Plant Design and Economics
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Lecture - 59
Linear Programming Problems

Welcome to lecture 59 of Plant Design and Economics. In this lecture, we will talk about linear programming problems. As of now we have talked about optimality criteria for unconstrained optimization problem, both single variable and multivariable optimization problem. We have also talked about equality constrained optimization problem. Today we will talk about linear programming problems.

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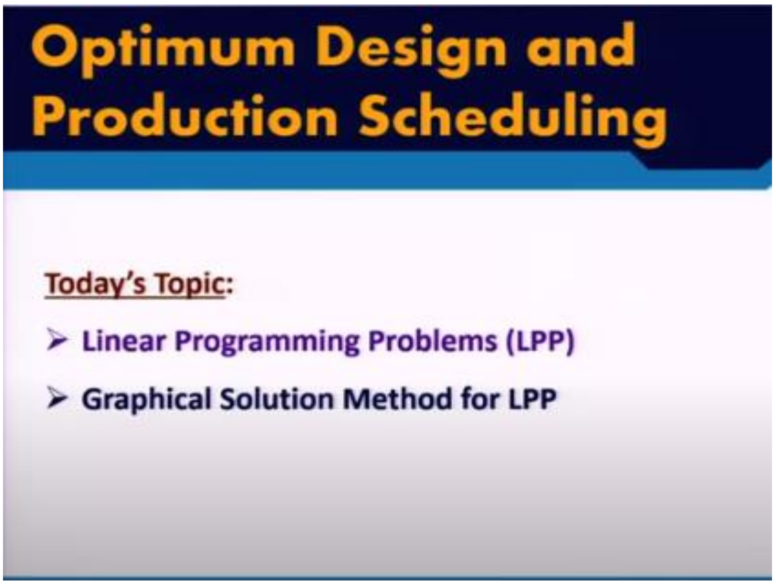

And we will also take some examples related to our plant design course. So we will first talk about how to formulate a linear programming problem. And then we will look at graphical solution method for linear programming problems. Note that since we will be using graphical solution method, this method is restricted to two variable problems only. Because beyond that, it will be very difficult to visualize.

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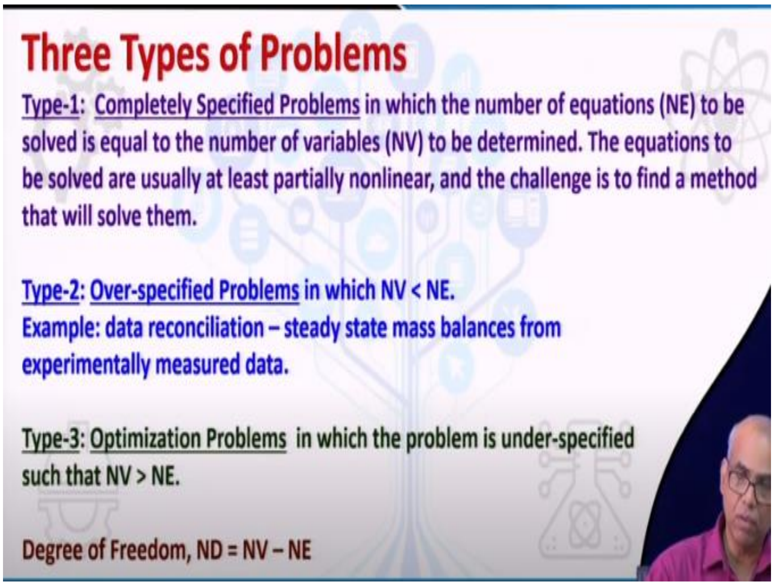

As chemical engineers, we encounter three types of problems while solving equations. So type one, they are completely specified problems in which the number of equations and number of variables are same, number of variables is equal to number of variables so that the equations to be solved are usually nonlinear. And the challenge will be here is to find a suitable method that solves them.

Second type of problems are over specified problems, where we have fewer variables than equations. So number of variables is less than number of equations. Such problems are common in experimental analysis. For example, suppose a steady state process is going on and you want to write down the mass balance equations. You want to experimentally verify.

So you measure say certain component florets, maybe. Now since these measurements will be in error, you will see that the mass balances are not exactly satisfied. So you have to do data reconciliation. So such problems will appear, such over specified problems will appear when we do experiments and take the experimental results and fit this into equations and so on and so forth.

Third type of problems are basically what we have been talking about in this module. So they are optimization problem in which the problem is underspecified such that number of variables is more than number of equations. And the degree of freedom is defined as the difference between number of variables and number of equations. So degree of freedom is equal to number of variables minus number of equations.

So for optimization problems, you must have degrees of freedom greater than zero. Note that if degrees of freedom equal to zero, there is no scope for optimization. So there will be scope for optimization only when degrees of freedom is greater than zero, meaning you have an unspecified problem such that number of variables is more than number of equations.

