Mathematical Modelling and Simulation of Chemical Engineering Process
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Lecture 18
Matched Asymptotics

Hello. Welcome to this class on the Asymptotics. Today, we are going to learn about the method of Matched Asymptotics. So, generally the method of constrained coordinates that is what we have just done in the last lecture, is not applicable or it is not capable of producing a uniform valid expansion where there is a sharp change in the dependent variables.

So, the problem generally comes when you encounter a sudden change or a sharp change in one of the dependent variables. And so tackle this problem what you generally need is that you need to have a sort of scaling or sort of magnified scaling in the region where you have sharp change and then you have a outer solution.

So, we segregate the solution into two part of the domain, one which is called generally as the inner solution or where we have a different scaling compared to what we are a scale of or a magnified scale system. And there is another portion which is known as the outer solution, where we have generally a different scaling of the coordinates.

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So, let us move ahead with the class today. And I will be as you know I will be talking about this idea of matched asymptotics or the condition under which these two types of solution in two different domain can essentially be matched together at their inner and outer limits. And of course, this involves the idea of the inner and the outer solutions. The inner solution is basically based on the idea of inflated or a magnified scale where you expect a sharp change in one of the variables. In both the cases, we will apply this idea of the asymptotics or the perturbation method.

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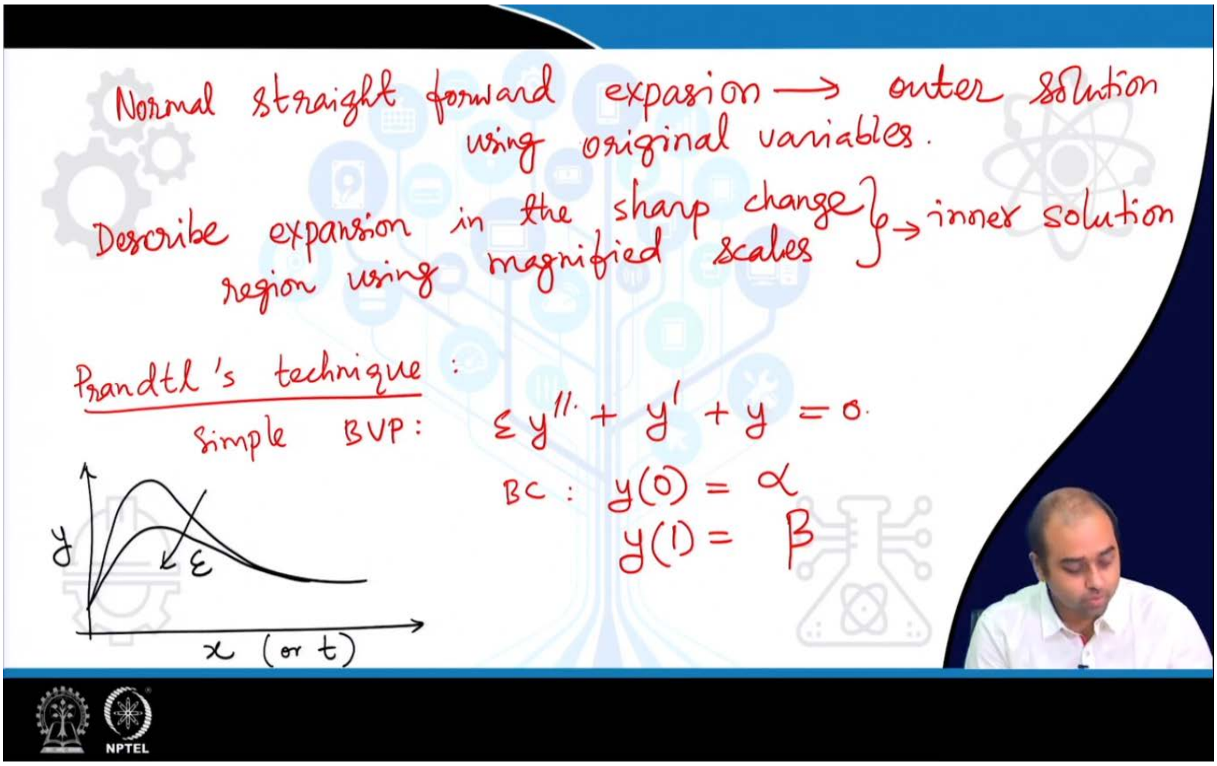

So, just to write down the idea is that straightforward, normal straightforward expansion, we will have in the outer domain. So, this idea or this concept of outer and inner is very important. And generally the outer solution is called or the part of the solution where we do not have sharp change in the variable or there is a slow change, slow change could be change with respect to time, with respect to space or gradual change that is what generally is described as the outer solution.

And using the, of course, the forward expansion using original variables, I mean non-modified and non-transformed variables. But, we describe expansion in the sharp change region of the variable, the sharp change region, using magnified scales. So, we have two different kind of these scalings for the inner and the outer solution. And they should be matched at each of their inner and outer limits. So, we will talk about them or we will see the mathematical condition for that.

