Plant Design and Economics

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Lecture No -18
Uniform Cash Flow and Continuous Cash Flows

Welcome to lecture 18 of plant design and economics. In this week, we have been talking about engineering economics part 2 today, we will continue our discussion on cash flow diagrams and we will discuss two important cash flows uniform series cash flow as well as continuous cash flow apart from that we will also discuss economic equivalence as well as summarize various compounding factors.

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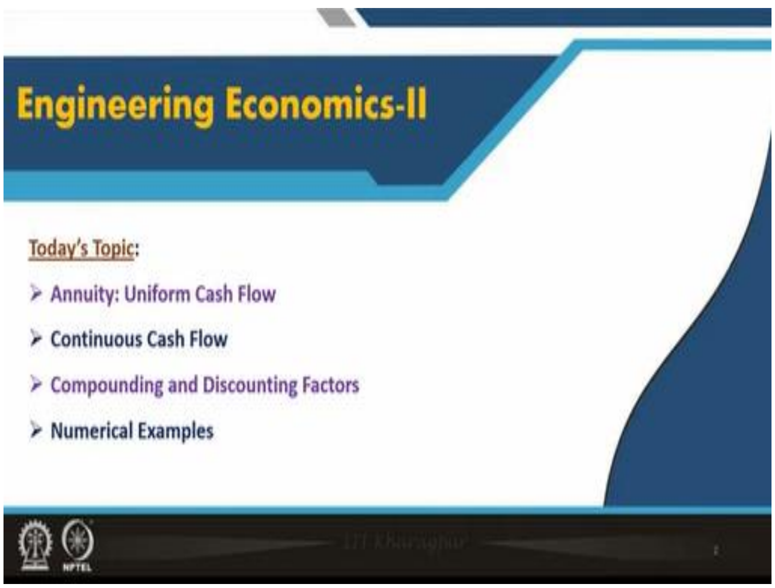

So this will be today's topic, uniform cash flow series will also called annuity, continuous cash flow then you summarize several compounding discounting factors and you also take few numerical examples.

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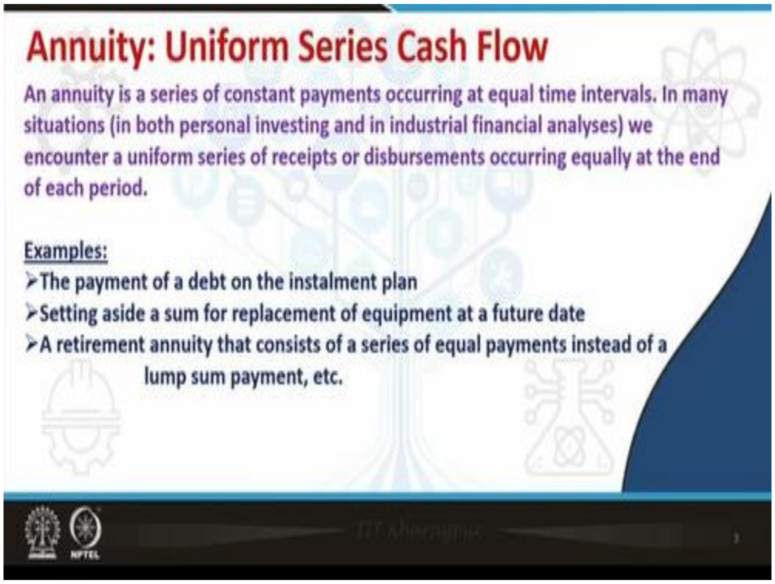

So annuities are basically uniform series cash flow and annuity is a series of constant payments occurring at equal time intervals in many situations in both personal investing and in industrial financial analysis we encounter a uniform series of receives or disbursements occurring equally at the end of each period, for example, the payment of a date on the instalment plan, setting aside a sum for replacement of equipment at some future date from now.

A retirement plan; where a determinant annuity that consists of a series of equal payments instead of a lump sum amount etcetera.