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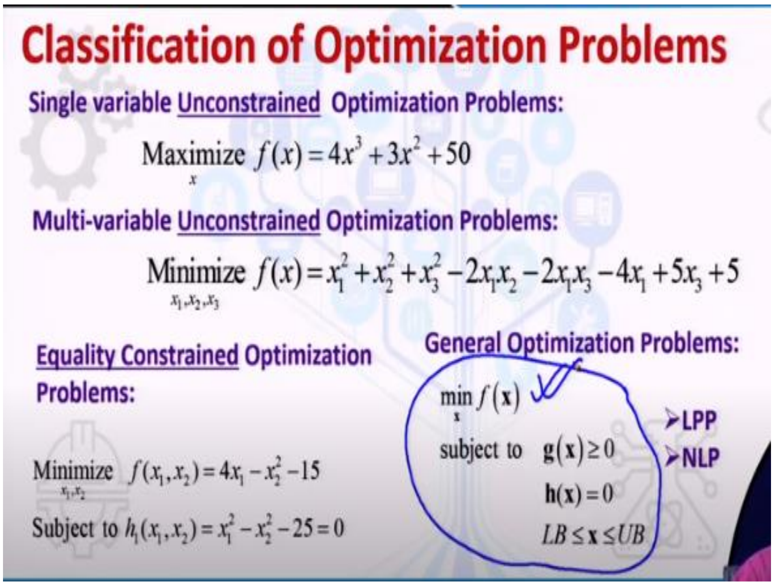

So we have talked about single variable unconstrained optimization problem. We have talked about multivariable unconstrained optimization problem. We also have talked about equality constrained optimization problems. And what you see now is a general formulation of an optimization problem, where your objective function may have n variables and then constraints maybe of both equality type as well as inequality type.

Note that constraints are also functions of those n decision variables. Now in a general optimization problem, we may have the objective function and or the constant functions all linear. We call that a nonlinear programming problem. Now in case of linear programming problem, our objective function will be linear and all the constraints will also be linear.

So for an optimization problem, when the objective function is linear, and all the constant functions are linear, we call that problem as linear programming problem or oftentimes it is called a linear program. So that is what we will discuss today.

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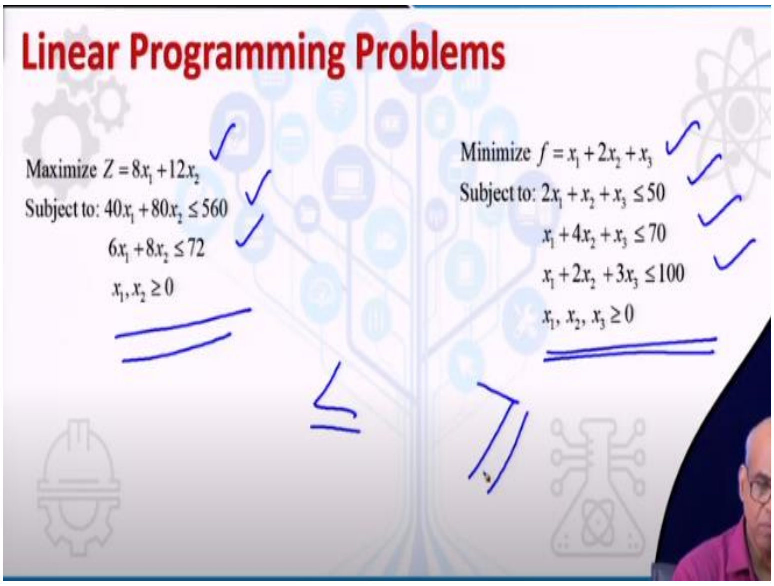

Note here your objective function as well as all the constraints are linear. So these two are examples of linear programming problems. While this has two decision variables, this has three decision variables. And it is not necessary all the inequalities will be of less or equal to type. You may have both greater or equal to type or less or equal to type.

Note that one is convertible to the other just by just taking negative sign. So multiply both sides by -1, your less or equal to will be converted to greater or equal to type.

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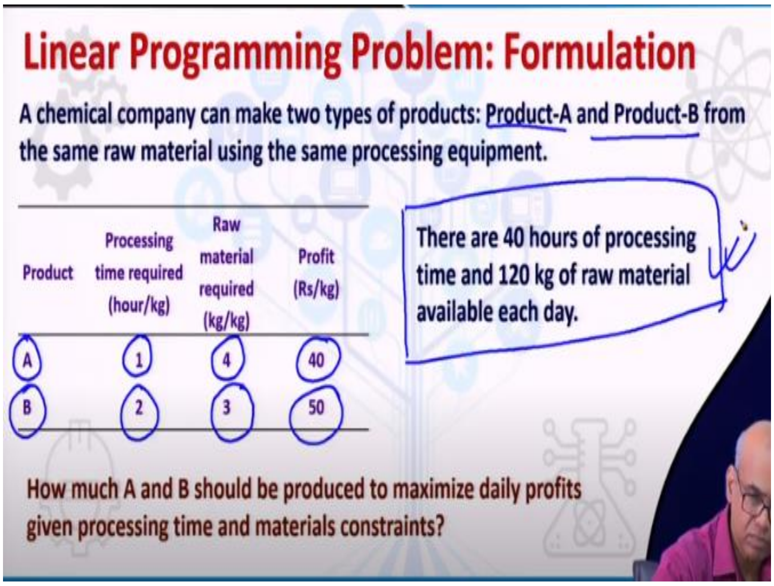

So let us take an example to understand the formulation and then subsequently solution of a linear programming problem by graphical method. So consider a chemical company which can make two types of products, product A and product B. From the same raw materials, use the same processing equipment. There are 40 hours of processing time and 120 kg of raw materials available each day.

Also the processing required time and raw materials required as well as profit are all given for products A and product B in this table. So how to read this table? Let us consider product A. The processing time required in units of hour per kg is 1 for product A whereas that for product B is 2. Similarly, raw materials required in the unit of kg raw materials per kg product A formed is 4. Similarly, that for B is 3.

