Plant Design and Economics
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Lecture - 58
Equality Constrained Problems: Lagrange Multipliers

Welcome to lecture 58 of Plant Design and Economics. In this module as of now we have talked about optimality criteria for unconstrained single variable and multivariable functions. We have also seen several examples on applications of these optimality criterion. Now today we will talk about the optimization problems with equality constraints. So we will talk about the optimality criteria for equality constrained optimization problems.

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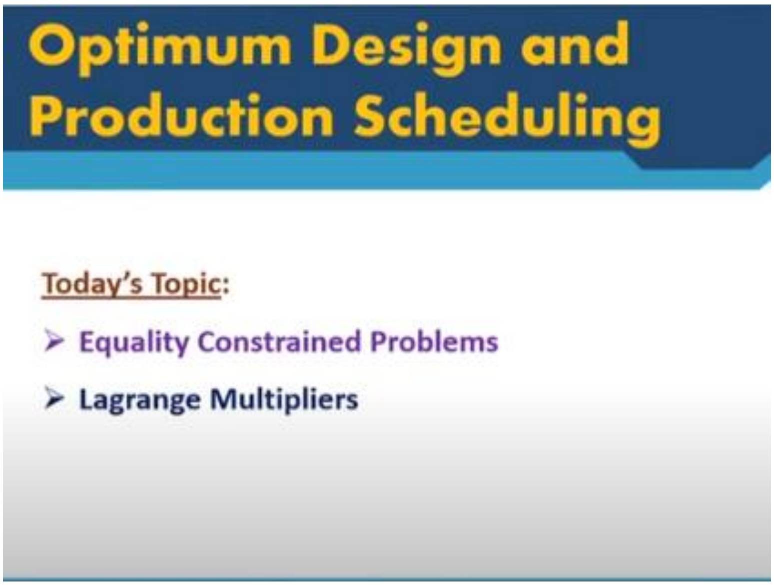

And the method for solutions of such equality constrained problems by Lagrange multipliers technique.

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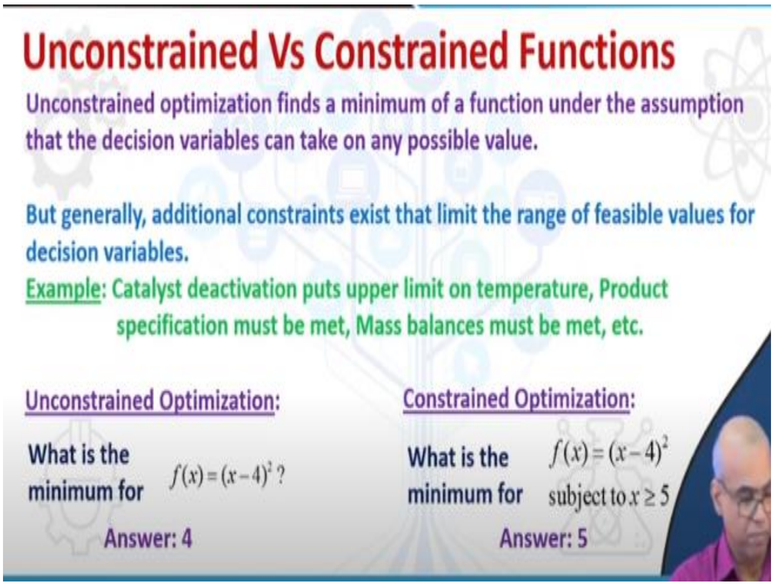

When you solve an unconstrained optimization problem, we assume that the decision variable can take on any possible value. But, this will in general be not possible because the additional constraints exist that limit the range of feasible values for decision variables. Suppose, you are designing a chemical reactor. So let us say to increase the rate of the reaction you would like to operate the reactor at high temperature.

But you are not free to take a temperature as high as you want because there may be operational constraint imposed on the process. If it is a catalyzed reactor, the catalyst may get deactivated. So a restriction or constraint will be imposed on the temperature. A product specification must be made. So constraints on the product specifications will be imposed.

Similarly, while solving an optimization problem related to a process, the mass balances must be respected. These mass balances will come as equality constraints. So in actual practice we will have constrained optimization problems. The solutions of constrained optimization problem will of course be different from solution so unconstrained optimization problem.

For example, let us take this simple example f x = x – 4 whole square. Now when the function is unconstrained, or the decision variables can take on any values, obviously the answer is x = 4. Because then at that case, the f x takes value equal to zero. And since f x is a square of a term, it cannot be less than zero. So x = 4 is the minimum of f x = x – 4 whole square.

Let us now put a constraint to the same objective function f x = x - 4 whole square and I say x now can take on values which are greater or equal to 5. So obviously, now the answer is x = 5. Any value greater than 5 will make f x value higher and we are minimizing the function. So when I say x = 5, the function value is 1. When I take x = 6, the function value is 6 – 4 = 2 square equal to 4.

But I can now take x = 4 because x has to be greater or equal to 5. So the minimum value for this constrained function now is x =5.