Now, you can argue that or we can, of course, think that to have this idea of matched asymptotes, we may need to have apriory idea of the solution. So, now that is of course true but that is to some extent intuitive, in the sense that if you, let us see if you are solving a velocity profile, you do expect the velocity profile to have a sharp change near the boundary, because of the boundary layer effects. Let us say you are talking about the thermal convection, then again near the boundaries, you do expect a sharp change in the temperature profile. So, there you need or there you need a different approach of the asymptotics, if you are approaching asymptotic solution compared to the straightforward expansion.

So, these are some of the cases where you would be having an idea on the when to choose these inner solutions or which zone would be the zone of the magnified scales. So, generally, if you are expecting some dirichlet type boundary conditions or a mixed boundary condition, do expect a sharp change in the variable there.

I mean, this is true for a numerical solution, for any solution or if you try to plot the profile of any solution, you will see that wherever there is a dirichlet condition, there is an expectation that the solution will change sharp from that point. So, let us and this this method of matched asymptotics was originally introduced by Prandtl, who is also a famous mathematician as well as a physician. I mean as a physicist. So, this is very nicely applicable for fluid boundary problems.

So, let us say a boundary, simple boundary value problem as like this. So, y prime, y double prime are the single and the double derivatives. So, the boundary conditions we have at x is equal to 0, it is alpha and at x is going to 1, we have beta. So, in both cases, we have the dirichlet type boundary condition.

So, the solution profile, let me draw it properly, the solution profile looks something like this. So, this is y versus x and as epsilon decreases, this solution has a different behavior. So, we can easily see that there is a zone close to x is equal to 0, either x or t whatever, goes to x is equal to 0, the solution changes very sharply. And there is a gradual change after some time interval.

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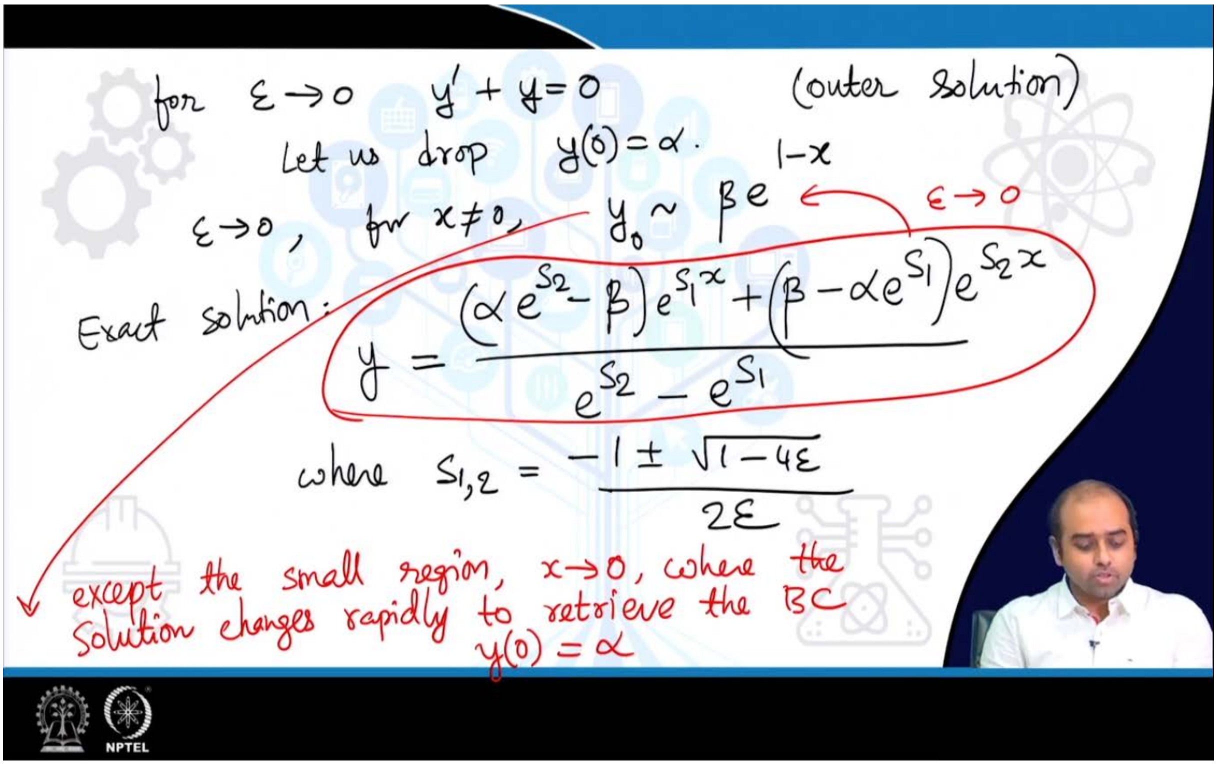

