Mathematical modelling and Simulation of Chemical Engineering Process
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Lecture 20
Stability of dynamical systems (Contd.)

Hello everyone. In this lecture, we are going to study about the analysis on dynamical I mean stability criteria, talk about the different bifurcation systems and then we try to explore that how the stability criteria can be achieved or the analysis can be done for the distributed parameter system.

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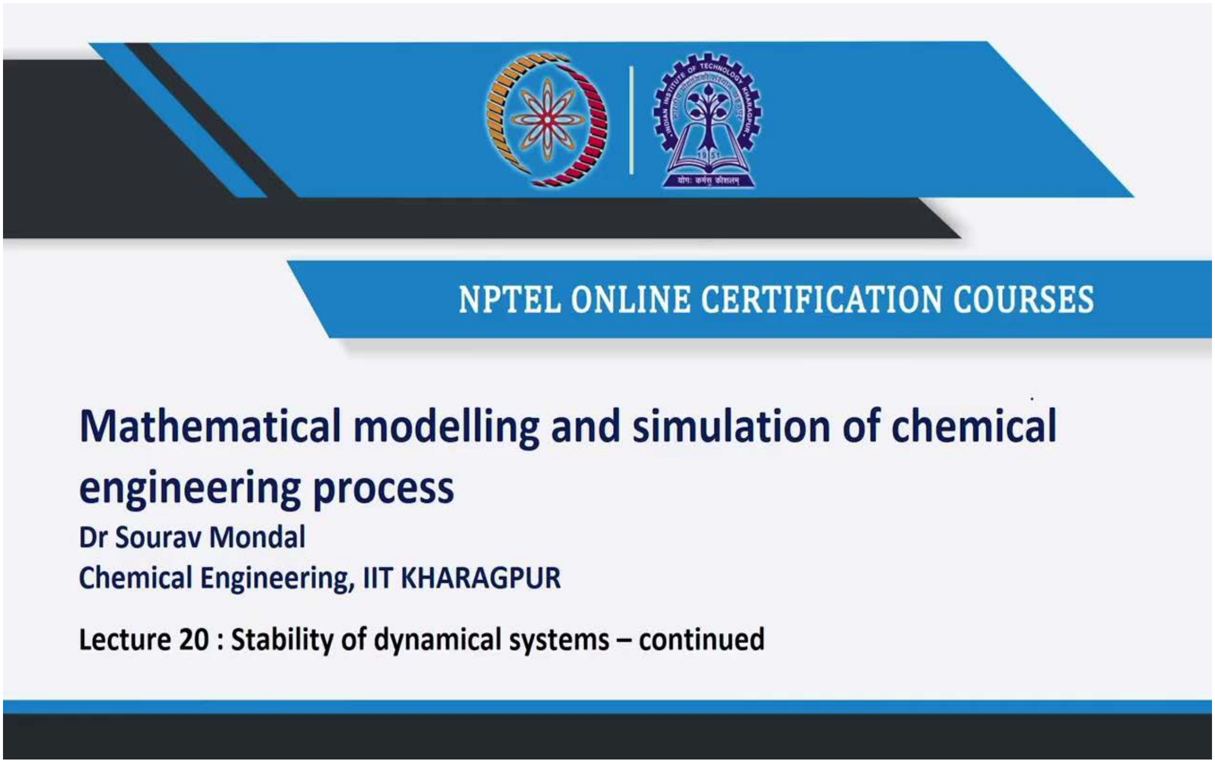