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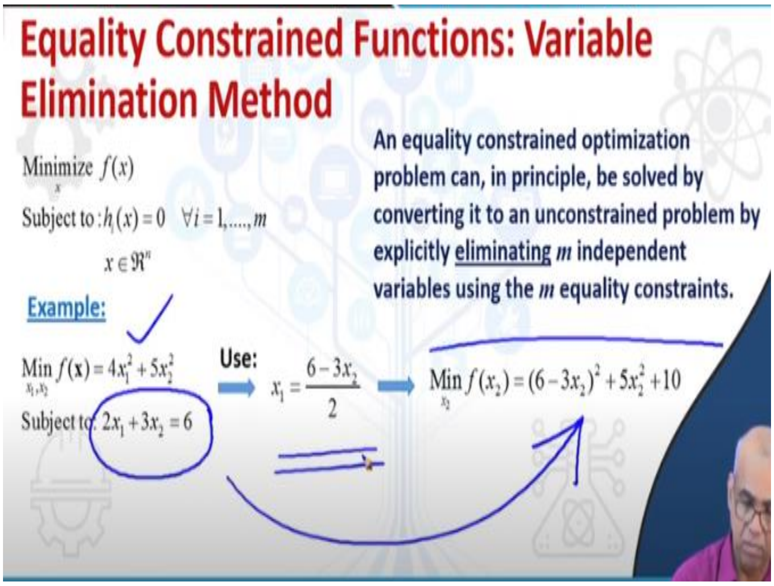

Let us consider this constrained optimization problem and we are talking only equality constrained optimization problem. That means the constraints are equality type. So we have only one type of constraints, all the constraints are equality type. There may be any number of such constraints, but inequality type constraints are not present, only equality type constraints are present.

So we are minimizing this function f x subject to m number of constraints. So which can be written as h i of x equal to zero. So h 1 = 0, h 2 = 0 up to h n = 0. And x takes values in the real space. So now the optimization problem is a constrained problem, number one. And the minimization of f x will be satisfied by these constraints. So the values of x that minimize f x must satisfy the constraints h (x) = 0 for all the constraints.

So h 1 x = 0, h 2 x = 0 up to h m x = 0. Now an equality constrained optimization problem, you can in principle solve by converting to an equivalent unconstrained problem. So how do you do that? If it is possible to explicitly eliminate these variables, the decision variables using the equality constraints, then it is possible. For example, let us consider this case.

I have an equality constrained optimization problem f x = 4x square + 5x 2 square subject to 2x 1 + 3x 2 = 6. What I say that an equality constrained optimization problem can be solved by converting it to an unconstrained problem by explicitly eliminating m independent variables using the m equality constraints. So now we make use of this constraint 2x 1 + 3x 2 = 6 to replace one of this decision variables x 1 or x 2.

Then it will be a decision variable or single variable. Then it will be an objective function of single decision variable. And also it will be unconstrained in nature. So let us say from this 2x 1 + 3x 2 = 6 we can obtain x 1 = 6 – 3x 2 by 2. So now I can rewrite my original objective function as a function of x 2 alone. And this becomes an unconstrained function.

Note that these constraint has now been incorporated in the objective function itself. You are still solving an unconstrained optimization problem by solving sorry you are still solving a constrained optimization problem by solving an unconstrained optimization problem. You basically have incorporated the constraint into the objective function.

Now it may not always be possible to explicitly eliminate these decision variables one by one. And if you have m number of decision variables, it may not be possible to eliminate you know each of these m - 1 decision variables using those equality constraints and express the objective function as a function of single decision variable.

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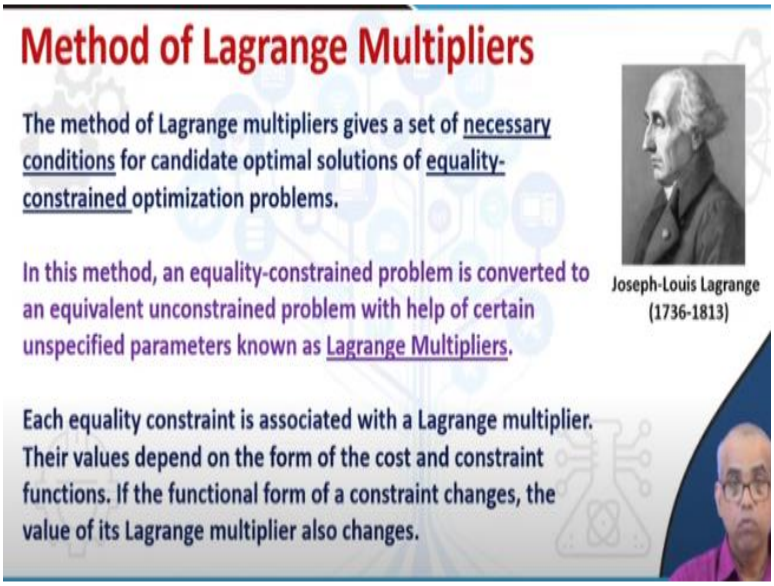

So we have this method of Lagrange multipliers for solutions of such equality constrained optimization problem. But for simple problems of course, you can make use of this variable elimination method to convert the equality constrained optimization problem to an equivalent unconstrained optimization problem.

The method of Lagrange multipliers gives a set of necessary conditions for candidate optimal solutions of equality constrained optimization problem. In this method, an equality constrained problem is converted to an equivalent unconstrained problem with help of certain unspecified parameters which we call Lagrange multipliers. So we will introduce Lagrange multipliers which are unspecified parameters.

And with help of those, we convert the equality constrained optimization problem to an equivalent unconstrained optimization problem. How do I do that? Suppose, I have an objective function and let us say two equality constraints. So I will introduce two Lagrange multipliers. So I will introduce one Lagrange multiplier for each equality constraint. Then, formulate an unconstrained optimization problem as follows.

Take the objective function then add to it a product of Lagrange multiplier and the corresponding equality constraint. So if I have two constraints say h 1 = 0, h 2 = 0. So both are functions of the decision variables. So and f x is my objective function. So I can convert these constrained optimization problem to an equivalent unconstrained optimization problem by formulating an unconstrained objective function as f x plus lambda 1 into h 1 plus lambda 2 into h 2.