Profit in the unit of rupees per kg product for A is 40 and that for B is 50. So this is how you have to interpret the data. Also there are 40 hours of processing time and 120 kg of raw materials available each day. So these are the constraints on the resources that I have. Now the question is how much A and how much B should be produced to maximize daily profits given processing time and material constraints.

So let us try to now formulate the problem. So to formulate the problem, we first have to decide what are my decision variables. Then we need to write the objective function which is basically an economic criteria and a function of these decision variables. And then, we will write the constrained function which are again functions of these decision variables. And as you understand that they will come from this constraints on resources.

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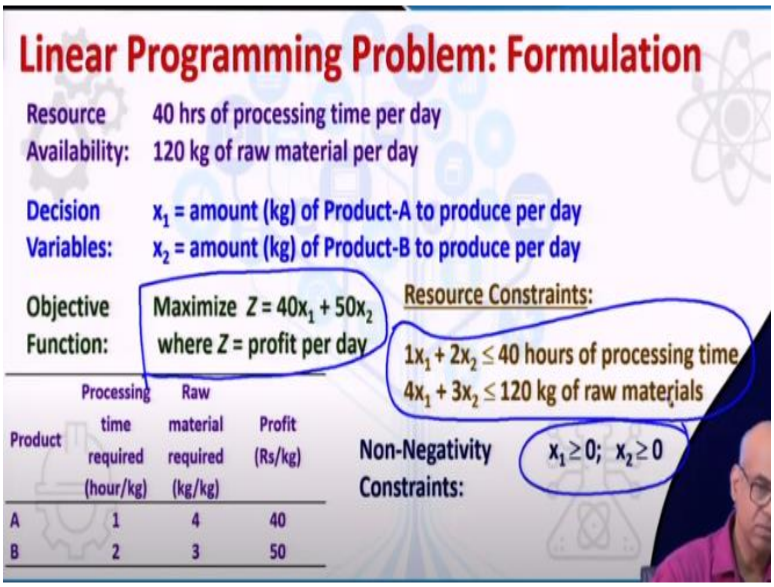

So now let us see. The resource availability it says that there are 40 hours of processing time and 120 kg of raw materials available each day. And we want to maximize daily profits. So you have to find out how much of A and how much of B should be produced so that we can maximize daily profit. So the natural choice for decision variables will be let x 1 be kg or amount of product A produced per day and x 2 equal to amount of product B to be produced per day.

So these are my decision variables, amount of product A to be produced per day, amount of product B to be produced per day. Now what should be my objective function? Objective function will be maximization of profit. So from the table we find out the profit in the unit of rupees per kg is for product A 40. So if x 1 kg of product A is produced per day, so profit that comes from product A is 40 into x 1.

Similarly, if x 2 is the amount of or kg of product B that is produced per day and the profit in the unit of rupees per kg for product B is 50 then the profit that comes from product B is 50 two x 2. So the total amount of profit per day will be 40x 1 + 50x 2. So I maximize a function in x 1 and x 2 which is Z = 40x 1 + 50x 2 where Z represents the profit per day. So we have to find out x 1 and x 2 such that Z = 40x 1 + 50x 2 is maximum.

Note that had it been an unconstrained problem, you would have chosen as high as possible for values of x 1 and x 2. But that is not the case here because you have constraints on resource availability which x 1 and x 2 must satisfy. So you cannot choose any value you want for x 1 and x 2 to make maximum Z. So let us now write down the constraints. First is 40 hours of processing time per day.

Now let us come to the table now. The processing time required in hour per kg for product A is 1 and that for product B is B sorry 2. So we are producing x 1 kg product A and x 2 kg product B. So that takes how much processing time? x 1 will take 1 into x 1 that means x 1 itself and B will take 2 into x 2. So the processing time required for x 1 amount of product A and x 2 amount of product B will be x 1 + 2x 2.

So that must be less or equal to 40 hours. So that gives me the first constrained equation. Next is 120 kg of raw materials available per day. Now let us come to table raw material required to produce 1 kg of product A is 4 kg and that for product B is 3 kg. So the raw materials required to produce x 1 kg product A is 4x 1 and x 2 kg product B is 3x 2. So 4x 1 + 3x 2 must be less or equal to 120.

So that gives me the second constrained equation. Is there anything else? So we have written down all the explicit constraints. Now there is some implicit constraints which are not stated clearly or which are not stated explicitly, but it is implicit in the problem. What is that? That is look at the decision variables. The decision variables are x 1 and x 2 which represents the amount or kg or mass of product A and product B.

So they cannot be negative. Note that they can be zero because you can say okay do not produce x 1, produce only x 2. That will maximize the profit, may be the case for some problem. So x 1 and x 2 must be greater or equal to zero. We call these non-negativity constraints or non-negativity restrictions on decision variables. So this is my problem formulation. What is my problem formulation then?

I have the objective function, I have the constraints, and I have the non-negativity restrictions. Look at the formulations, all are linear. So we have a linear programming problem in hand.