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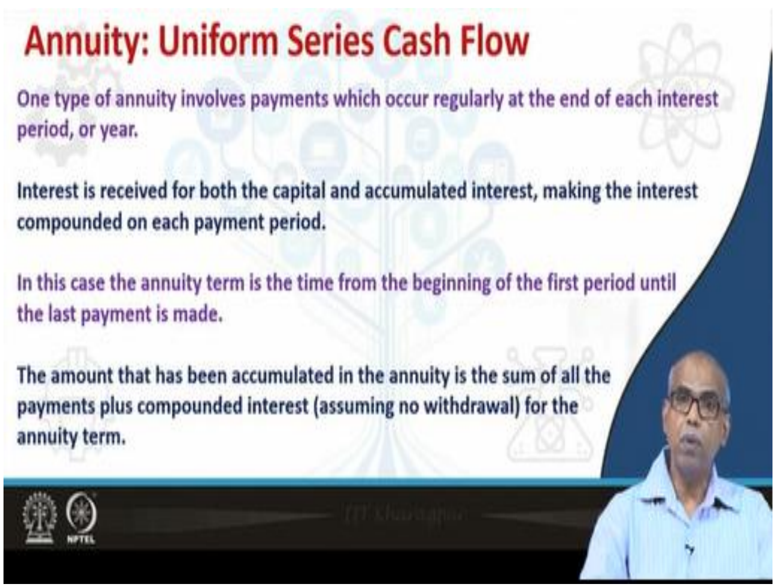

One type of annuity involves payments which occur regularly at the end of each interest period or year. Interest is received for both the capital and accumulated interest, making the interest compounded on each payment period. In this case, the annuity term is the time from the beginning of the first period until the last payment is made. The amount that has been accumulated in the annuity is the sum of all the payments plus compounded interest assuming that no withdrawal has taken place for the entire annuity term.

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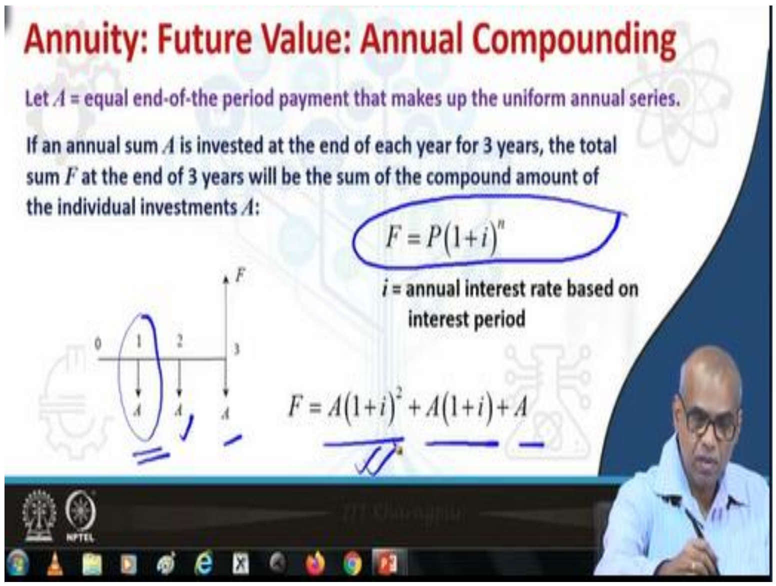

Now, let us see how to calculate the future value of uniform series cash flow or annuity, we consider annual compounding. Let A represent equal end of the period payment that makes up the uniform annual series. If an annual sum A is invested at the end of each year say for 3 years the total sum F at the end of three years will be the sum of the compound amount of the individual investment A, what is meant is? Seen by this cash flow diagram.

So we have a 3 years timeline where the first years second year as well as third year, you are investing and annual sum of A, so what will be the future value? You can make use of the well known formula F equal to P into 1+ i to the power n, for the amount that you invest in the first year it runs for two interest periods, so the contribution for this towards F will be A into 1+ i square.

Similarly, for the amount that you invest in second year runs for one interest here. So it will be A into 1+ i and in the third year it is also we are investing amount A. So the future value in this case for a three year timeline will be F equal to A into 1+ i square + A into 1+ i + A. So if it was for four years it would have been A into 1+ i cube + A into 1+ i square +A into 1+ i + A. So if you extend for n years, you can say that this term will be A into 1+ i to the power n -1.