This we call as Lagrangian. Let us we look at this in the next slides. So each equality constraint is associated with a Lagrange multiplier. Their values depend on the form and the form of the objective function as well as on the form of the constrained functions. If the functional form of the constraint changes the value of the Lagrange multiplier also changes. So remember that for each equality constraint we will associate one Lagrange multiplier.

And the values of this Lagrange multiplier depend on the form or the functional form of the objective function and the constrained functions.

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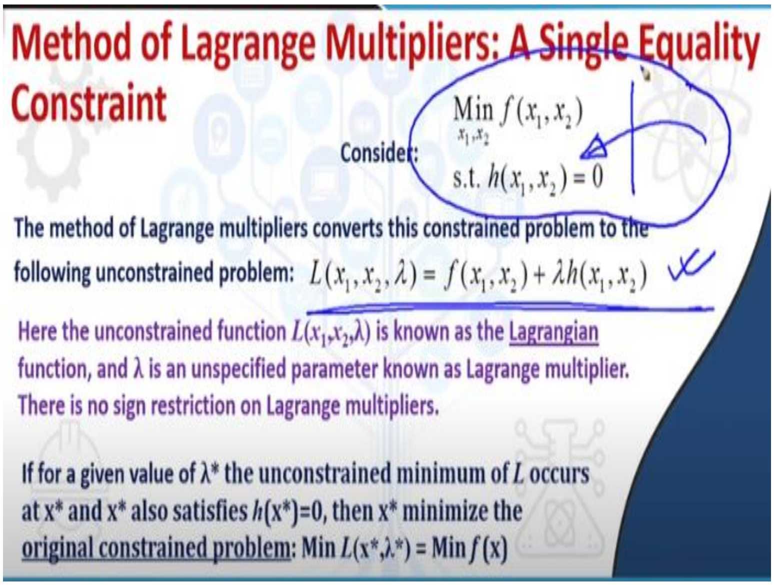

Let us look at this equality constrained optimization problem. We have an objective function of two variables and we have a single equality constraint which is of course a function of these two decision variables. So what will be my Lagrangian? So what is Lagrangian? The method of Lagrange multipliers converts this constrained problem to an equivalent unconstrained problem. How?

We consider one Lagrange multiplier for each constraint. I have one constraint here. So I assume only one Lagrange multiplier, let us say lambda. So I formulate the unconstrained optimization problem as f x. So this is f of x 1 and x 2 plus lambda into h. So this new function which is a now function of the original decision variables as well as the unspecified parameter that we have introduced Lagrange multiplier lambda.

So this function let us call L is a function of x 1, x 2 and lambda. We call this function as Lagrangian function. So the first step towards solving equality constrained problems using Lagrange multipliers is to formulate this Lagrangian. So how will you formulate the Lagrangian? You will specify one Lagrange multiplier for each equality constraint. Then, multiply the equality constraint with its corresponding Lagrange multiplier.

Then add this with the objective function. Repeat for each equality constraint you have. So then you obtain a function like this, which we will call as Lagrangian. So L is the Lagrangian and lambda is the Lagrange multiplier. Note that the Lagrange multiplier is unspecified parameter. So the value of the Lagrangian multiplier will also be determined optimally along with the value of x 1 and x 2.

So once you have this unconstrained function now I can apply whatever I have learned as optimality criteria for unconstrained problems. So it will always be multivariable problems. So whatever we have learned as optimality criteria for unconstrained multivariable functions now I can apply those criteria on this which is an unconstrained function, the Lagrangian is an unconstrained function.

So I will apply the optimality criteria for unconstrained multivariable functions to this Lagrangian. And that way thereby I will solve the equality constrained optimization problem. Note that there is no sign restriction on this Lagrange multiplier. So Lagrange multiplier can take values with any sign. Now this is my Lagrange multiplier. This is my Lagrangian in which the constraints have been incorporated.

Now let us apply the optimality conditions for multivariable functions to these Lagrangian. Now if the solution that I get, that means let us say I am minimizing this problem. So the solution that means x 1, x 2 and lambda that minimizes these Lagrangian function also satisfies this constraint then the solution for this Lagrangian and the solutions for this original problem is same. I repeat this.

I want to minimize this problem which is an equality constrained problem. I formulate this Lagrangian. Now let us consider that I have obtained the minimum point for this

Lagrangian. Let us consider that as x 1 star, x 2 star and lambda star. So x 1 star, x 2 star and lambda star minimizes this Lagrangian function. Now if that x 1 star x 2 star also satisfies these equality constraint then the x 1 star and x 2 star also solves this optimization problem. So this is the principle of Lagrange multiplier techniques.

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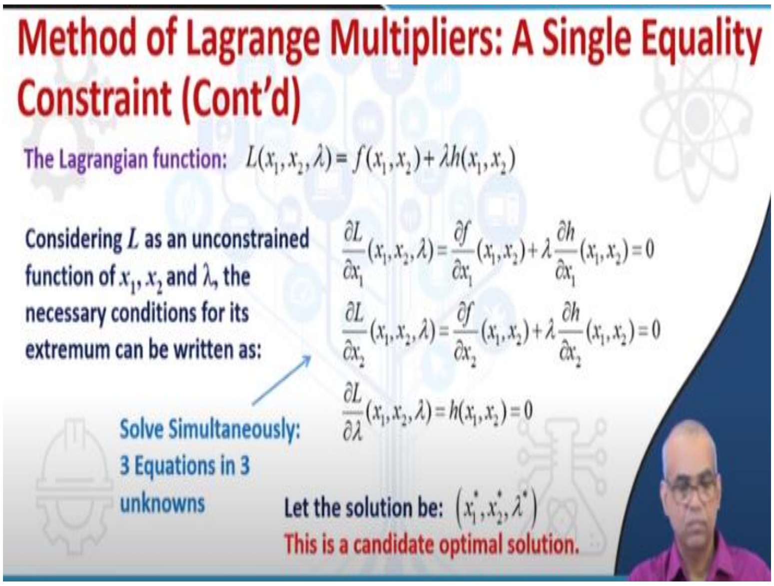