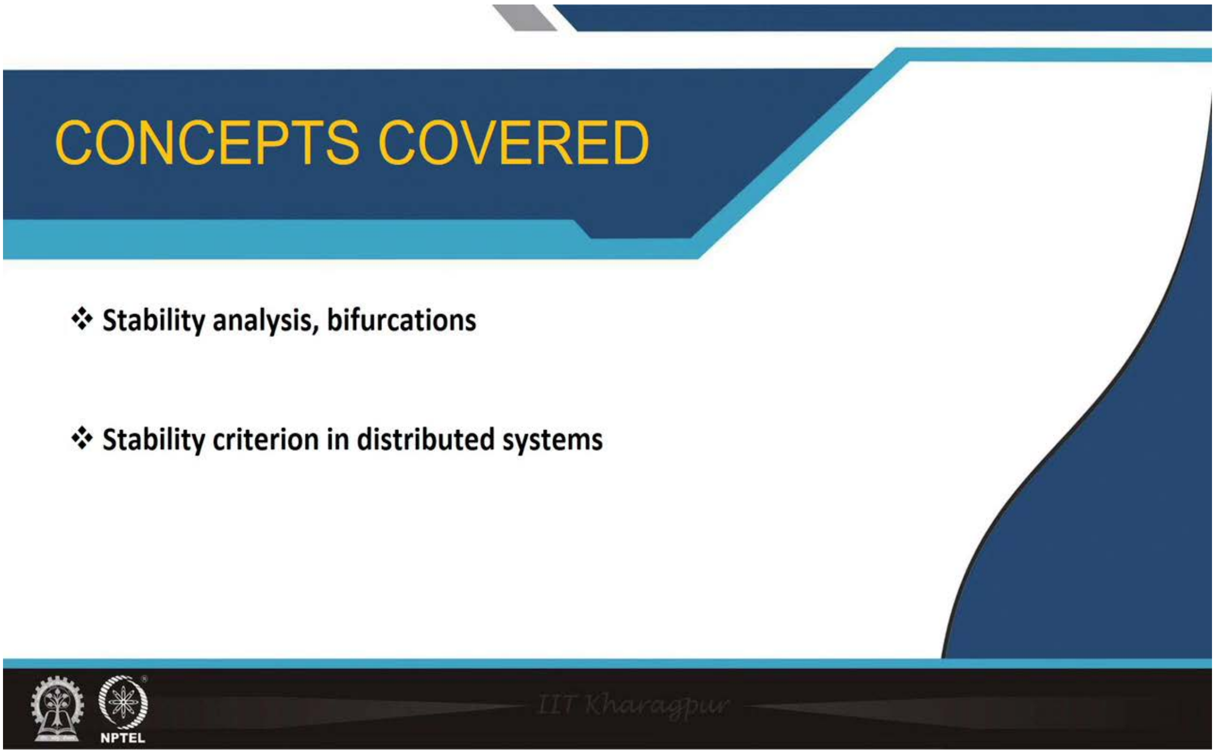

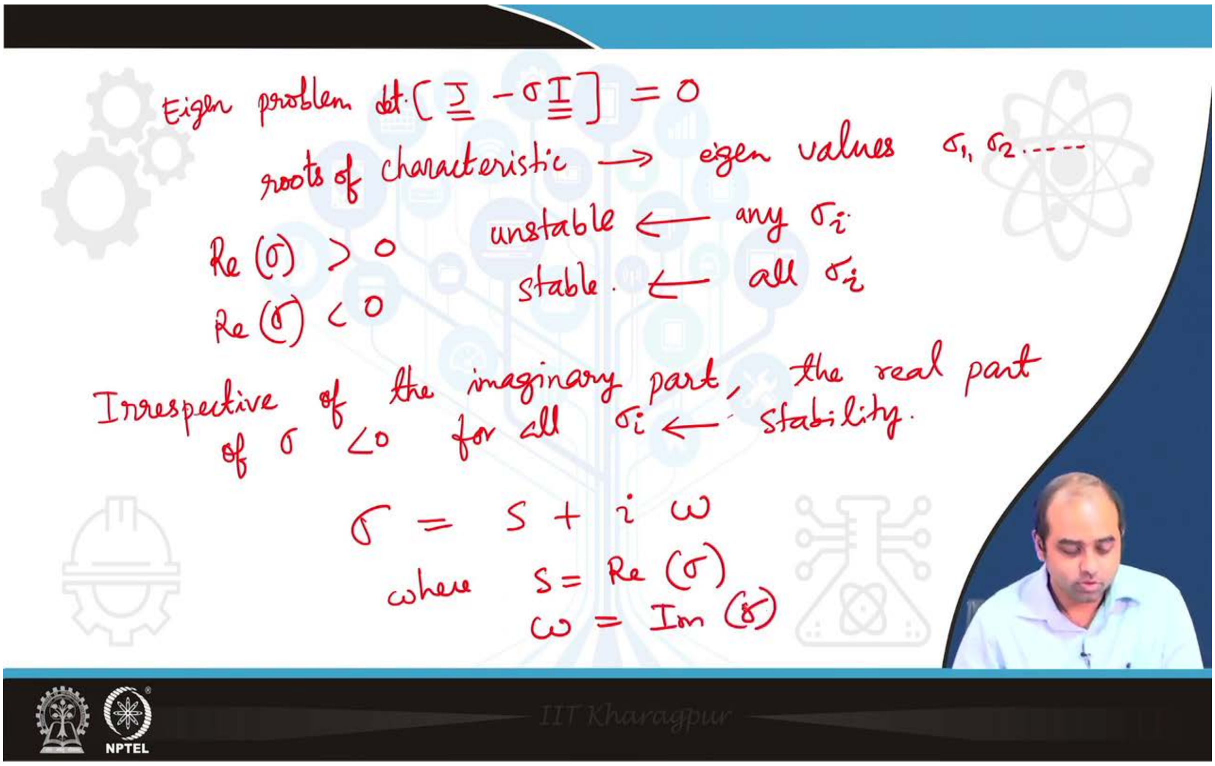

So, if you recall in the last class we talked about the eigenvalue problem and how the Eigen values of the stability equation or the characteristic equation tells us about the stability criteria. So, essentially we have the eigenvalue equation. So, the Eigen problem that we had there is something you recall, was set to 0. The determinant of this was set to 0 and the characteristic equation gave us the, I mean the roots of the characteristic equation tells us about the Eigen values.

So, it could be sigma 1, sigma 2 like for n number of system and then we said that the real part of this sigma if it is greater than 0, then we have unstable condition or unstable situation because the growth of the disturbance increases with time but if it is less than 0 then it becomes stable. And please note that these instability criteria that we have here, is should be, I mean maybe true for any variable, any sigma but whereas, for stability all the sigma values has to satisfy this criteria that their real part is negative.

So, irrespective of the imaginary part always maybe I should write this down, irrespective of the imaginary part, the real part, the real part of sigma should be less than 0 for all sigma i's. So, this is the criteria for stability. Now, the critical situation arises that let us say I represent that my sigma as s plus i Omega. Where s is the real part of the Eigenvalue and Omega is the imaginary part of the Eigenvalue.