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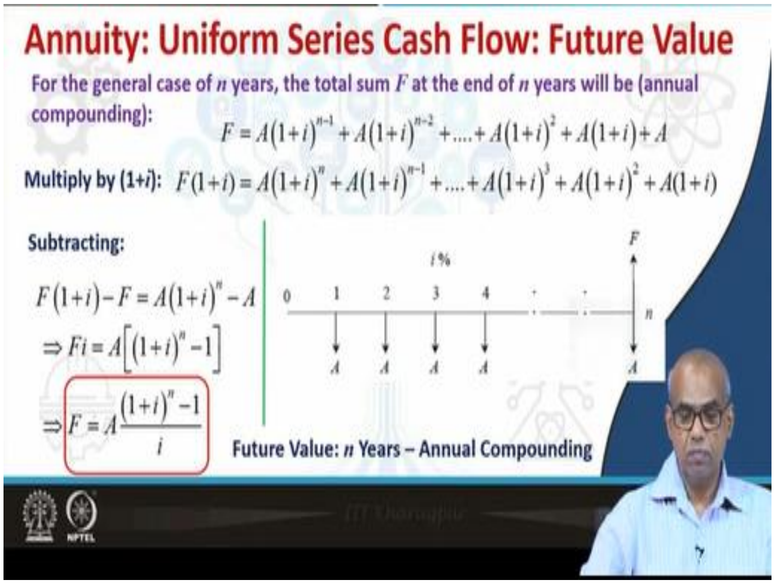

So we can write for the general case for n years, the total sum F at the end of n here will be this. Remember for three years this term was A into 1+ i square, so for n is we can say that this will be A into 1+ i to the power n -1. So this is the pictorial presentation of the cash flow diagram. Now, Let us do some algebraic manipulations. So what i do is multiply this series by the quantity 1+ i.

So if you multiply this by 1+ i what will happen? This term will be A into 1 + i to the power n because you have multiplied this by 1 + i. So you get A into 1 + i to the power n. Similarly this term will be A into 1 + i to the power n - 1 so on and so forth. Last term was a here. So it will be A into 1 + i. Now subtract these 2, see if we subtract you will have F + F into 1 + i -F. We are subtracting this from this we are subtracting the first line from the second line.

So F into 1 + i - F will be equal to this term will remain and only this term will remain, note that all this term will get cancelled. So what we will get is F into 1 + i - A equal to A into 1 + i to the power n - A. You can now rearrange and finally get if equal to A into 1 + i to the power n - 1 divided by i. So, this is the future value for uniform series cash flow that is occurring for n years and it is being compounded annually, this is an important relationship. So, please try to understand the derivation and you can also try to remember the expression.

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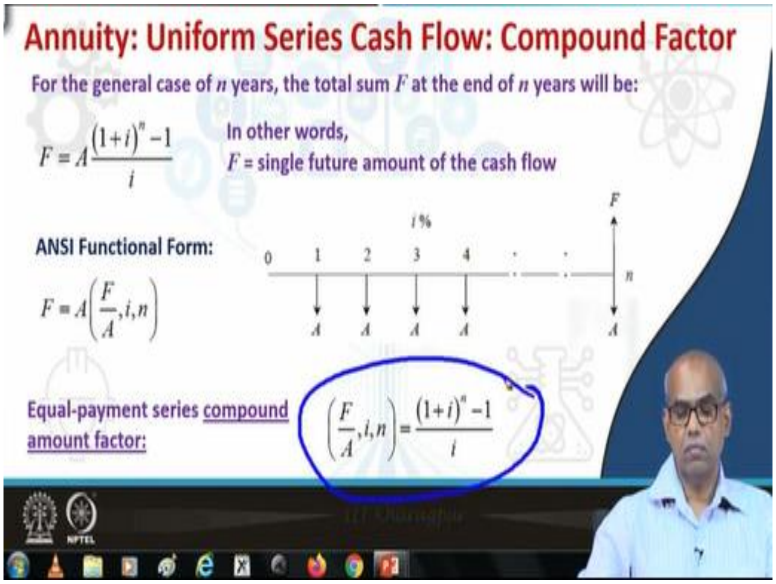

So now we will see what will be the compound factor. We have seen that if the future value is equal to A into 1 + i to the power n - 1 by i. So if can we call single future amount of the cash flow in the functional form we can write as F equal to A into F by A, i, n where F by A, i, n is 1 +I to the power n - 1 divided by I. So this quantity is known as equal payment series compound amount factor.