So now obtaining the optimality criteria will be straightforward because we will straightaway apply the optimality conditions for unconstrained multivariable function. So what we do is we set del L del x 1 equal to zero. Set del L del x 2 equal to zero. And set del L del lambda equal to zero. Note that in the Lagrangian all x 1, x 2 and lambda are now variables.

Lambda is the unspecified parameter whose value also has to be found out optimally along with x 1 and x 2. So Lagrangian is a function of x 1, x 2 and lambda. So the optimality criteria for unconstrained optimization problems will require you to set del L, del L equal to zero. Gradient of the Lagrangian equal to zero. So that will be del L del x 1 equal to zero, del L del x 2 equal to zero and del L del lambda equal to zero.

So these three conditions will give you three equations and three variables x 1, x 2 and lambda. So you can solve simultaneously. If the solution is x 1 star, x 2 star and lambda star so then this becomes a candidate optimal solution.

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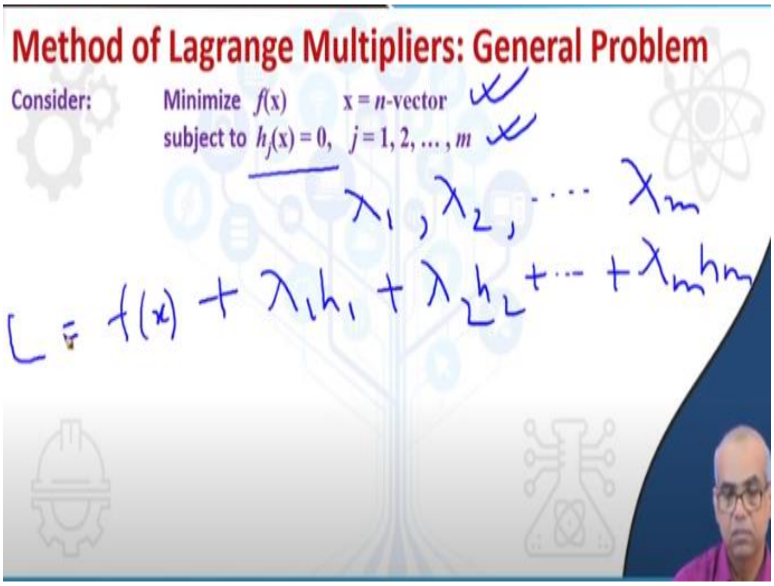

Now we saw the examples with one equality constraint. You can just extend it you can extend it to any number of equality constraint. Let us consider I have a function, objective function in n variables and I have m number of equality constraints. So how do I formulate the Lagrangian? So for m equality constraints I define m Lagrangian multipliers; lambda 1, lambda 2 up to lambda m.

Then take lambda 1 into h 1 plus lambda 2 into h 2 up to lambda m into h m and also add the function, objective function f x. So that gives me the Lagrangian.

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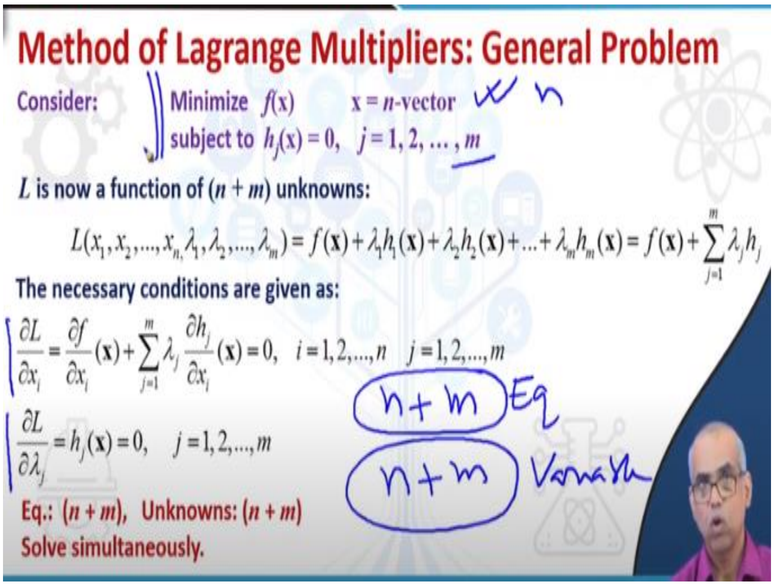

So this is how we will do for the general problems where objective function has n variables and you have m number of equality constraints. So this can be written more compactly using the summation notation. So the necessary conditions will be obtained as taking the derivatives of this Lagrangian and function with respect to each decision variables and each Lagrange multipliers.

So there are n variables and m number of equality constraints. So you will have del L del x i where i equal to 1 to n equal to zero and then del L del lambda j where j equal to 1 to m for the Lagrange multipliers. So note that this will give you n + m number of equations. So you have n number of decision variables. So del L del x 1, del L del x 2, del L del x 3 up to del L del x n equal to zero. That gives me n number of equations.