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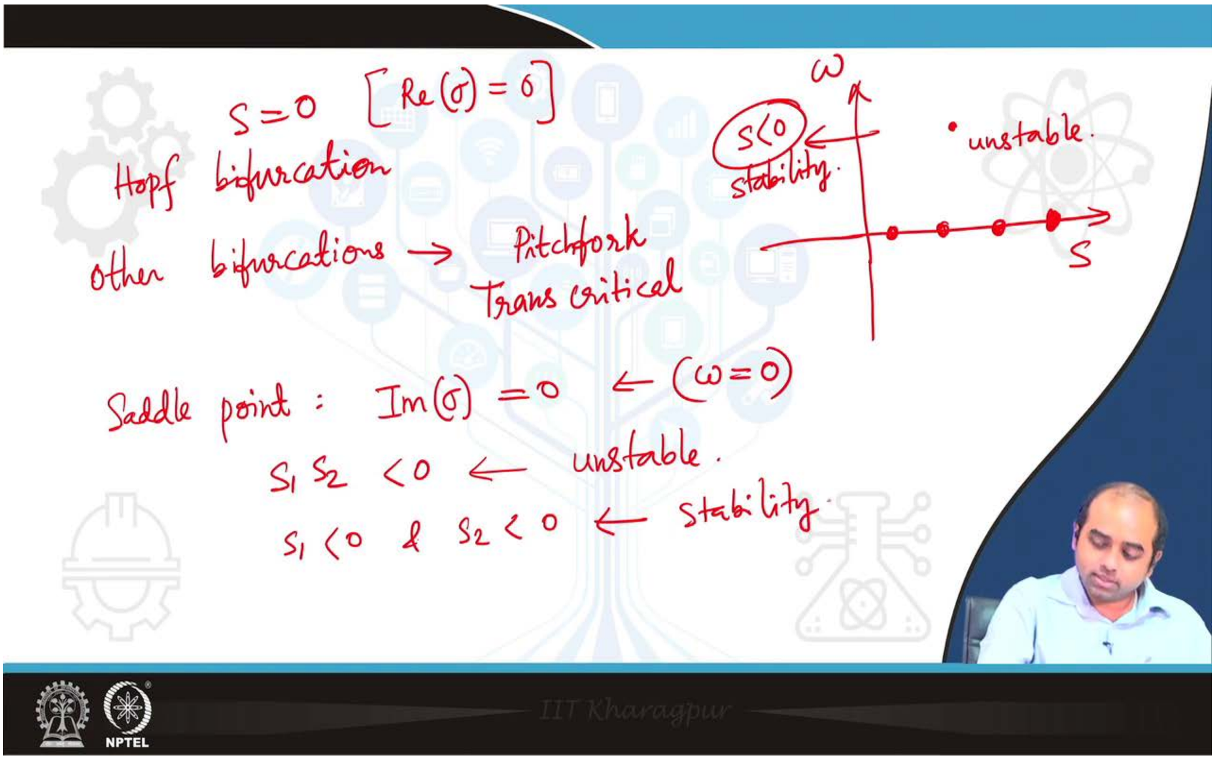

So, in a complex plane, I can draw this, the real part S and the imaginary part W and I can mark in the complex plane where my point lies. So, if it is somewhere here, so, this is unstable for stability always it should be across this point or S should be greater than less than 0 for stability. Now, what happens? What happens when the real part is 0, there is no real part it is only imaginary. So, the real part of sigma or this Eigen value if it is equal to 0, then we have a condition known as the Hopf bifurcation.

So, what essentially bifurcation means is that from that point onwards or at that conditions these are sort of critically stable situations where the solution of the wider system can proceed in either direction. So, there is a possibility that it can take 2 roots of its evolution. So, that is why we call them to be Hopf bifurcations or essentially we call them as bifurcations.

There are some other bifurcations and possibly we will not dig deep into bifurcations in this class and possibly is beyond the course syllabus, but if you are interested you can also explore them this pitchfork bifurcation, there is also something known as transcritical bifurcation and each one of them has their own physical significance and there are certain conditions when these arise, there is also something known as the saddle point.

So, this in the case of the saddle point there is no imaginary component. So, the imaginary component of the sigma is equal to 0 or essentially here the W is equal to 0. So, this means that all possible values will only lie on the x-axis of this complex plane, if it lies on the y-axis then there is a possibility of bifurcation and if it lies on the x-axis, which means imaginary component is 0 these are called saddle points.

Now, even if the imaginary component is 0, the criteria remains still the same. So, if you have two, this component if both of them if one of them I mean the product of them is let us say less than 0, if one of them is of different sign, which essentially means it is unstable. So, the criteria still remains the same.

So, unless both of them, both of them are negative it is not stable. So, if they are different sign and the easiest way to check if there are two Eigen values you take the product, if the product is negative, which means one of them is of course positive and one of them is negative there is no other way. So, this is the of course, the situation when you have unstable state.

So, for stability both of them has to be negative, both of them has to be negative. So, and individually we have to check that s1 is less than 0 and s2 is also less than 0 for stability. Now, what is the condition or the generalized criteria for the saddle point or to achieve or to have a Hopf bifurcation.

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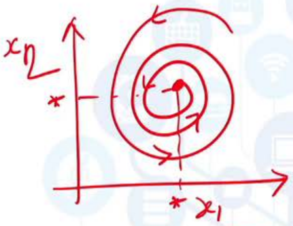
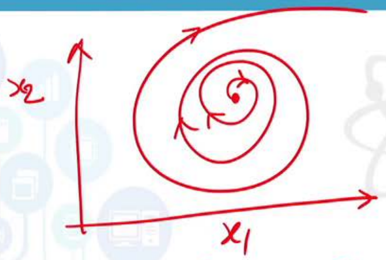
Foci





$$\underline{\underline{J}} = \begin{bmatrix} f_{1x_1} & f_{1x_2} \\ f_{2x_1} & f_{2x_2} \end{bmatrix} \quad \therefore \quad \underline{\underline{J}} - \sigma \underline{\underline{I}} = \begin{bmatrix} f_{1x_1} - \sigma & f_{1x_2} \\ f_{2x_1} & f_{2x_2} - \sigma \end{bmatrix} = 0$$

$$\Rightarrow (f_{1x_1} - \sigma)(f_{2x_2} - \sigma) - f_{1x_2} f_{2x_1} = 0$$

$$\Rightarrow \sigma^2 + \sigma(f_{1x_1} + f_{2x_2}) - [f_{1x_1} f_{2x_2} - f_{1x_2} f_{2x_1}]$$

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$$\Rightarrow \sigma^2 - (\text{tr}\,\underline{\underline{J}})\sigma + \det(\underline{\underline{J}}) = 0$$

Hopf bifurcation : $Re(\sigma) = 0 \Rightarrow \text{tr}(\underline{\underline{J}}) = 0$ (no real part)

stability $\text{tr}(\underline{\underline{J}}) > 0$

Saddle point : $Im(\sigma) = 0$ (no imaginary part)

$(\text{tr}\,\underline{\underline{J}})^2 > 4 \det(\underline{\underline{J}})$ [necessary cond.]