So here if you know the factor F by A and A is known then you can obtain the future value F, note that this is dimensionally, all right? So this kind of functional form tells you that A is known and F is unknown for interested i and years n. So if you know if by A we have seen how to obtain this quantities from table then by multiplying this quantity with A the known annuity you can find out the future value F.

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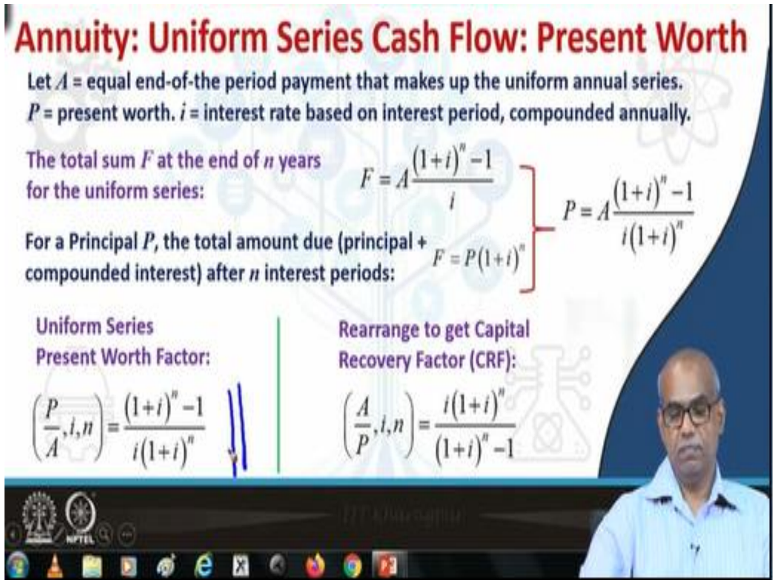

Now, let us see how to compute the present worth or the present value of uniform series. Let A be the equivalent of the period payment that makes up the uniform annual series P equal to present value or present work and small i represent interest rate based on one interest period compounded annually. We have seen that the future value expression is this F = A into 1 + i to the power n - 1 divided by i, for uniform series cash flow occurring for n years.

Now, for a principal P that you invest today the total amount that means principle plus compound and interest after n interest periods will be F equal to P into 1 + i to the power n. So i can combine these two expressions and can get P equal to A into1 + i to the power n - 1 divided by i into 1 + i to the power n. Note that we are equating this and this and obtaining P the present value are present worth of the series.

So what will be the present worth factor? The present worth factor will be this quantity. So P by A, i, n the present worth factor will be 1 + i to the power n - 1 divided by i into 1 + i to the power n. Now, you can rearrange to get capital recovery ratio or A by P, so P by A is uniform series present worth factor, whereas A by P is uniform series capital recovery factor. So just rearrange this. So A by P, i, n will be just reciprocal of the present worth factor.

So in case of present worth factor, you can find out the unknown present value if A is known. Similarly, from the capital recovery factor A by P you can find out A given P and of course interest rate and number of years should be known. So that you can find out A by P from the table.