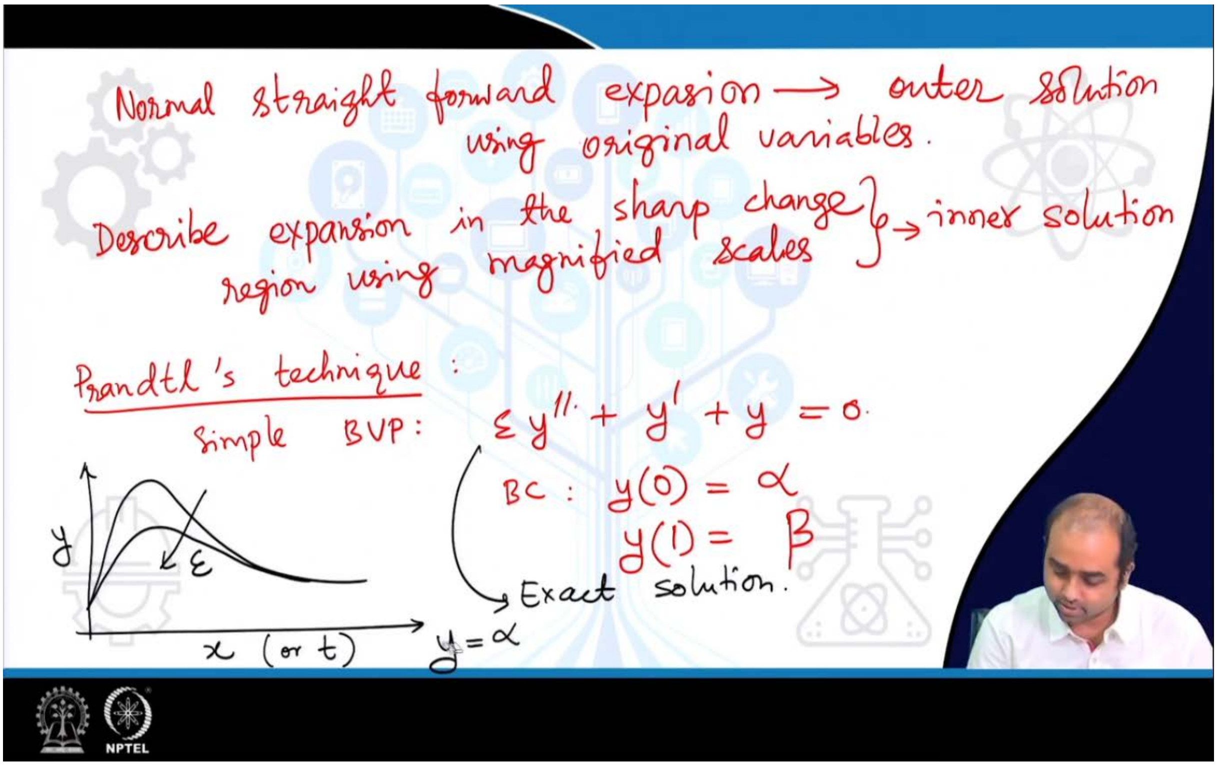

So, for epsilon tending to 0, you have y prime plus y is equal to 0. So, this is a solution which is valid when this epsilon is small or epsilon, epsilon is let us say setting to 0 and this represent the outer solution. So, this represent the outer solution. And in this case let us drop, so, this is the first order system, so, let us drop the boundary condition, y is equal to 0 is equal to alpha, this boundary condition let us drop. So, for epsilon equal to 0, we have the solution as y is equal to beta e to the power 1 minus x, this is the solution.

And of course, this is the first order solution. There is I mean I can write this to be as the outer solution. And this is also sort of the zeroth or solution to the problem also. So, this is the solution of the reduced equation in the sense that when epsilon is 0 and epsilon is essentially it matters, I mean, the term related to the epsilon matters or the second order term in this problem matters for a very small zone and for rest of the part of the problem, you can very nicely have this exponential decay.

So, from the solution this is the solution or approximate solution to this problem which does not capture the effects close to the boundary x is equal to 0. This boundary, this solution very nicely satisfies the condition at x is equal to 1, but it does not satisfy the boundary condition, as x is equal to 0. So, this is the solution or the reduced or simplified solution of the reduced equation which does not capture the effects near the wall or near x is equal to 0.

And if you just try to compare the so, for this equation, this is a simple equation, is a simple linear equation and you can have an exact solution. So, let us say the exact solution I can write to this equation, I will write it, try to write it here, maybe not here, this there is no not enough space. So, I will write the exact solution.

So, exact analytical solution is possible in this case and that can be used to compare the results. So, where s1 and s2 can be written like this. So, all of how we can solve linear ODEs. Now if you see this exact solution, you will see that for epsilon very small or small epsilon in the limit of epsilon close to 0, these exact solutions will be approaching this solution.

So, the exact solution will be approaching when epsilon tends to 0. So, the final solution also approaches to this outer solution, I mean to this solution when epsilon tends to 0 and this also tells us that this, the behavior of the solution or the behavior of the y profile close to the boundary x is equal to 0, is not captured by this simplified equation.

And the solution close to x is equal to 0 is different from this setting this parameter epsilon close to 0. So, it means that epsilon parameter or the role of this term which contains the epsilon parameter is important or that contributes the solution only to the part close to the wall. So, for close to the wall problem, we have a different this approach.

So, except in this region, x is equal to 0, where the solution, so, except this solution is valid everywhere, except for the zone or except the small region when x approaches to 0, where the solution changes quickly or rapidly I would say, to retrieve the boundary condition that at y, x is equal to 0 is equal to alpha. And this is a very common phenomena that is observed in fluid mechanics or any transport phenomena problems due to the presence of the boundary layer effects.