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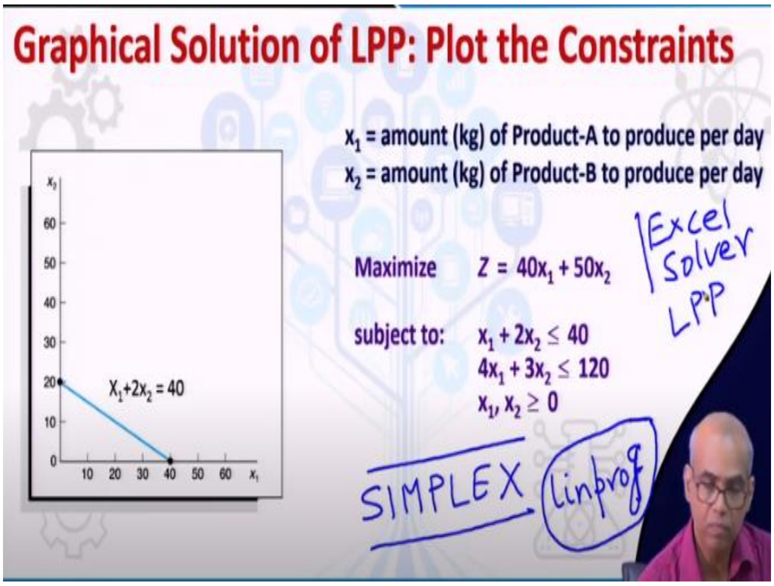

Now since it is a two variable problem, we can solve it using graphical method. Note that, if the problem is three variables or the problem has four variables or more number of variables, we cannot use graphical method for the simple reason that we cannot visualize it. But for two variable problems, graphical solution method is a powerful method and we can easily understand what is going on in the problem.

For general nonlinear programming problem for any number of variables, we have methods such as simplex method, which almost all softwares implement. But this is a rather much broader subject. So for the purpose of this class, we will restrict ourselves to graphical method of solution for linear programming problem with two variables. If you have access to MATLAB, there is a command called linprog, there is a function called linprog, which can be used to solve a linear programming problem.

Also the Excel solver, Microsoft Excel solver which comes loaded with your Excel but you have to activate it actually. So that also has linear programming problem solver. So you can also use Excel solver to solve linear programming problem.

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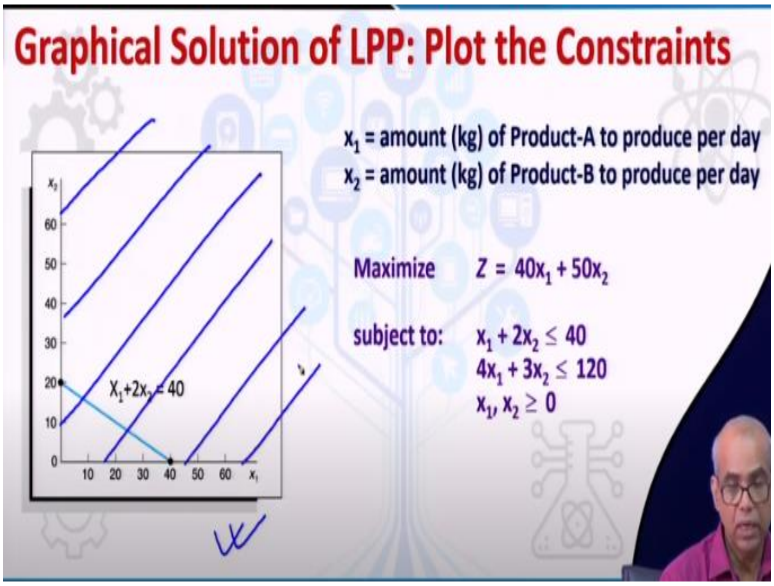

So now let us look at how to solve linear programming problem by graphical method. The first step is to plot the constraints. Before that of course, we consider x 1 and x 2 this axis, okay. Note that both x 1 and x 2 are greater or equal to zero. x 1 is my amount or kg of product A to be produced per day and x 2 is kg of product B to be produced per day. So both x 1 and x 2 are greater or equal to zero. So my region of interest is this where x 1 and x 2 are greater or equal to zero.

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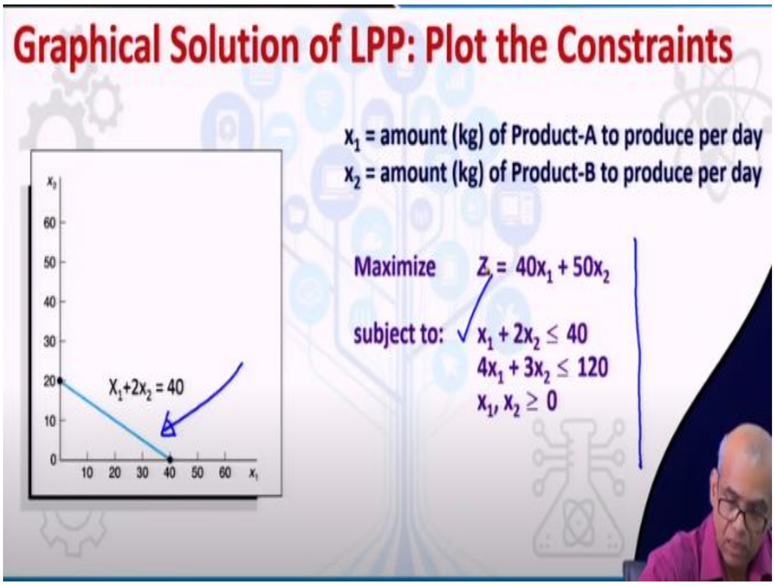

So this was the problem formulation. So first thing is to plot the constraints. So we will plot both the constraints x 1 + 2x 2 less or equal to 40 and 4x 1 + 3x 2 less or equal to 120. We will plot these constraints, which are of inequality type equation as equation. So basically we will plot x 1 + 2x 2 equal to 40 and 4x 1 + 3x 2 equal to 120. Note here the first constant x 1 + 2x 2 less or equal to 40 has been plotted and it has been plotted as equality.