And you have m number of equality constraints. So m number of Lagrange multipliers. So you said del L del lambda 1 equal to zero, del L del lambda 2 equal to zero up to del L del lambda m equal to zero. So this gives me m number of equations. So n + m number of total equations I get and my variables are also n number of decision variables and m number of Lagrange multipliers.

So I have n + m equations and n + m variables. So it is possible for me in principle to solve and we can obtain the values of x 1 to x n as well as lambda 1 to lambda m. That minimizes or solves my optimization problem.

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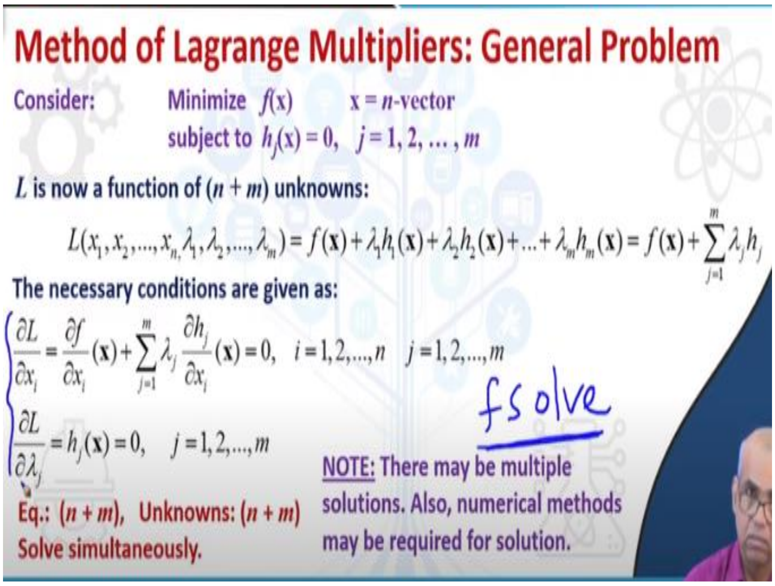

See note that there may be multiple solutions. Also it may not always be possible to solve these equations by hand. Often times this when you said del L x i equal to zero, del L lambda j equal to zero the equations that you get may be nonlinear in nature and you can make use of any software that solves nonlinear equations for that. For example, if you have access to MATLAB, you can use f solve to solve the resulting nonlinear equations if it happens.

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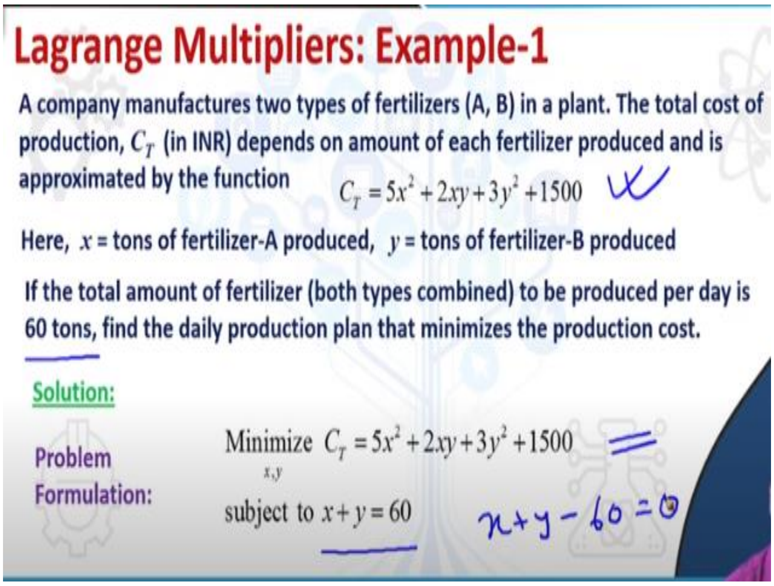

Now let us take one example. A company manufactures two types of fertilizers A and B in a plant. The total cost of production C T in Indian rupees depends on amount of each fertilizer produced and is approximated by the function C T equal to 5x square plus 2xy plus 3y square plus 1500. x equal to tons of fertilizer A produced and y equal to tons of fertilizer B produced.

If the total amount of fertilizer both types combined A plus B to be produced per day is 60 tons, find the daily production plan that minimizes the production cost. So how much of fertilizer A and how much of fertilizer B, that means how many tons of fertilizer A and how many tons of fertilizer B we have to produce per day so that we can minimize the daily production cost.

So we will solve this using Lagrange multiplier. So let us formulate the problem first. We minimize this cost function. So I assume that x equal to tons of fertilizer A to be produced and y equal to tons of fertilizer B to be produced per day. So C T equal to as given and it must satisfy the equality constraint x + y = 6 because the daily production must be equal to 60 tons, exactly 60 tons. So x + y = 60. You can also write as x + y – 60 = 0.

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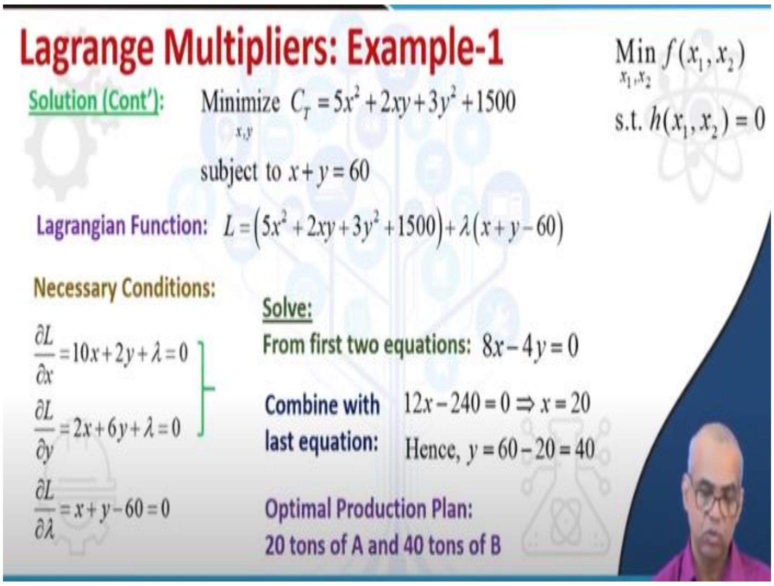