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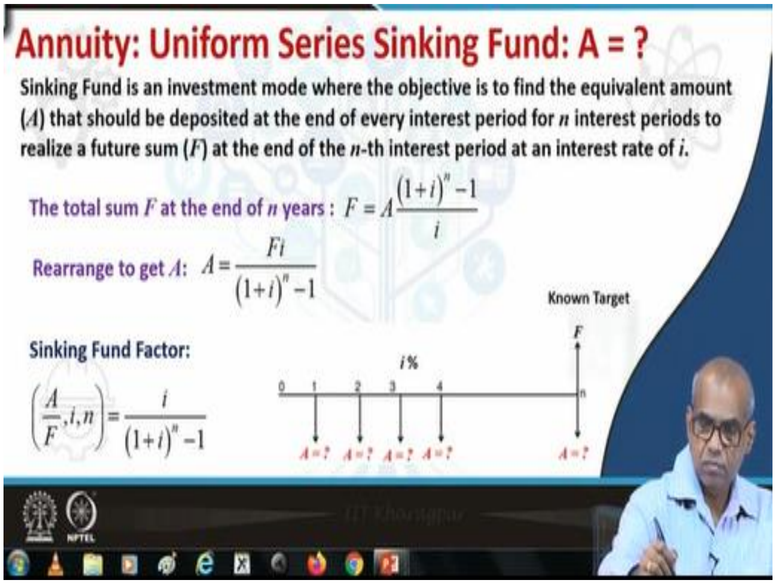

Now, we will talk about uniform series sinking fund. Sinking fund is an investment mode where the objective is to find the equivalent amount of A that must be deposited at the end of every interest period for n interest periods to realize the future some F at the end of the nth interest periods at an interest rate of i. So basically if you look at the cash flow diagram for a known target F, I want to find out A.

The equal constant payments that is being made starting from your end; 1, 2, 3 up to n, this is known as uniform series sinking fund, so here you want to know what will be the value of a that equal amount investment at the end of every interest period. So that I can accumulate a known target F at the end of n interest periods. I know the expression for the future value at the end of n years, which is F equal to A into 1 + i to the power n - 1 by i.

If you rearrange this I get A. So A is F into i divided by 1 + i to the power n - 1. So sinking fund factor which is A by F will be i divided by 1 + i to the power n - 1, you can read it from the expression for A. So from the sinking fund factor, for known F you can find out A by multiplying the factor A by F with the known F.

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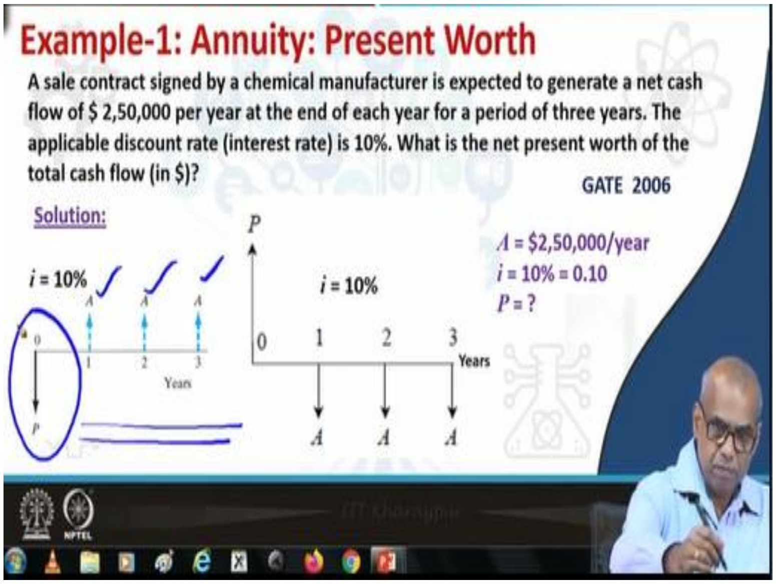

Now, we will take few examples to understand the annuity or uniform series flow uniform series cash flow. A sale contract signed by a chemical manufacturer is expected to generate a net cash flow of 2,50,000 dollar per year at the end of each year for a period of three years. The applicable discount rate or interest rate is 10%. What is the net present worth of the total cash flow?

So let us first try to understand the cash flow diagram any one of these diagrams you can use. One is mirror image of the other. So it is generating a net cash flow of 2,50,000 per year for a period of three years. So let us consider this, so you know these quantities A. So we have to find out what is P? The other figure is just mirror image there of this. So if we can think like this one is from borrowers perspective, another is from lenders perspective.