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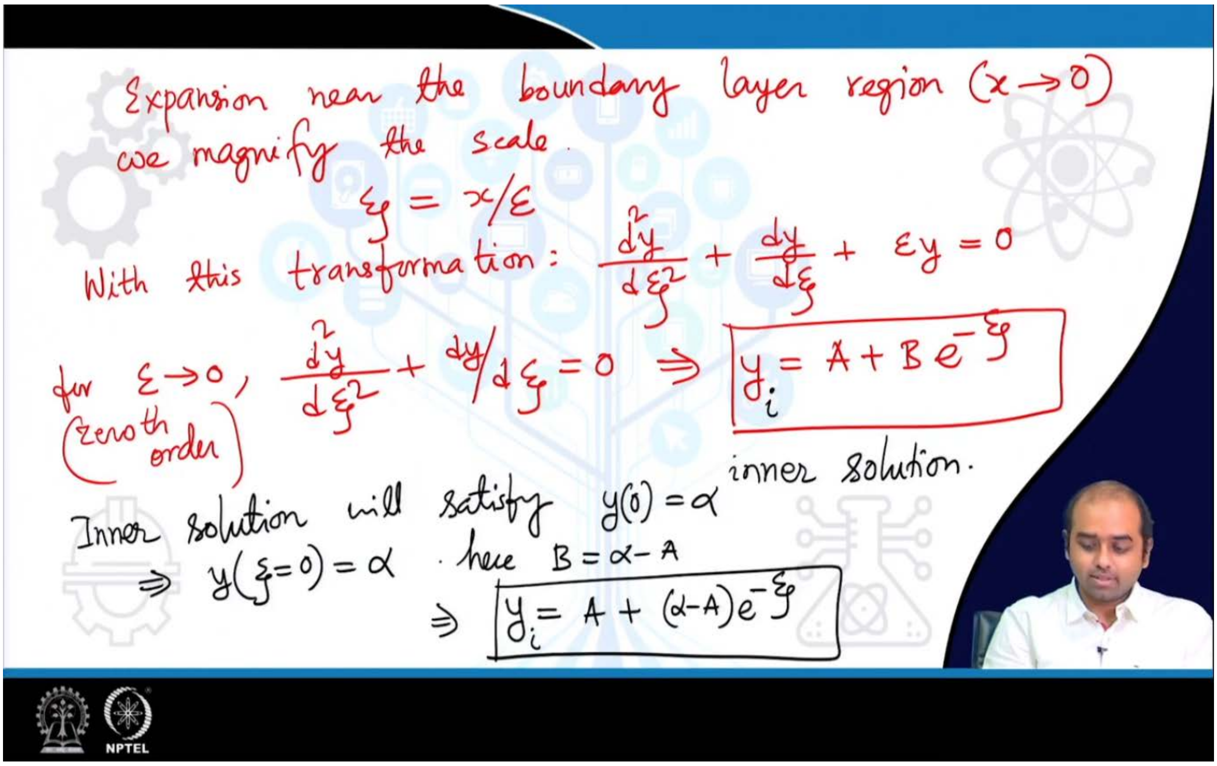