$\det(\underline{\underline{J}}) = 0$ sufficient condition.

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$$\sigma_{1,2} = \frac{tr(\underline{\underline{J}}) \pm \sqrt{tr(\underline{\underline{J}})^2 - 4\, det(\underline{\underline{J}})}}{2}$$

$$\Rightarrow \sigma^2 - (tr\,\underline{\underline{J}})\sigma + det(\underline{\underline{J}}) = 0$$

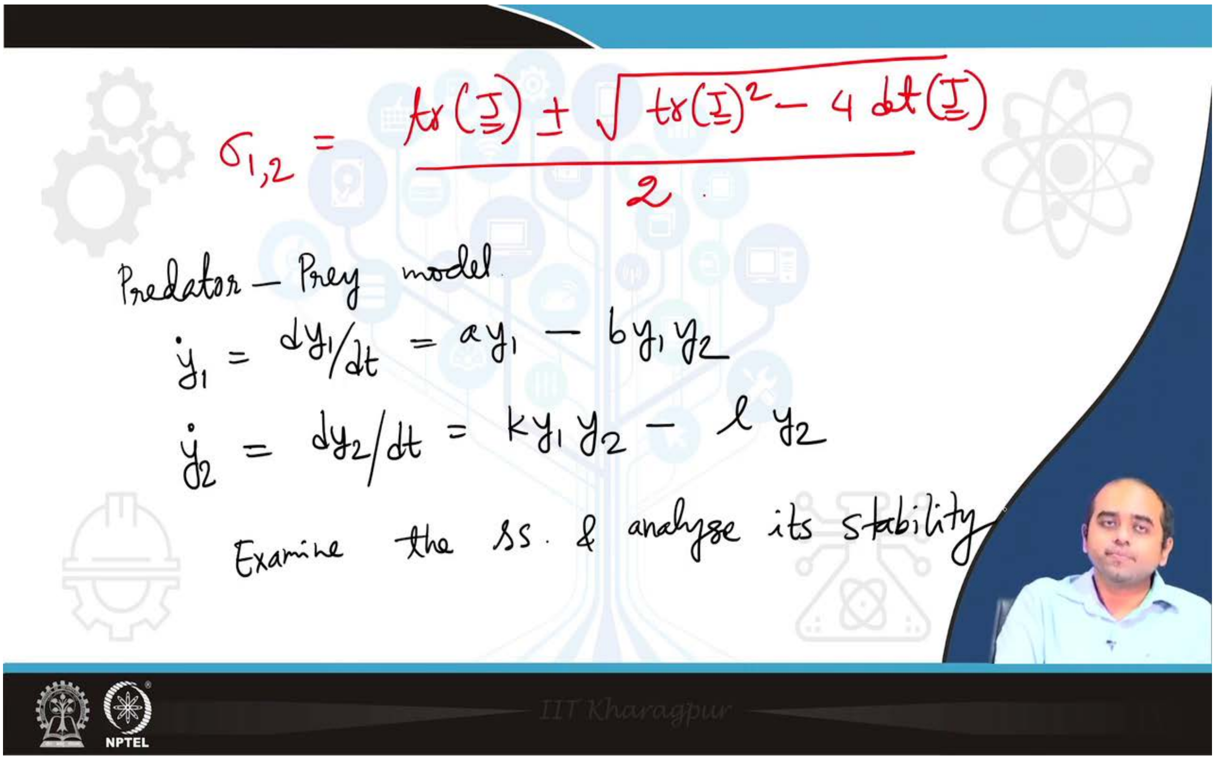
Hopf bifurcation : $Re(\sigma) = 0 \Rightarrow tr(\underline{\underline{J}}) = 0$ (no real part)

stability $tr(\underline{\underline{J}}) > 0$

Saddle point : $Im(\sigma) = 0$ (no imaginary part)

$(tr\,\underline{\underline{J}})^2 > 4\, det(\underline{\underline{J}})$ [necessary cond.]

$det(\underline{\underline{J}}) = 0$ sufficient condition.



But before that let me also give you some this Foci diagrams these are very popular in terms of stability analysis, which tells you that if you try to make a plot of the two state variables around your, this stability point, so, this is the star point.

So, if it is a stable focus or if it is a stable parameter space then any disturbance even if there is there will still evolve towards the stable situation. But if you have an unstable steady state then the system will actually deviate from its set point or the stable point because most continuous operations need to be operated at a steady state condition. So, if that steady state is not stable, then it is impossible to control the system.

So, any fluctuations in the process variable will lead to instability in the system. So, it is very essential that the steady state or the steady operating conditions which has chosen to attend the steady state should be stable. So, any process fluctuations whatever is there before the control system tries to act on the system should not deviate away further from there and it should not be uncontrollable. So, that is the reason why the stable steady state is the preferred choice.

Now, let us look into the case that we have this, the Jacobian and try to understand the generalized condition for saddle point or Hopf bifurcation, for the two space, two state variables, this is what we have. So, let us write this in the shorthand notation for the subscript represent the derivative with respect to x1 or x2, so, this one looks like this.

So, it is not product, it is there in the subscripts that is the derivative and here you have f 2x1, f 2x2 minus sigma and this characteristic equation is set to 0. So, if you work this out you can expand this. So, we have f 1x1 minus sigma f 2x, and if you work out the rest of the this algebra equation you will get something like this sigma square plus I mean the intermediate steps I am not writing. This is something you will get.