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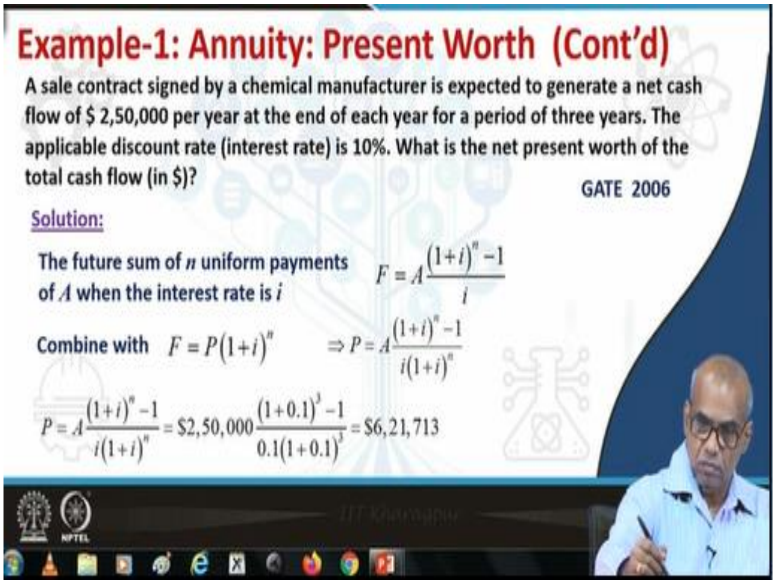

So the future sum of n uniform payments of a when the interest rate is i, you know the expression for the future value A into 1 + i to the power n - 1 divided by i. Combine this with F equal to P into 1 + i to the power n by combining this 2 we can obtain the expression for the present value or present worth P. Now in this expression you substitute the value of A and i, the value of A is 2,50,000 and i is 10%, so 0.1.

Do the computation and you will get the present value as 6,21,713 dollars. So, that is the present worth of the total cash flow.

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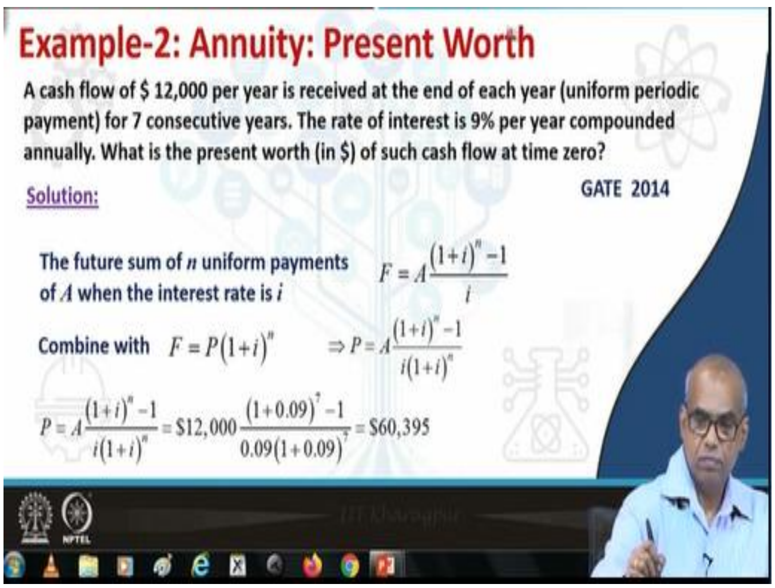

Let us take another example from annuity; this is also about present worth of the cash flow. A cash flow of 12,000 dollar per year is received at the end of each year uniform periodic payment for 7 consecutive years. The rate of interest is 9% per year compounded annually. What is the present worth in dollar of such cash flow at time 0. So this is also very much similar to the previous problem state forward application of the formula.

So we get the expression for present worth P and then substitute the values A equal to 12,000 and i is 9 %, which is 0.09. So if you do the computation you get the present value of the present worth of the cash flow at time 0 as 60,395 dollar.