So now let us solve it using Lagrange multiplier method. Formulate the Lagrangian function. So introduce one Lagrange multiplier lambda. So the objective function f x or CT here plus lambda into the equality constraint. So the Lagrangian gives me the unconstrained function to be minimized. So set del L del x = 0, set del L del y = 0 and set del L del lambda = 0.

You have three equations and three variables x, y and lambda. All are linear equations, very easy to solve these equations. For example, from the first two equations, if you simply subtract the second equation from the first one, you will obtain 8x – 4y = 0. Now combine this with the last equation. Combine this with this last equation. Simply multiply by this del x + y - 60 = 0 by 4 and then add it up with this.

And you will obtain 12x – 240 = 0 which gives x = 20. So total is 60 tons. So the y will be 60 - 20 = 40. So 20 tons of fertilizer A, 40 tons of fertilizer B. So that should be my optimal production plan.

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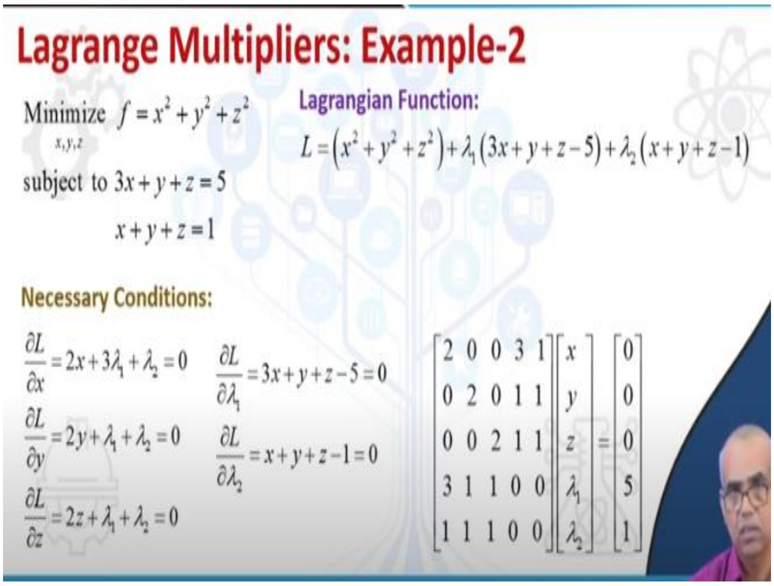

Another example minimize f equal to x square plus y square plus z square subject to 3x + y + z = 5, x + y + z = 1. I have now two equality constraint. Formulate the Lagrangian. So introduce two Lagrange multipliers lambda 1 and lambda 2. Set the necessary conditions del L del x = 0, del L del y = 0, del L del z = 0. And also del L del lambda 1 equal to 0 and del L del lambda 2 equal to 0.

Now we have five variables x, y, z, lambda 1, lambda 2 and five equations. You can solve it. In fact you can, all are linear equations, so you can write this x equal to b form.

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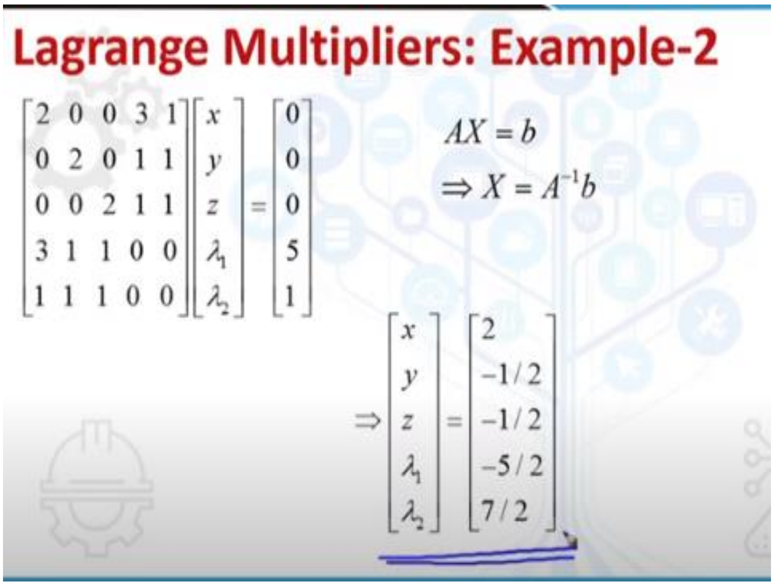

And the solution will be x equal to A inverse B. So by matrix inversion method or Kramer's rule, you can obtain this solution. So this is possible only for linear equations like this. So had it been nonlinear, so it perhaps will be necessary to make use of a software to solve the resulting nonlinear equations.

For large nonlinear large number of nonlinear equations, solutions by hand will be really time consuming and may be difficult and cumbersome.