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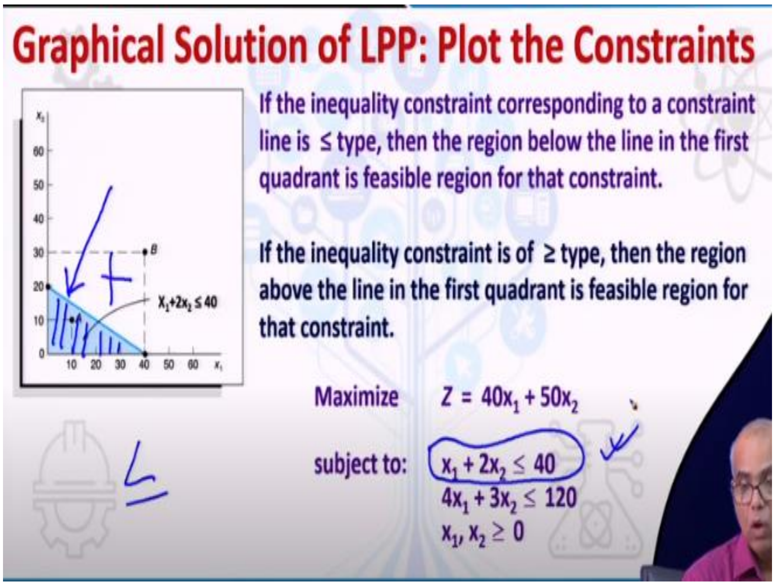

Now since the inequality is less or equal to type, then the region below the line which represents the equality in the first quadrant is the feasible region for that constraint. So first I plot this which represents the first constraint, plotted this as equality. And then since this is less or equal to type, the region below this line is feasible, outside is not feasible. So you must look for solution to the problem which lies below this line.

There may be points out on the other side of this line, which can give higher value of the objective function, but that will be an infeasible solution. That will not satisfy the constraint. Since this is less or equal to type, so the region below the line is my region of interest or because that is the feasible region. Similarly, if the inequality is greater or equal to type, we look for the region above the line.

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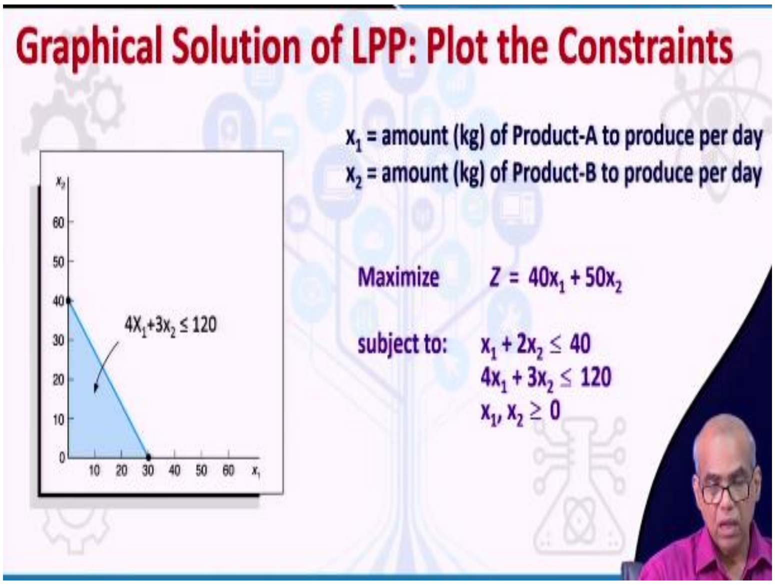

So now this way you write you plot the other constraint 4x 1 + 3x 2 is less or equal to 120. So again it has been plotted as equality.

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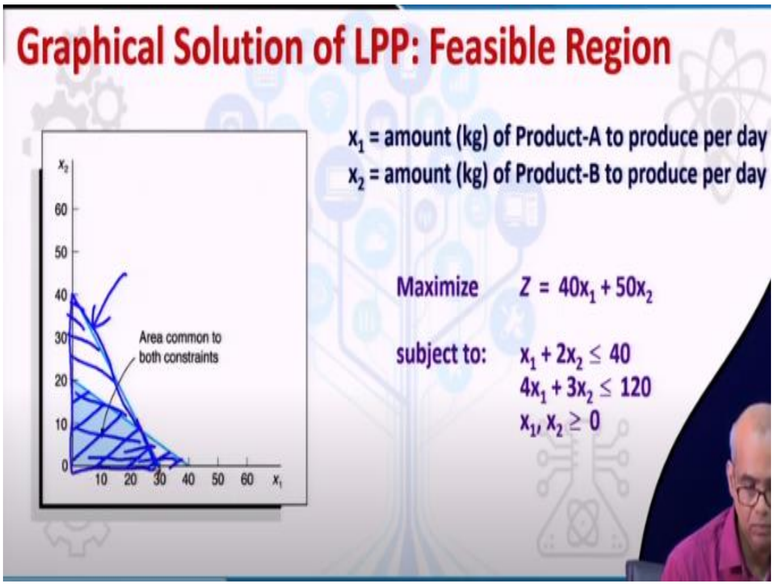

And you can now see how it looks like when you plot both. So now the area that is common to both constraints is highlighted. Note that the feasible region for this constraint is this part, which is shaded like this and for this constraint the entire region below this line that means, this up to this.