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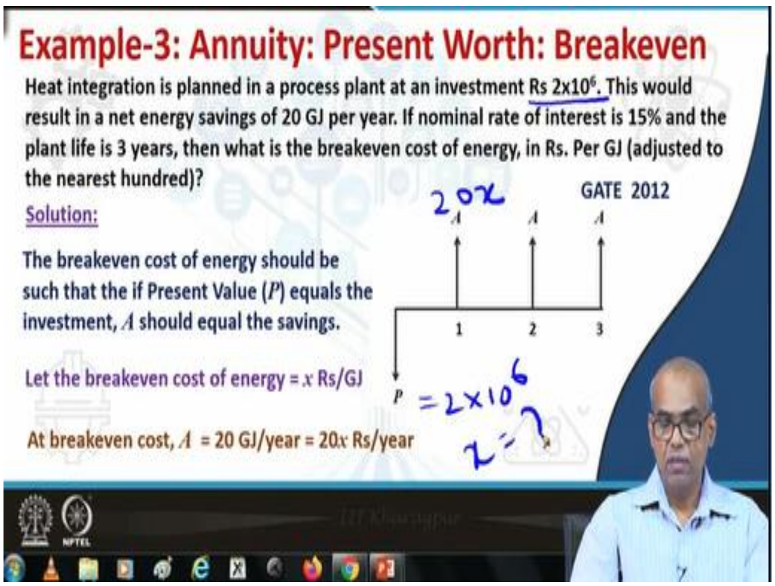

Let us take another problem, which is based on annuity or equal payment series involving present worth and break even analysis. Heat integration is planned in a process plant at an investment of rupees 2 into 10 to the power 6. This would result in a net energy savings of 20 giga joule per year if nominal rate of interest is 15% and the plant life is 3 years then what is the breakeven cost of energy in rupees power giga joule adjusted to the nearest 100?

So let us first draw the cash flow diagram. So you have Plant life specified as three years and each year, you have Net energy savings of 20 giga joule per year, so these A's are nothing but energy savings of 20 gigajoule per year. So, breakeven cost has to be such that if P equals 2 into 10 to the power 6, A will be equal to the net energy savings. Now, this energy savings is given in unit of 20 giga joule per year gigajoule per year and P is in rupees.

So each A has to be expressed in rupees per year. So the becoming cost of energy should be such that if present value P equals the investment A should equal the savings. Let the breakeven cost of energy be x rupees per giga joule x rupees per giga joule. So at breakeven cost a equal to 20 giga hertz per year multiplied by x rupees per giga joule, which is 20 x rupees per year. So now the problem simplifies to that P is given as 2 into 10 to the power 6.

And A is given as 20 x, each A is 20 x, so what is the value of x? So we can we know the expression for the present worth for uniform series, so let us make use of that.

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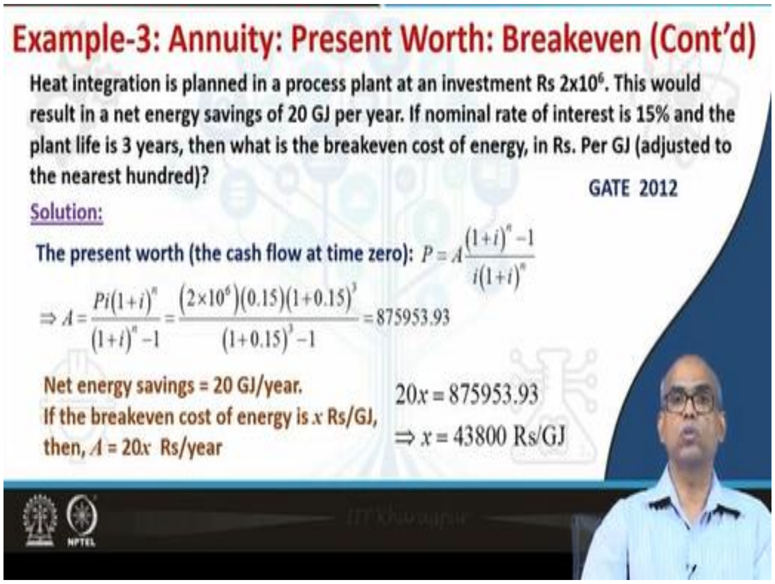

So this is the expression for present worth that means cash flow at time 0 for A uniform series cash flow diagram from this you compute once you compute A equate A to 20 x and then find out the value of x's 43800 rupees per gigajoule.