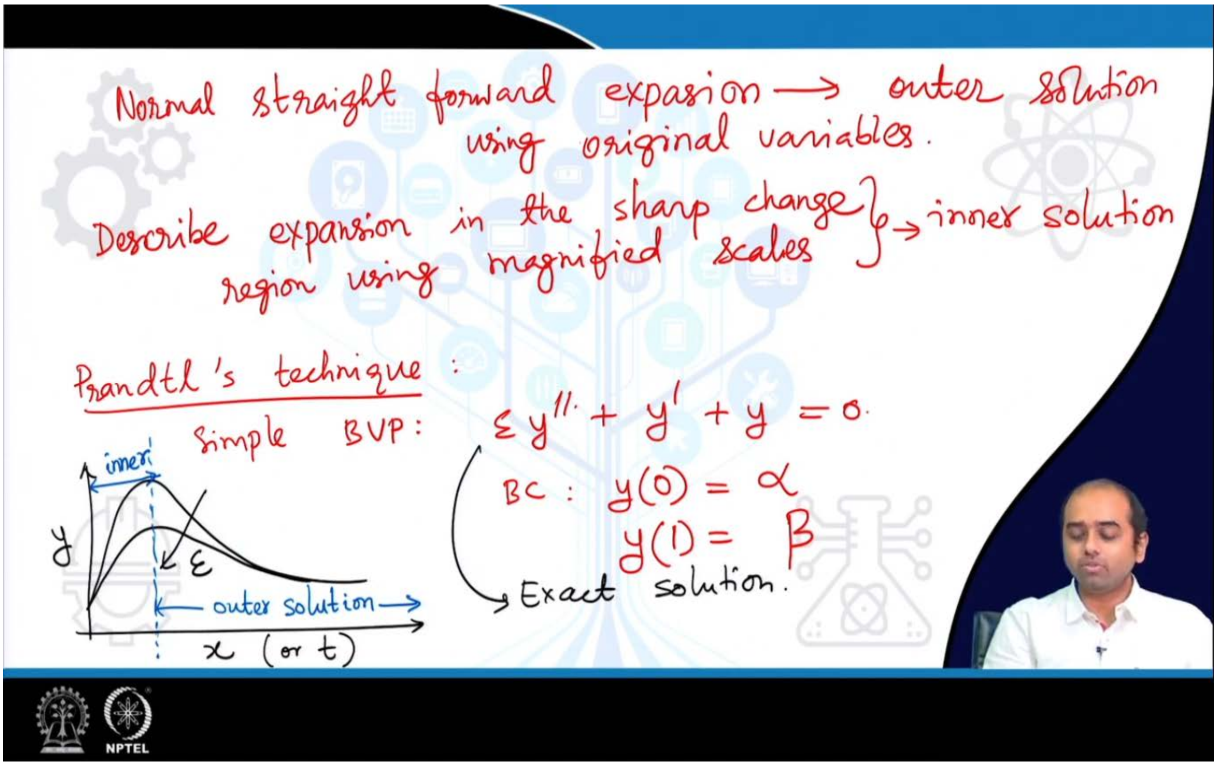

So, near this boundary region we are doing an expansion, expansion near the boundary layer region. I mean this x close to 0. In that case, x is very small, so, we magnify the scale as let us consider a new parameter xi as x by epsilon. And this this epsilon which plays a major role in determining the solution close to the wall.

So, with this transformation the modified ODE, this equation turns out to be like this. This is not zeta is epsilon. So, in this problem epsilon close to 0, so, if I set my epsilon close to 0 in this transformed equation, I will be getting d2y d zeta square dy dzeta is equal to 0. This is the case. And the solution to this problem can be written down as A B e to the power minus zeta.

So, in both cases we are considering this zeroth order solutions. In the previous case also we did the zeroth order solution and in this case also we are doing the zeroth order solution. I mean and you will see at the end that zeroth order solution is good enough to approximate the final solution. But only the situation here is that we are treating the two zones of the solution in two different type of scalings.

In the case of the outer solution there is the normal scale because x is quite large but in this problem when x is very small, we try to magnify it with the sort of this new transformed independent variable like x, x by epsilon. And now we see this new equation where we get to get the zeroth order solution, we can obtain it by setting this epsilon to be close to 0 that is the zeroth order solution.

So, in both these cases, we are getting the zeroth order solution. So, this part of the problem is the inner solution to this equation. So, that is the reason I am substituting this I mean instead of writing as y, I am writing that as yi. So, this is the inner solution. Of course, this inner solution, we can use the boundary condition at x is equal to 0. So, it will of course satisfy.

So, inner solution, inner solution will satisfy the boundary condition as y 0, x is this is equal to alpha. So, which means that y zeta is equal to 0, you have this as alpha. So, hence B is equal to alpha minus A. And this will give you the solution y is equal to A plus alpha minus A, where A is of course still an unknown. And let us call this as the inner solution.

Now, from this curve of the solution, I think all of you now have a fair idea that this is the part, so of the something known as the outer solution that we are dealing with, where there is a change of the gradual change of the variable and this is the part close to the wall. This is the part close to the wall, where we are seeing as the inner solution.

Now, this inner part can be very small also and this is normally the case. And that is where we see a sharp change in the variable and you can also follow that the outer solution is a sharp decay, exponential decay. Now, how do you determine you are A? So, the idea is that this criterion of the matched asymptotes is applicable now, which tells you that the outer limit of the inner solution should be equal to the inner limit of the outer solution.

So, both of these solution at their respective edges, so, for the inner solution that edge is at the end or at the right exterior of this, sort of this zone or this dotted line, that is the outer limit of the inner solution. And again that part is also the inner limit of the outer solution. And of course since the final profile is continuous, these two had to be equal. So, the outer limit of the inner solution should match with the inner limit of the outer solution then only you will have a continuous profile.

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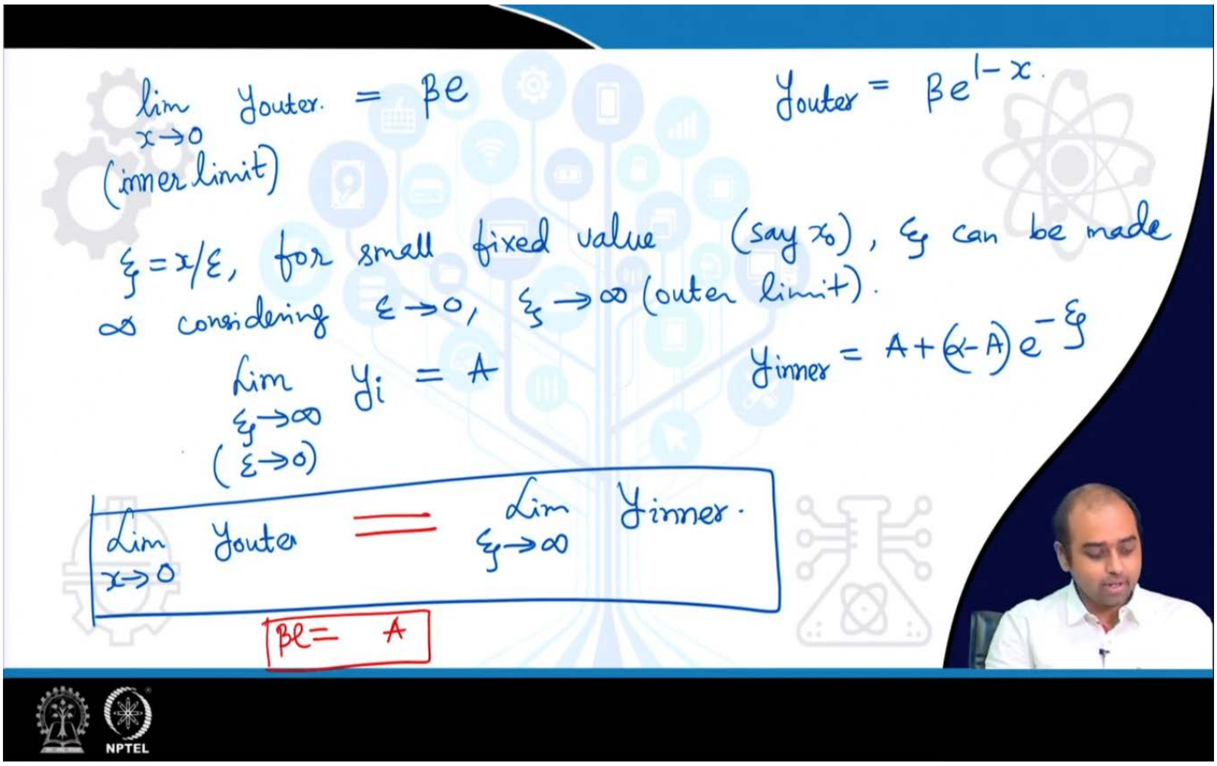