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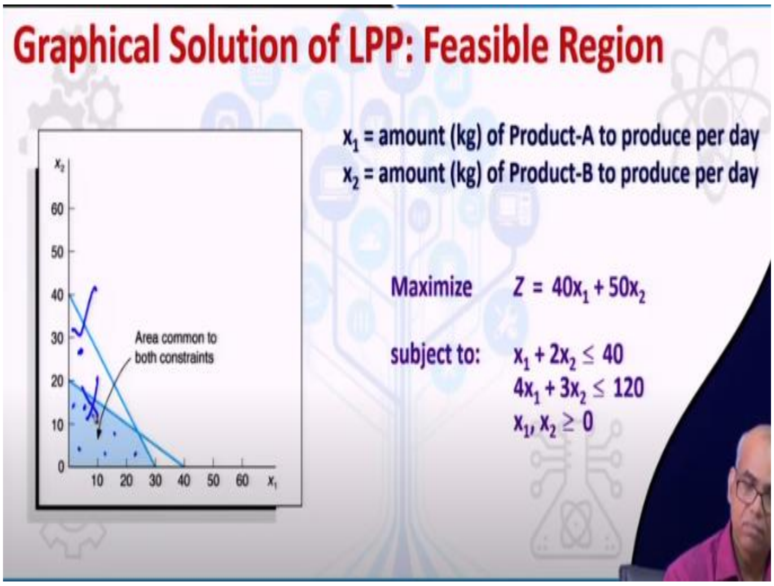

So the region which is feasible for both is the area which is common to both constraints and that is now shaded. So this is the feasible region for the linear programming problem, because this linear programming problem has two constraints and the region which is common to both the constraints is the feasible region. That means, any point within this feasible region will satisfy the constraints. See point here satisfy this, but does not satisfy this.

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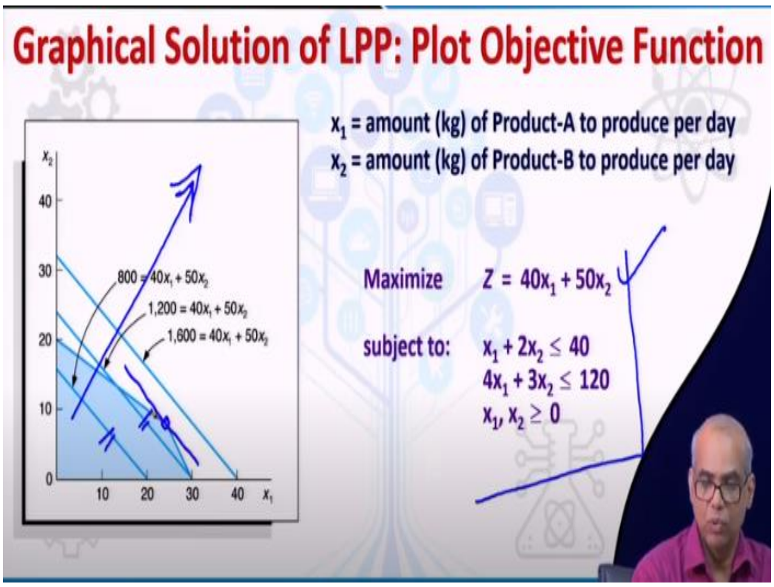

So once we have plotted the constants, now we are ready to solve the problem. So to do that, let us plot objective function line. We call it objective function contour. What is my objective function? Objective function is my Z = 40x 1 + 50x 2. Note that, this is all these parallel lines are objective function contour lines, objective function lines. So this is 800 = 40 x 1 + 50 x 2. What does it mean?

Any point on this line you take, read x 1 and x 2, put this here, they will give you 800. How will you plot this? So this is, this passes between this and this. Note that if you want to say 800 = 40x 1 + 50x 2, you pass it through x 1 = 20, x 2 = 0, and x 2 = 16, 16 into 50 is 800 so 16, x 1 = 0 and x 2 = 16. Similarly, you draw all other objective function lines. So this one is for 800.

All points on this objective function line represents the value of Z = 800. Similarly, this is 1200 and similarly this is 1600. If you notice, you see that as you move in this direction, the objective function value increases from 800 to 1200 to 1600. And I want to maximize my objective function so I would like to shift the objective function contour line in this direction. But how far can I go?

See it can I have to be within the feasible region to locate a point, which is solution to the linear programming problem from within the feasible region. So I am looking for a solution of linear programming problem, which maximizes this Z. So I am finding out, I am trying to find out x 1 and x 2 that maximizes this and also satisfies the constraint.

That means, I must find out x 1 and x 2 which lie within the feasible region, that means the shaded region, the shaded in blue. Now I know that the objective function line if I move in the direction shown, I am basically increasing the value of Z and that is my objective, I want to do that. But how far can I move the objective function line? Say from this 800 I can definitely can go to 1200.

But and there is still I am still lying within the feasible region and there is still scope for pushing this objective function line in the direction shown. And still I will be within the feasible region. If you notice that will be up to this point. So beyond this, I escape the feasible region. So I will shift the objective function line up to that point. And then this point will be my optimal solution.