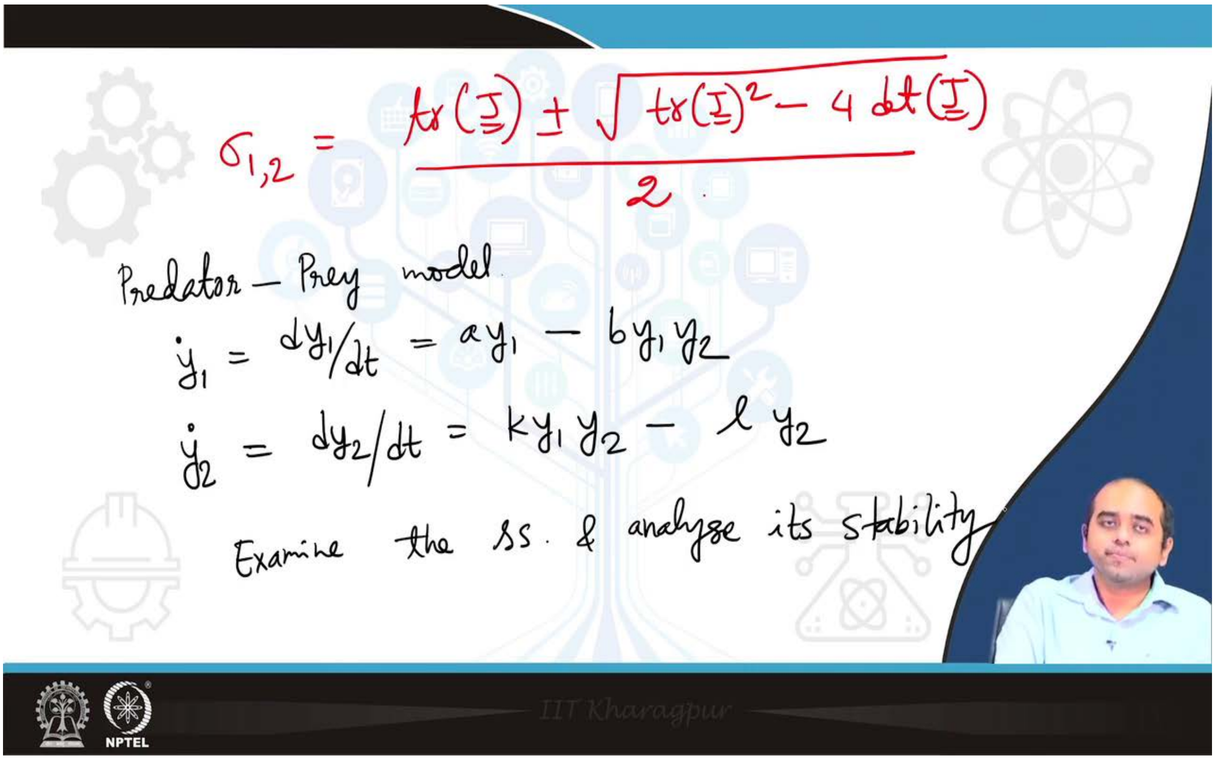
So, if you just look into the general case, this can be generalized as the coefficient to the second term is nothing but the trace of the Jacobian. And the last term if you look carefully, it is nothing but the determinant of the Jacobian. So, this part so, this is nothing but the trace of the Jacobian and this part is nothing but the determinant of the Jacobian.

So, that is something we write this in the generalized form here. Now, this is a quadratic equation and we all have studied about quadratic equation back in our high schools. So, for a Hopf bifurcation which tells you that the real part of sigma has to be equal to 0 and this necessarily implies that the trace of the matrix, I mean the Jacobian has to be equal to 0 and this will satisfy that there is no real part. This is the criteria for no real part that the trace of the matrix should be equal to 0 and of course, a stability criterion tells you from here also you can get that the trace of the matrix has to be greater than 0, this is the stability criteria we all get it.

What about saddle point? So, for saddle point, the imaginary component of this Eigen values has to be 0. So, no imaginary part and how is that possible? So, for the necessary condition is that or no imaginary part that the trace of the matrix should be greater than 4 times of the determinant of the Jacobian. So, this is the necessary condition, and of course, the sufficient condition is the determinant of the J is equal to 0. So, this is the sufficient condition.

Of course, the sigma 1 comma 2 can be evaluated as, this is something we get the Padmana Siddartha Acharya’s formula for quadratic equation and from there you can easily relate out that if the trace is equal to 0 then we have a Hopf bifurcation and if the determinant is equal to 0, then we definitely have a saddle point. So, that is these are sufficient conditions for to achieve or whether we are going to have a Hopf bifurcation or whether we are landing to a saddle point.

So, before we move to the next part on this spatial temporal system, so far we have been talking about temporal systems and dynamics and stability of temporal systems, but there is also a part related to the spatial stability and we will talk about that in the sense of distributed parameter system. I also want you to try a small problem as a simple work at home exercise something like a Predator-Prey model.



So, this is a very popular model of process variables let us say, you have two process variables as y1 and y2 and they vary in this way so, these a b k l all these are sort of constant. So, you need to examine I mean the work at home exercises that please examine the steady state of this system sorry examine the steady state and assess or analyze its stability.