So, with this idea what we get is that the we try to calculate first the inner limit of the outer solution. So, the inner limit is obtained by setting x close to 0 or as you are moving close to the wall, for the outer solution that x close to wall is good enough to its starting value and this you can get as beta into e. So, remember, outer solution was beta e to the power 1 minus x.

Similarly, with this transformation for zeta is equal to x by epsilon, let us say the idea is that for any small fixed value say x 0, not s zero, x 0 zeta can be made infinity, considering epsilon close to 0. So, if I set my epsilon close to 0, for any finite value of x it is possible that I can set my zeta to infinity and this zeta setting to infinity is the outer limit condition of the inner solution.

So, you will reach the outer limit for the inner solution. So, what we are trying to do is that limit zeta tending to infinity, which is same as saying epsilon tending to 0, in this case, for yi will give you A. And remember that y inner solution was A plus alpha minus A e to the power minus zeta. And these two must match; these two limiting conditions must match.

So, this limit of x tending to 0 for the outer solution should be equal to limit zeta tending to infinity of the inner solution. This has to match. Let me write this in much bolder way. So, this has to match. So, x tending to 0 of the outer solution represents the inner limit or the inner edge of the outer solution.

Similarly, for the inner solution, it is operated in the transformed independent variable which is zeta and zeta tending to infinity is like almost reaching the edge of the inner solution. So, it is the outer limit of the inner solution. This two has to match. So, this tells you that your beta e is equal to A. So, this is how you can calculate A.

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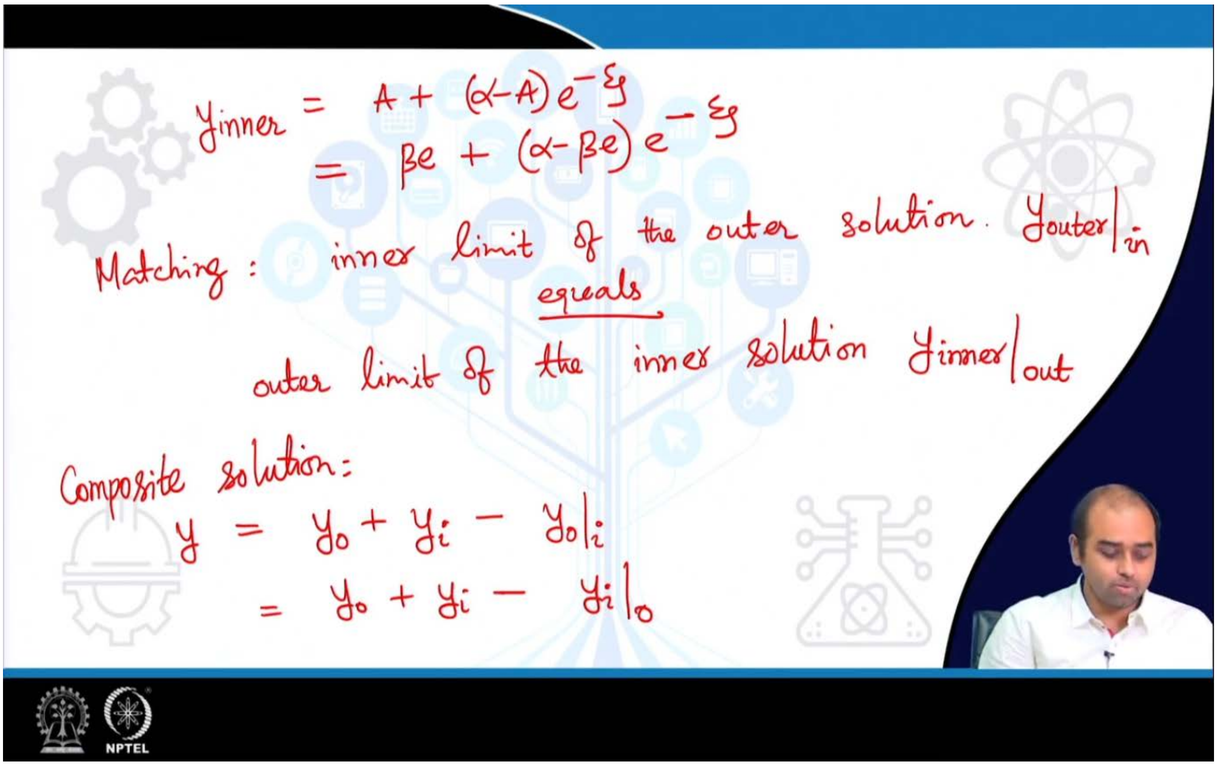