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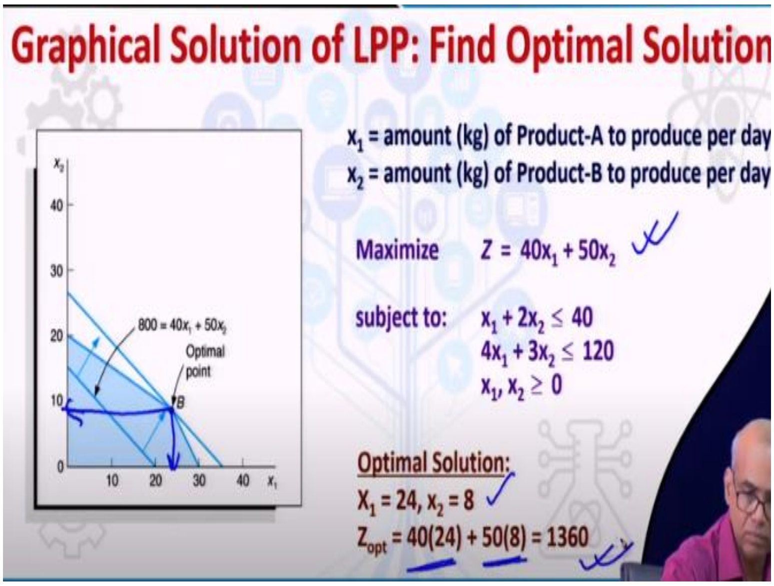

So this is what is shown here now. Note that the I started with the objective function line or objective function contour with 800 = 40x 1 + 50x 2 and then I start pushing it in this direction, as in this direction the Z value is increasing. And then this is the objective function line which gives me the optimal point B.

Note that if I draw a parallel line again, that means another objective function line that will have higher value of Z, but problem is this that this will not give me a solution within the feasible region. So that will give me an infeasible solution. So I can push only to the extent of point B. So point B is my optimal solution. So this point B corresponds to x 1 = 24 and x 2 = 8.

So I will produce 24 kg of product A per day and 8 kg of product B per day and that will give me how much profit? You just put value of x 1 and x 2 in this equation. So 40 into 24 plus 50 into 8. So this gives you 960 and this gives you 400. So that gives me 1360. So this is how you can solve a linear programming problem in two variable.

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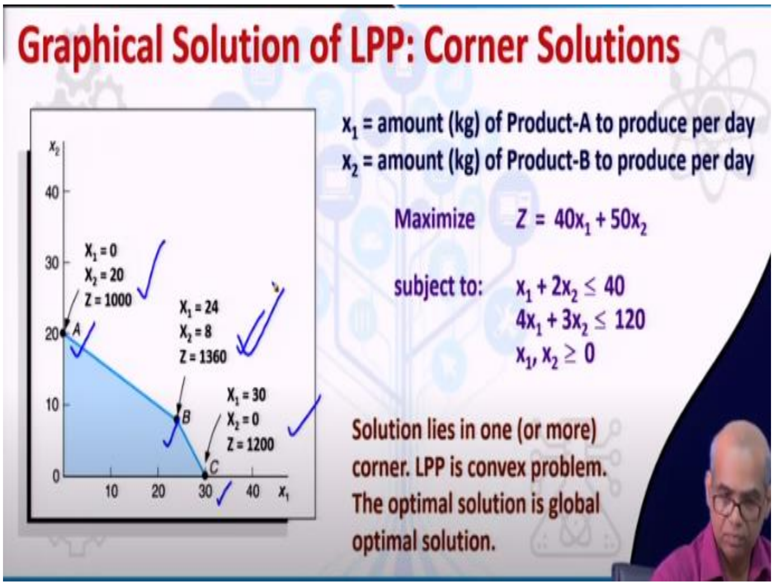

Note the shaded region is the feasible region. Geometrically it is known as polytope. It is the convex polygon. The theory of the linear programming problem says that the optimal solution will lie on one or more of this corner. One of the corner values will be the optimal solution. So for a very simple problem, we can just find out the values of x 1 and x 2 at each corners and just compare these values and you know that that will give me the optimal solution.

That is what you see here also. This corner corresponds to Z = 1000. Point B corresponds to 1360 and point C corresponds to 1200 and this is the maximum. So solution of a linear programming problem will always lie on one corner.

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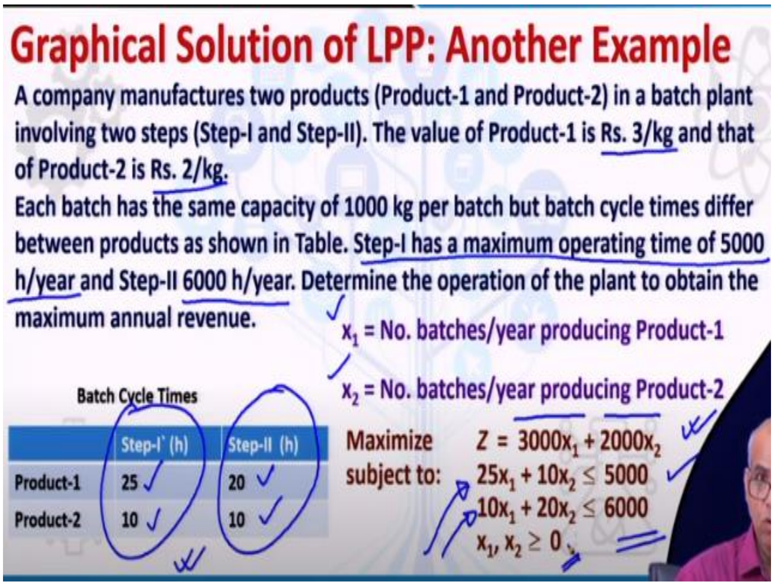

Now let us take another problem quickly. A company manufactures two products. Let us call them product A, sorry product 1 and product 2 in a batch plant involving two steps step 1 and step 2. The value of product 1 is Rs. 3 per kg and the value of product 2 is Rs. 2 per kg. Each batch has the same capacity of 1000 kg per batch. But batch cycle times differ between products as shown in the table.