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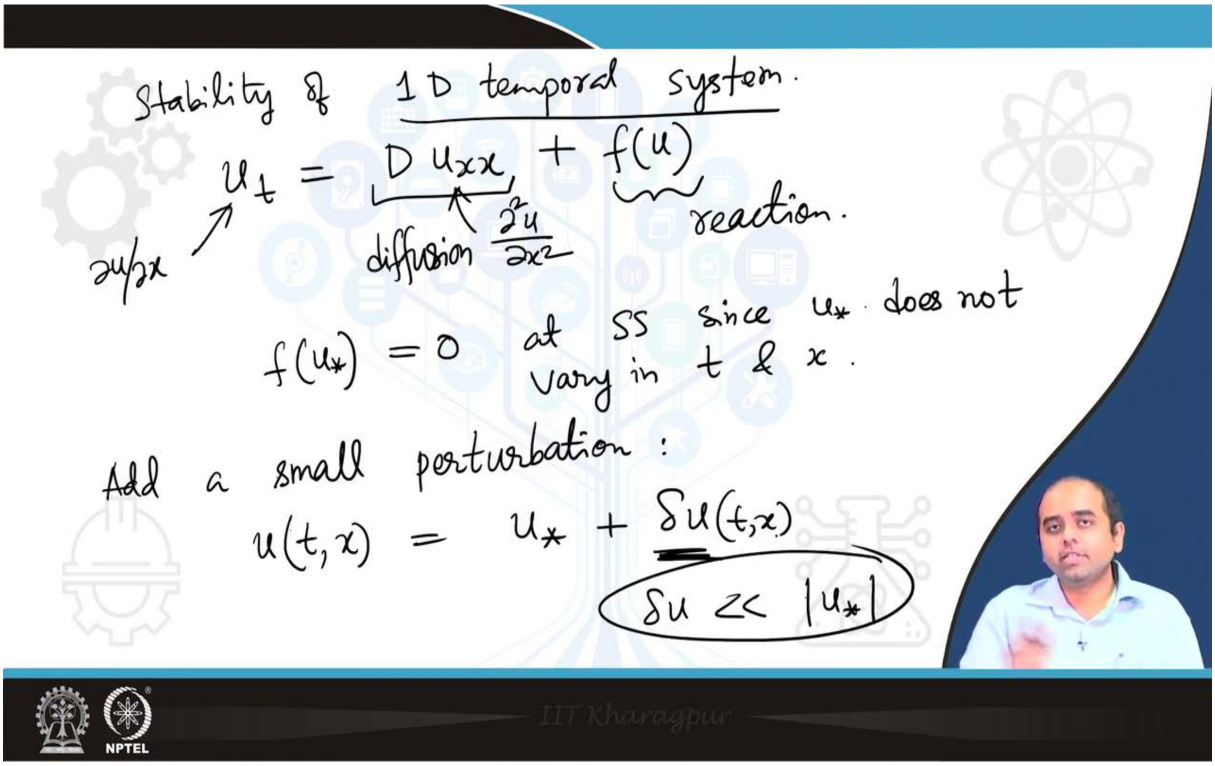

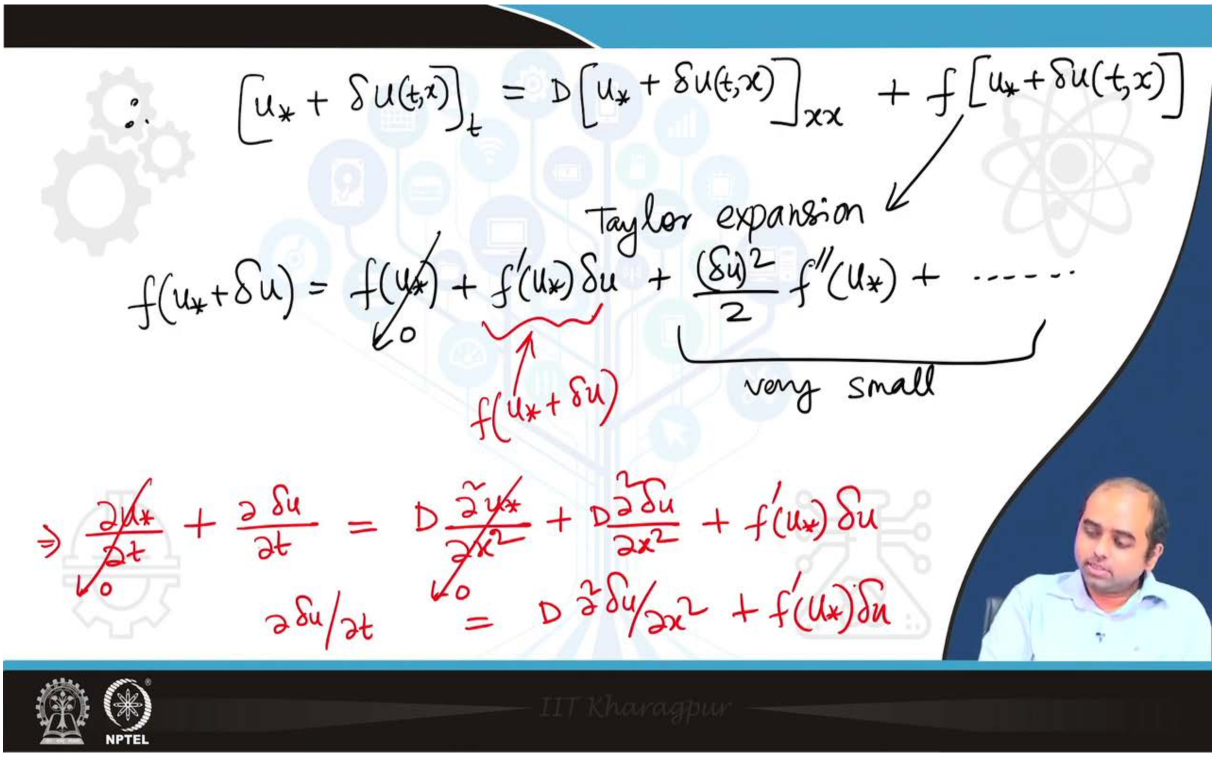

Next we move to the case of you know stability for let us say 1 dimensional transient system stability of 1D temporal system, let us say it is sort of a reaction diffusion problem without any conviction. So, you can easily understand this is the U subscript t is del U del t and this is del2 U by del x square.

So, this is the diffusion term the first term on the right hand side is corresponding to diffusion and this is a sort of reaction term. So, the first step to identify this analysis of stability is to identify the steady state condition and we consider that steady state the process variable in this case it is U, I mean you can relate this to a concentration of solid concentration does not change time means and space. So, at the steady state condition the process variable U does not change, does not vary with time and space both.