So, once you calculate A, this inner solution whatever we have is equal to A plus, so, inner solution now becomes completely defined. So, you know your a. So, it is beta e plus alpha minus beta e, e to the power xi. So, what is the criteria of the matching that we just wrote down and this is something that I want you to remember always that the inner limit of the outer solution, let us call that as like this equals the outer limit of the inner solution.

So, we call this as y inner but the outer limit. So, this is the matching criteria that needs to be satisfied for a continuous solution. So, now what how do you write the complete solution? So, the composite solution stands as, let us not write yc, just the composite final solution, the outer solution plus the inner solution.

And now since both of them has the condition that satisfies their inner and, their respective inner and respective outer limits, this should be subtracted with either of them, either the inner limit of the outer solution or you can also write them to be the inner limit of the outer solution. So, please, note that these two are same. And of course this composite solution, you can tell clearly that these also represent, I mean the inner limit and the outer limits of the composite solution, does satisfy the respective inner and outer limits. How?

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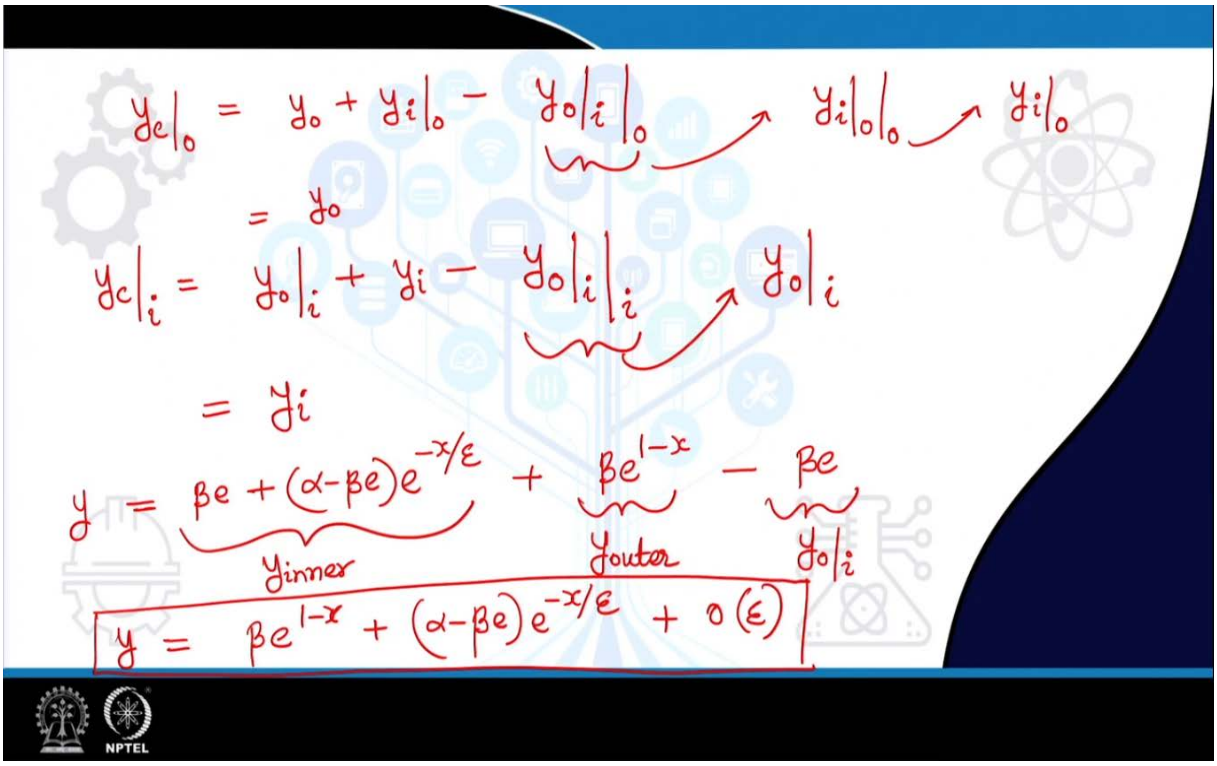