Step 1 has a maximum operating time of 5000 hour per year and step 2 has a maximum operating time of 6000 hour per year. Determine the operation of the plant to obtain the maximum annual revenue. So how do I formulate this problem? Let us consider first the decision variables. We have to find out the operation plan for the plant so that we get maximum annual revenue. What does this mean?

It means, how much you will produce product 1, how much you will produce product 2. In the context of this problem it means how many number of batches, how many batches per year you will be producing product 1 and in how many batches per year you will be producing product 2. So let us call let us say x 1 equal to number of batches per year producing product 1 and x 2 equal to number of batches per year producing product 2.

Then we can formulate this problem. Let us explain this. What will be my objective function? I am producing x 1 number of batches per year. Now each batch has same capacity of 1000 kg. So x 1 batches per year means 1000 into x 1 kg per year. Now product 1 value is Rs. 3 per kg. The value of product 1 is Rs. 3 per kg. That means, the x 1 batches per year gives me value of 3000 into x 1.

Similarly, if x 2 is the number of batches per year producing product 2, it means I am producing 1000 into x 2 kg product 2 per year. Since the value of the product 2 is Rs. 2 per kg so the corresponding value will be 2000 into x 2. So I am maximizing the revenue. So my objective function is maximized, Z = 3000x 1 + 2000x 2. Now there is operational constraints, what is that?

That the process, the batch process has two steps, step 1 and step 2. And step 1 has a maximum operating time of 5000 hour per year and step 2 has 6000 hour per year. And the batch cycle times is different for product 1 and product 2 and they are given here. So let us now look at for step 1. The step 1 has a maximum operating time of 5000 hour per year. So let us now come to table and look at this column which represents batch cycle time for step 1.

So this gives me 25 into x 1 + 10 into x 2. So 25 into x 1 + 10 into x 2 will give me the operating time for the step 1. So 25x 1 + 10x 2 must be less or equal to 5000. Similarly, let us look at step 2 and the maximum operating time for step 2 is 6000 hour per year. So this x 1 and x 2 batches will take how much operating time for step 2? 20 into x 1 + 10 into x 2. So 20 into x 1 + 10 into x 2 should be less or equal to 6000.

And of course non-negativity restrictions on x 1 and x 2 which are greater or equal to zero. So this is how you can formulate this problem. Again it is a two variable optimization problem we can easily solve by graphical method.

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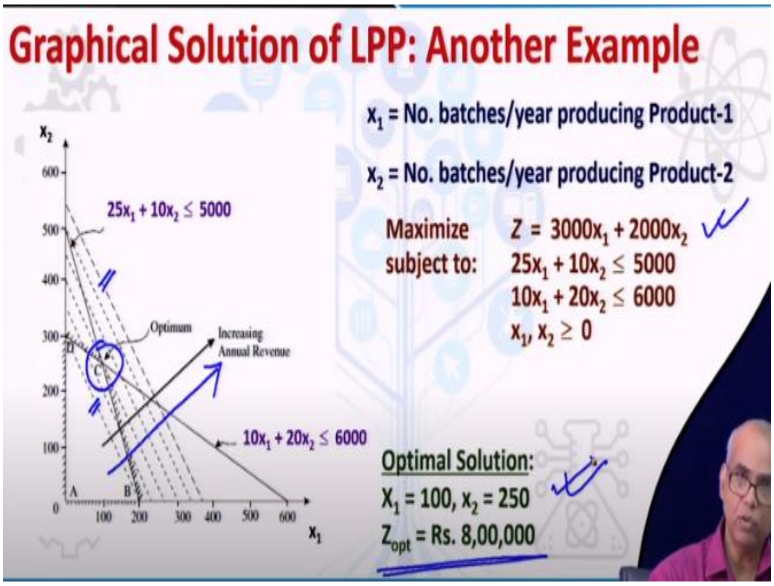

So let us plot. So both the constraints are plotted which then this gives me this as the feasible region. Note that both are less or equal to type. So the region below these lines are representing feasible region and the area that is common to both the constraints is my feasible region. So we have to find the optimal solution in this region. So again draw the objective function lines.

These all these parallel lines represent objective function contour lines. Each line represents the constraint objective function line. And you see that if we move in this direction, objective function value is increases. So following the previous example, we find that this is the optimum point. This corresponds to x 1 = 100 and x 2 = 250. If you put these values in this objective function, we get that the maximum revenue will be Rs. 8,00,000.

Note this for this particular problem the decision variables was number of batches. So fortunately it came as whole numbers. But if the numbers was not whole numbers even then the solutions will be optimal and we can interpret in the following way. That the remaining part will be you cannot do a fractional batch in a year. So the remaining part we will be doing in the following year.

So this is how we can solve a linear programming problem in two variables using graphical method. As I told you, there are several softwares which implement this, such as you have access to MATLAB or Excel solver and then there are several other powerful solvers such as CPLEX from IBM. Then there is LINDO. There are so many different types of commercial solvers available for solutions of large scale linear programming problem.

And one of the popular method is simplex method. But there are variations of simplex method as well. And in this class, we could only show you the graphical method for solutions of two variable problems. With this, we stop our discussion here.