So, if that is the case, so, then this f of U star, where U start is the steady state value of the process variable is also equal to 0, at steady state since, U star does not, does not vary in time and space. So, either you could be a single variable it could be a multiple component or like a vector and then for each of the cases you need to do this solution separately. Now, add a small perturbation inspiration is same as in the case of the previous part of the purely dynamical systems and here we are trying to work with spatial temporal systems.

So, in this case whatever we are going to have this u t sorry t is written down as something like this this is a small perturbation part where we say that delta U is much much smaller than U star. This is a small perturbation we added to the, to the state variable that is U. So, what does it imply that at steady state if I am going to evaluate, I mean substituting it back into the main equation, this is what we have substituting it back into the main equation.

Now, please note that this last term we can do a Taylor expansion, so, what does the Taylor expansion tell us that f of U star plus delta U is equal to f of U star plus f star delta U plus delta U whole square by 2 f double dot U star and so on so, if you only consider the terms I mean delta since delta this perturbation itself is very small.

So, these components are very small to be ignored. So, we only consider the first two term the after the first two term, so, this one is 0 at steady state. So, this f, I mean this component here I mean everything is replaced by this so, this is the value of f at this U star plus delta U condition.

So, what we get here so, please note that if you try to do the partial derivatives of your, this U star with respect to t. So, if I write something like this, this is straight away equal to 0 because of steady state so, I have to write these exist then on the right hand side the first one of course, again is 0. So, we have this like this and the last term is f prime U star delta U.

So, what we have this equation now we have like this, so, this is again we get a PDE in terms of the perturbation variable or the disturbance variable delta U.

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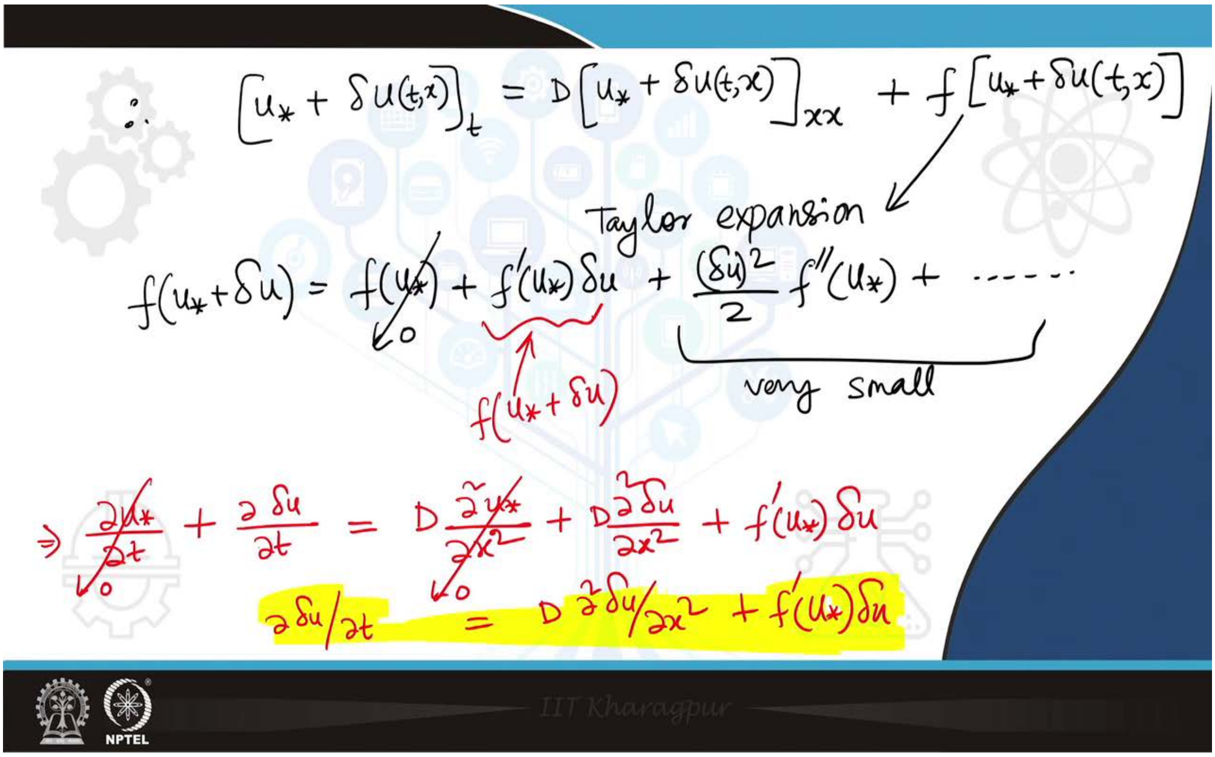