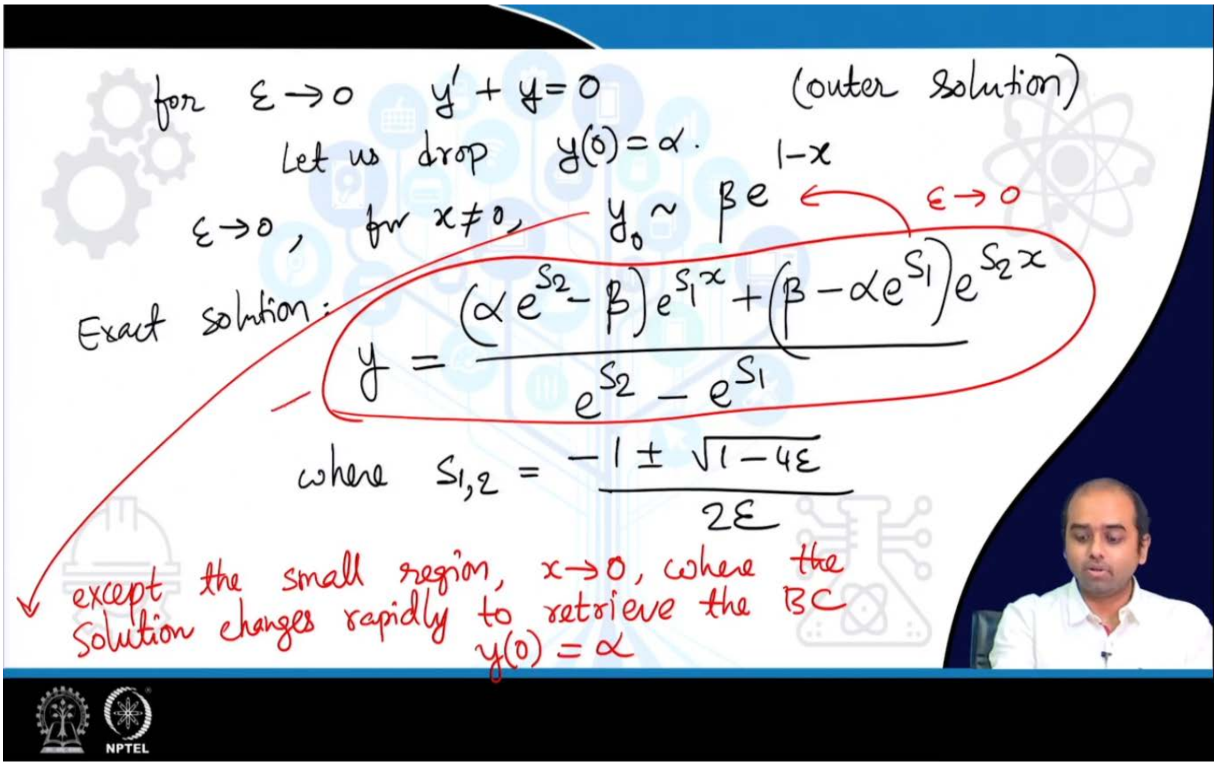

So, this composite solution, let me write this as y c. So, it will not confuse you. So, if you take the this outer limit of the composite solution, you get something like this minus y 0 i, i, and this quantity can also be represented as y i 0 and then I can also write it in this way. So, this ultimately means nothing but y i 0. So, which these two will cancel out.

Similarly, I can also write the inner solution as y 0 i plus y i minus y 0 i i and this we can be represented as y i 0. This is nothing but can be represented as y 0 i. So, these two will cancel out and this will represent y i. So, sorry just a correction, I mean, this outer solution of this this first one I made it wrong here.

So, this is 0 here. So, which means this will be 0 and this is 0. So, this is the respective inner and outer solution of this composite form is nothing but will solve their respective values. So, the composite solution or the final solution y is equal to beta e plus alpha minus beta e, e to the power minus zeta and let me write zeta as x by epsilon.

So, this part is the inner solution and beta e to the power minus x. So, this is the outer solution and minus beta e, this is the inner limit of the outer solution or the outer limit of the inner solution. So, you just can, you can easily understand this beta e beta e can be cancelled out. So, this can be written down very easily something like this. So, this is just based on the zeroth order.

So, there will be additional, equal additional terms due to that may be present due to the higher orders of your epsilon. So, this is just to compare having the inner and outer solution of the zeroth order form and if you want further refinement or further accuracy, you can go for the first order expansion of your epsilon and then do the matching of the first order terms.

So, this is so far we have done the matching of the zeroth order terms of the inner and outer solution. You can also do a similar matching of the inner and the outer solutions corresponding to the first order solutions. So, this taking is generally quite helpful to solve turbulent boundary layer profiles past sphere or past body.

And you can see that this solution this simple analytical expression very closely approximates this exact solution. This exact solution that we have written down, can be very nicely matched by this approximate solution, even with the zeroth order condition. And of course further accuracy can go with the first order expansion.

So, I hope all of you had a nice idea on this technique of asymptotics or essentially the approximate solution methodology and these are very powerful techniques for analyzing or finding the solution of very complex systems and often where direct analytical solutions are very difficult or numerical solutions are very complex to get a fair idea of the the solution profile or to get a first hand information on the variation of the profile asymptotics.

And these perturbation techniques are quite useful. Also please, note that from the approximate relation, you can very well get a fair idea of the inverse problem. I mean no numerical solution will give you the inverse solution that if I want to get the value of y corresponding to, I mean, I want to get a value of x where the value of y will be something like this, this is the inverse problem.

Instead of having a solution y as a function of x, if I want to get x as a function of inverse of y, I mean, function inverse of y that sort of solutions can only be obtained from some analytical relations. And that is impossible to get in numerical solutions. So, these are some of the benefits even with the approximate solution and asymptotic techniques is one of the very powerful methods in analytical mathematics. I hope all of you like this lecture. Thank you.