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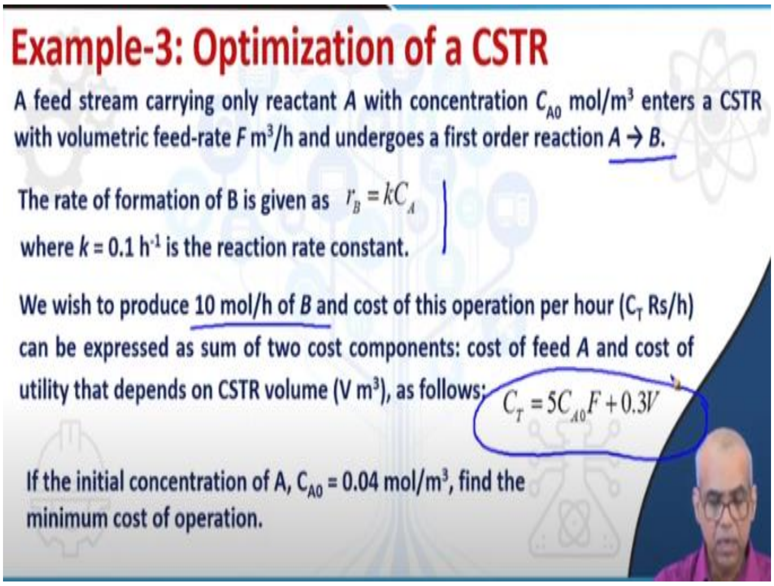

Now let us take one more example, a slightly more involved example. It is about optimization of a CSTR. A feed stream carrying only reactant A with concentration C A0 mole per meter cube enters a CSTR with volumetric feed rate F meter cube per hour and it undergoes a first order reaction A to B within the CSTR. The rate of formation of B is given as r B equal to k into C A where k equal to 0.1 hour inverse is the reaction rate constant.

We used to produce 10 mole per hour of B and the cost of this operation per hour C T rupees per hour can be expressed as a sum of two cost components, cost of feed A and cost of utility that depends on CSTR volume V as follows C T equal to 5C A0 into f plus 0.3 into V. If the initial concentration of A, C A0 equal to 0.04 mole per meter cube find the minimum cost of operation.

So to summarize you have a CSTR where a first order reaction is taking place. The rate constrained is given. We want to produce exactly 10 mole per hour of B and the cost of production or cost of this operation is given as a function of C A0 feed rate as well as volume. C A0 is specified as 0.04 mole per meter cube, okay? So we have to find the minimum cost of operation. So how do I solve this problem?

So you have to minimize this cost of operation C T equal to 5 into C A0 into f plus 0.3 into V. Note that C A0 is specified as 0.04.

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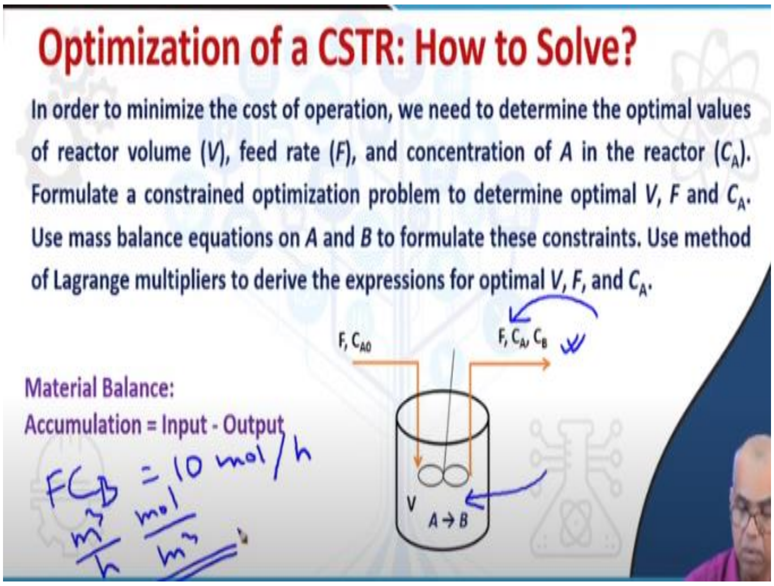

Now in order to minimize the cost of operation, we have to determine the optimal values of reactor volume B, the feed rate F and also the concentration of A in the reactor. So concentration of A in the reactor and the concentration of A in the effluent is same. And we have to do this such that the amount of B right, the moles per hour of B in the outlet is exactly 10. So 10 moles per hour of B is obtained.

So F into C B will be 10 moles per hour. So the concentration within the reactor is same as concentration outside the reactor effluent stream here because of CSTR. So in the objective function there was no C A or C B. But note that this will be this has to be taken care of. Because this determines the fact that we have to find out the operating condition such that we have the B as 10 moles per hour.

So F is volumetric feed rate. So that will be like meter cube per hour and C B the concentration of B will be moles per meter cube. So it will be moles per hour. So F into C B must be 10 moles per hour.

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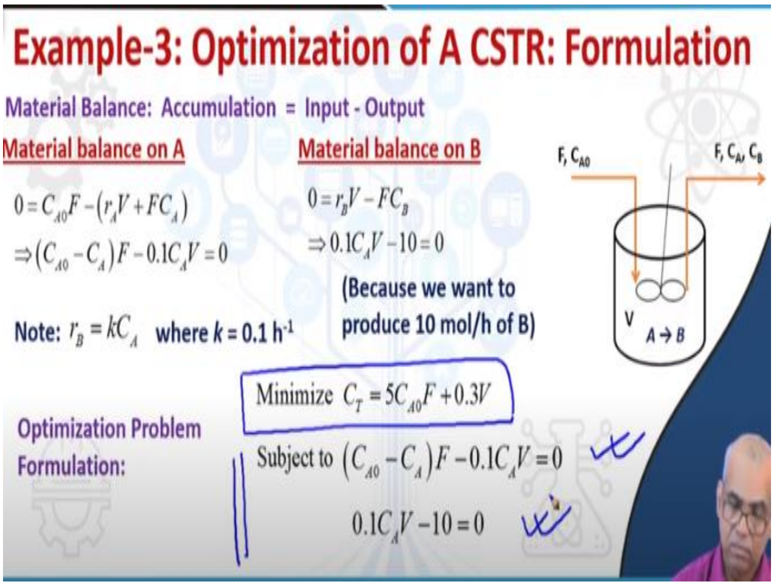

