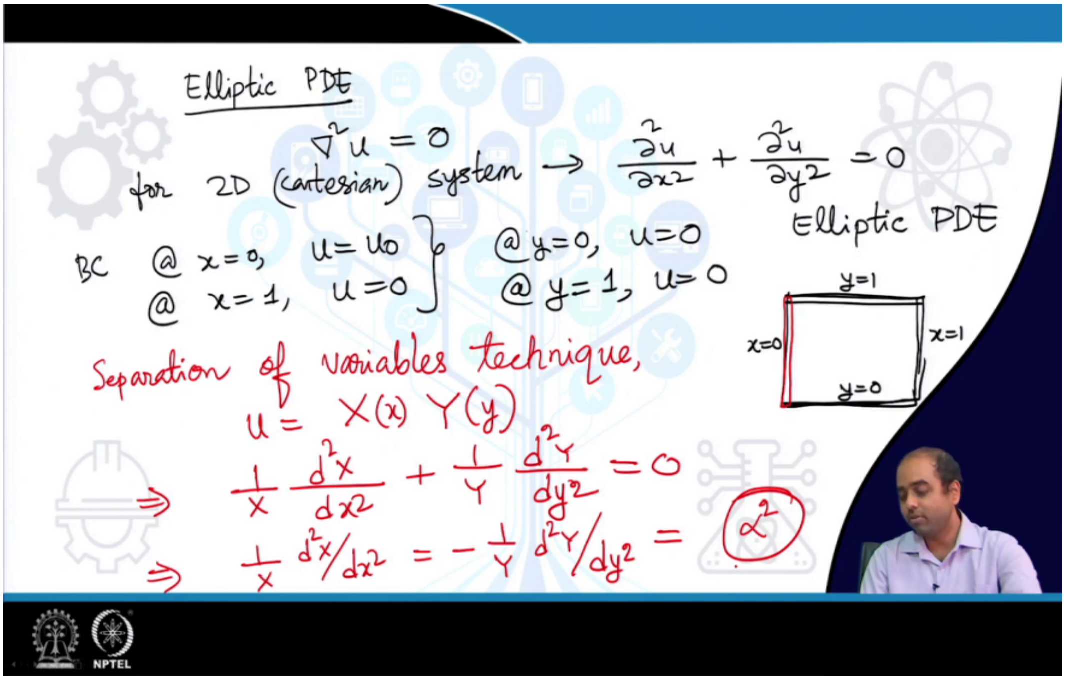

So, if you work out the integrations you will end up with the expression of Cn has u0 exponential n pi by sinh n pi minus cos of n pi. Hence together, now if you club them together the final solution will look something like this is u0 exponential sinh n pi 1 minus x, sin n pi y. So, this is the complete solution for two dimensional problems two-dimensional Cartesian steady state problem. One thing I would like to your attention now is that, when we try to evaluate or choose this constant to be positive. What would have happened if the constant is negative?

So, by now you all of you, I mean have realized that if the constant is negative, then we are likely to get the exponential functions. The exponential function in term of y and sin and cosine functions in terms of x and if that is the scenario, if that is the scenario then we cannot use, I mean the sine hyperbolic functions are not orthogonal in nature. So, the idea of orthogonality cannot be utilized at the later stage for a non-zero boundary condition, utilizing that. So if the non 0 boundary condition, so what is the idea to determine whether you should chose positive lambda or negative this alpha positive alpha negative alpha is that the case, if you if you see that whichever boundary condition in non-zero, at least from this part of the example where we do not have any Neumann conditions, whichever boundary condition is non-zero, you should ensure that the solution or the solution to the auxiliary ODE for that you know, coordinate or that independent variable do have the trigonometric or do have a Eigen function, which we for which we can use the orthogonality functions. So, in this case, I mean the alpha should be chosen in such a way that we that we land up, is not it we land up for the case, when we can apply the orthogonality function in the y direction.

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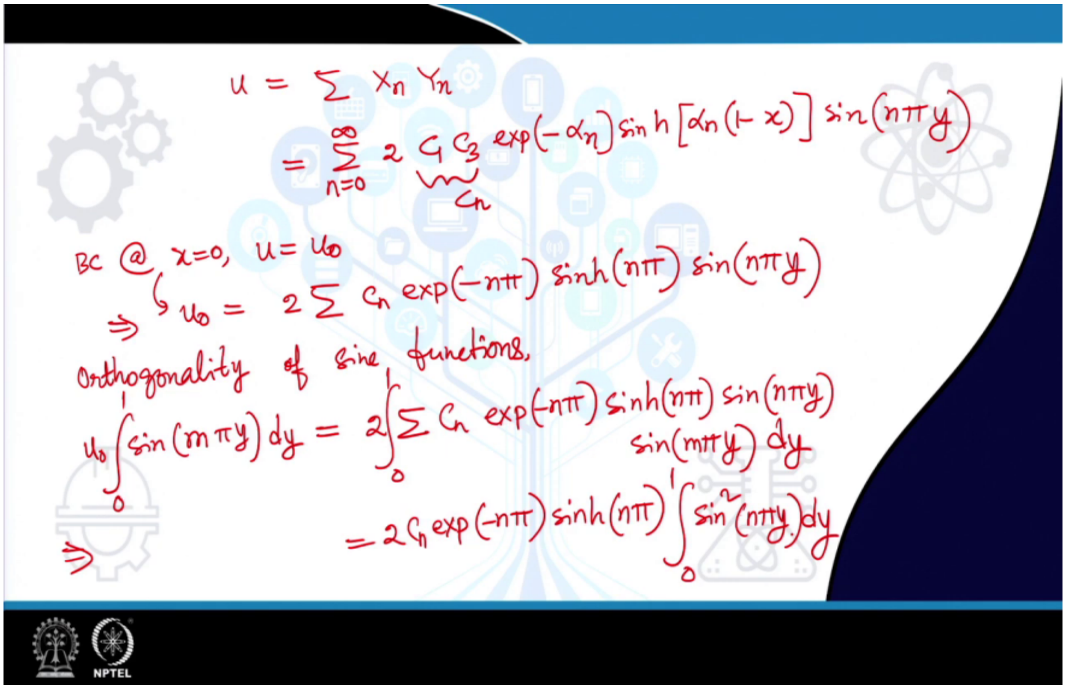

So, with this thank you for this your attention to this lecture and I hope you have learnt about the separation of variable technique and this is something which will be quite useful, later on for the solution of different PDEs, both you know parabolic and elliptic PDEs. In the next class we will talk about the integral transform techniques for solving partial difference equations. Thank you. Thank you for your attention.