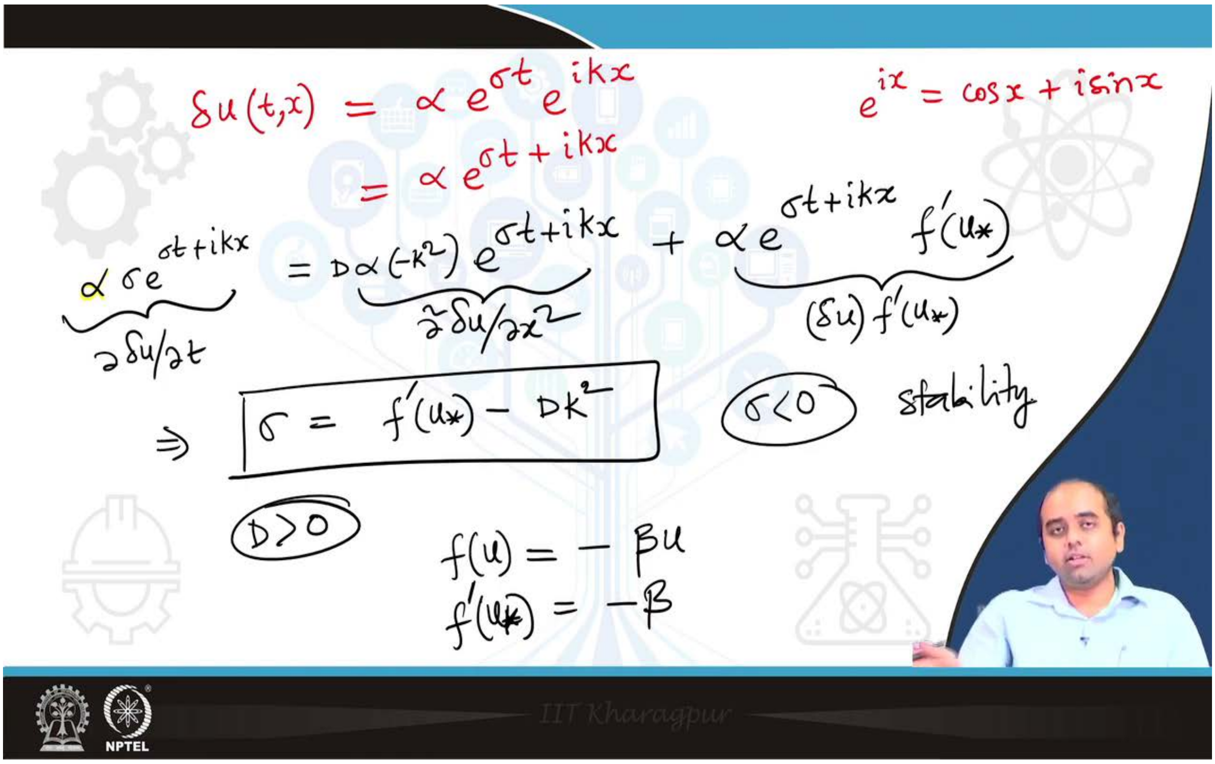

Now, consider this delta u which is a function of both space and time. So, this is a both a time as well as spatial disturbance. So, it is I mean the space it is the, I mean the time part is e to the power sigma t, but for the space part it is represented as a form of the Fourier series component.

So, e to the power ix can be converted to a I mean e to the power ix is nothing but cos x plus i sin x, so, this is a combination or the Fourier transform of the disturbance in the space variable and that is the reason why we choose them in the form of i k x. Where, k is the wave number for the spatial disturbance.

So, with this we substitute back to our, this perturbation equation if we choose that. So, if you if you try to choose or you try to substitute the delta U in the form of the exponential sigma t plus i k x, what you get is, let us write that down.

So, on the left hand side you have this on the left hand side you have alpha sigma e to the power sigma t plus i k x, right hand side is D alpha minus k square, to the power sigma t plus i k x plus alpha e to the power sigma t plus i k x into f star f prime of U star. So, this is d delta u by dt this part is the spatial derivative or the diffusion term and this is the reaction term.

So, that is delta U f star U dash. So, on both sides I can cancel out e to the power sigma t plus ikx and this gives me the resultant equation as sigma is equal to f prime U star minus Dk square. So, please note here that for generally for this sort of diffusion problems D is always positive. D is always positive as diffusion cannot be negative.

So, in the absence of any chemical reaction any spatial system or spatio temporal systems or distributed systems is inherently sustable and the more rapidly the perturbations varies spatially the larger is the wave number. The large is the wave number this k the shorter will be the wavelength and more negative would be this Dk square. So, higher wave numbers of the perturbations are less likely to cause a disturbance because it is minus Dk square.

So, if k so, if k increases, so, what does this k mean, it is the different modes of the perturbations in the Fourier space domain. So, if for higher mode of disturbances or disturbance having higher modes are generally not likely to cause an issue in the instability it is the lower modes that create a problem for smaller values of k because it is always has a negative sign, in general first order systems, I mean first order reaction systems, where we can write f U is equal to like something like this, f U is equal to minus Beta U, if f is something like that, then f of U star is equal to minus Beta and Beta could be a rate constant. So, that will be positive. So, even first order reaction systems are also stable, interesting thing comes when we start to have the second order systems. So, in the case of second order higher order systems a f of U could be negative sorry, could be a, could also have that f prime U sorry this is f prime, this is this one. So, f prime could have the concentration in that and so, the stability would also depend on the value of the concentration at the steady state condition is not it? If it is a second order system.

So, these are some of the important interesting consequences of the analysis from the stability in spatio temporal domain and of course, when whenever we talk about the spatial domain the wave number plays a very critical role here and higher modes of the perturbations generally do not cause instability, but it is the lower mode of the perturbations or when k is smaller that creates a stability criteria. So, you need to find out I mean different wave numbers can have different their amplitudes.

So, that is something you need to work out and find out that for different amplitudes of this wave number which is the dominant mode of the instability? and generally if that is stabilized, the rest of the spatial fluctuations or the oscillations in space will also be suppressed.

So, the key conclusion or the key take home message is that for simple diffusion systems, things are inherently stable in spatially as well as temporally. But for reaction systems, it depends on the order of the reaction as well as the reaction rate equation.

So, any this sort of reaction equation in the picture, you need to work out the, this particular value of your sigma and tell us that whether it is negative or positive. So, here also the criteria is same that for stability sigma needs to be less than 0, then only the stability happen you cannot only suppress the spatial oscillations the temporal oscillations also need to be suppressed.

So, I hope all of you get a clearer idea about the stability problem, both in the lumped system as well as in the distributed systems. There are several examples of spatial fluctuation and spatial stability issues. For example, when you have this, this problem of Saffman Taylor instability in a, when that is the interfacial instability between low viscous liquid trying to push through a high viscous medium the interface is often unstable.

So, that instability also changes both in space and time and you can clearly see the different modes of this instability in the space domain. And there are also several other examples, both that happened in the physical world as well as something that you generally also can encounter in special situations. I hope all of you have liked the lecture today. I hope you find it useful. Thank you.