So to obtain those constraints, we have to write down the material balance on A and material balance on B. So what is the mass balance on A? The A enters with the feed stream. How much? C A0 into F moles per hour because F is meter cube per hour and C A0 in mole per meter cube. So this amount enters to the reactor and what goes out is F into C A and also it gets consumed. It is converted to, it gets reacted to B.

So r A into V. So r A is 0.1 into C A. So after putting these values you can obtain this. So this is one constraint that I obtained from material balance on A. This must be respected. Similarly, we can we have to also write material balance on B. There is no B entering, but B leaving as F into C B and B is being formed due to reaction within the reactor that is r B into V. So that is 0.1 into C A into V.

And then FC B is 10 because we want to produce 10 mole per hour of B. So this is another equality constraint. So this is one equality constraint and this is another equality constraint. So now I am in a position to write down the problem. So I formulate the optimization problem as follows. Minimize C T equal to F into C A0 F plus 0.3 into V as given.

But then I have these two equality constraints. One from material balance on A, another from material balance on B.

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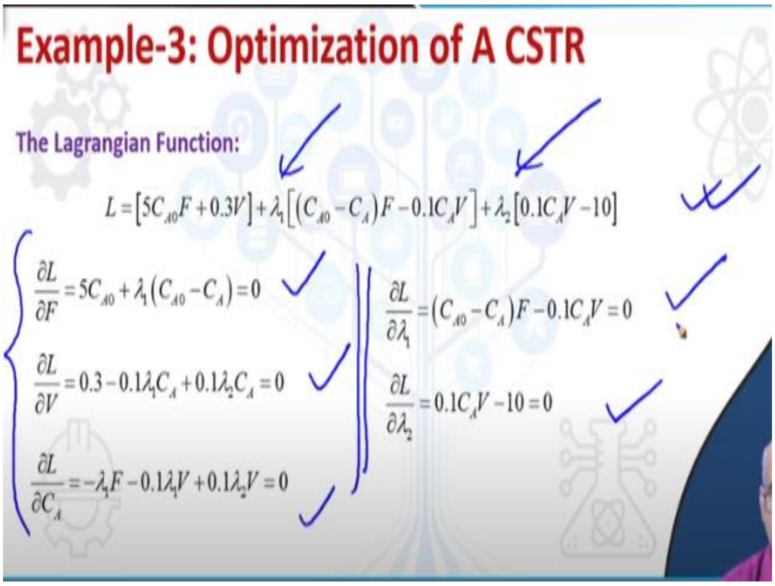

So make use of Lagrange multipliers method. We have two equality constraint introduced to Lagrange multipliers, lambda 1 and lambda 2. Formulate the Lagrangian function. So you have now five variables actually. Three variables as flow rate feed flow rate, volume of the reactor, and concentration of A within the reactor and the other two are Lagrange multipliers lambda 1 and lambda 2.

So set del L del F = 0, del L del V = 0, del L del C A = 0. Del L del lambda 1 = 0, del L del lambda 2 = 0. Five variables five equations you will be able to solve.

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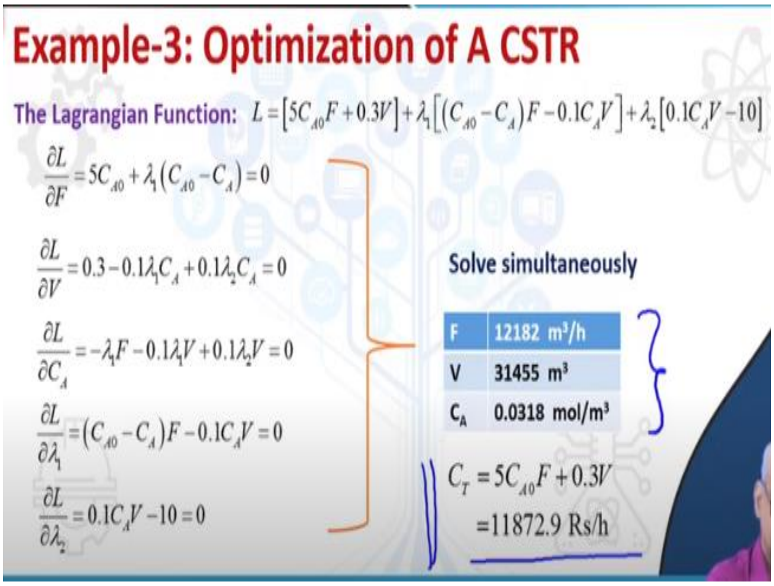

So you have to solve these equations simultaneously. Here MATLAB's Fsolve was used for this and we obtain the solutions as F = 12182 meter cube per hour. V = 31455 meter cube and C A = 0.0318 mole per meter cube. So these are the optimal values of feed flow rate, volume of the CSTR and the concentration of A in the reactor which minimizes the cost of operation given as $C_T = 5C_{A0}F + 0.3V$.

Putting all the values and putting the values of C_{A0} as given we obtain the minimum cost as 11872.9 Rs/h. With this we will stop our discussion here.