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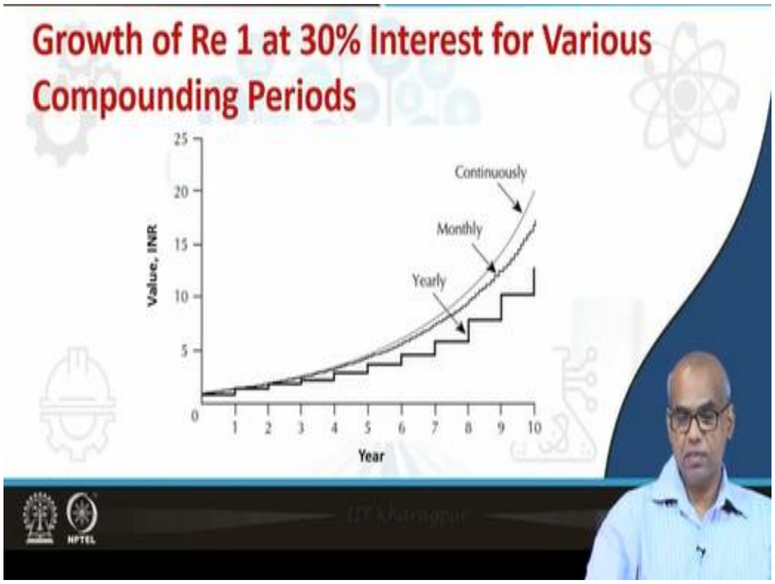

So this schematically presents the growth of rupee 1 at 30% interest for various compounding periods. Yearly compounding see the staircase is that large enough, monthly compounding happens more frequently, so small, small staircases and continuously compounded. So you get a smooth curve.

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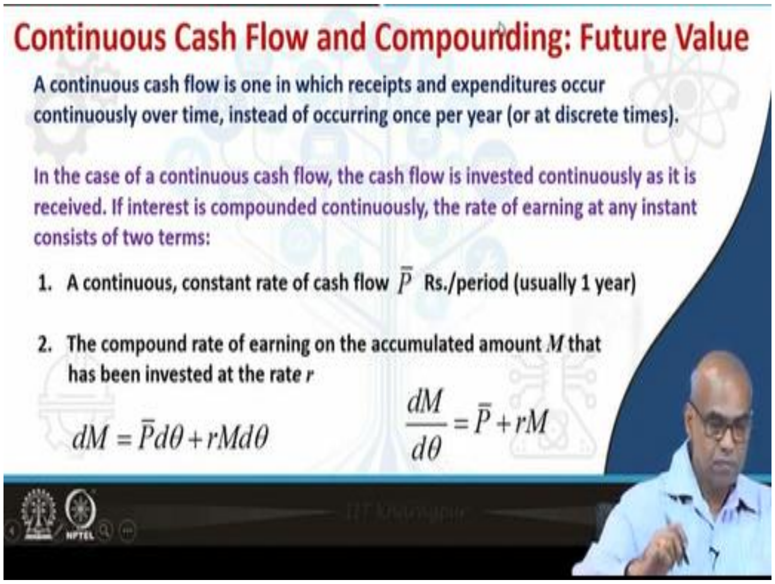

So now we will talk about continuous cash flow and compounding, so the cash flow is continuous. As well as it is being compounded continuously, so how to compute the future values such a cash flow diagram? A continuous cash flow is one in which receives an expenditures occur continuously over time instead of occurring once per year or a discrete times. In the case of continuous cash flow the cash flow is invested continuously as it is received.

If interest is compounded continuously, the rate of earning at any instant consist of two terms: one A continuous constant rate of cash flow P bar, which is rupees per period, the period may be usually one year. So rupees per year and the compound rate of earning on the accumulated amount m that has been invested at the rate r. So in case of continuous cash flow the cash flow is invested continuously as it is received.

Interest is compounded continuously the rate of earning will have two terms one is the continuous, continuous constant rate of cash flow P bar, then the other contribution will be from compound rate of earning on the accumulated amount M that is as that has been invested at the rate r. So the accumulation d M over a period d theta will be P bar into d theta + r M into d theta or rate of change of accumulation with time d M d theta will be P bar there is the continuous rate of cash flow plus r into M where r is the rate of interest and M is the accumulated amount, so d M d theta will be equal to p plus r M.

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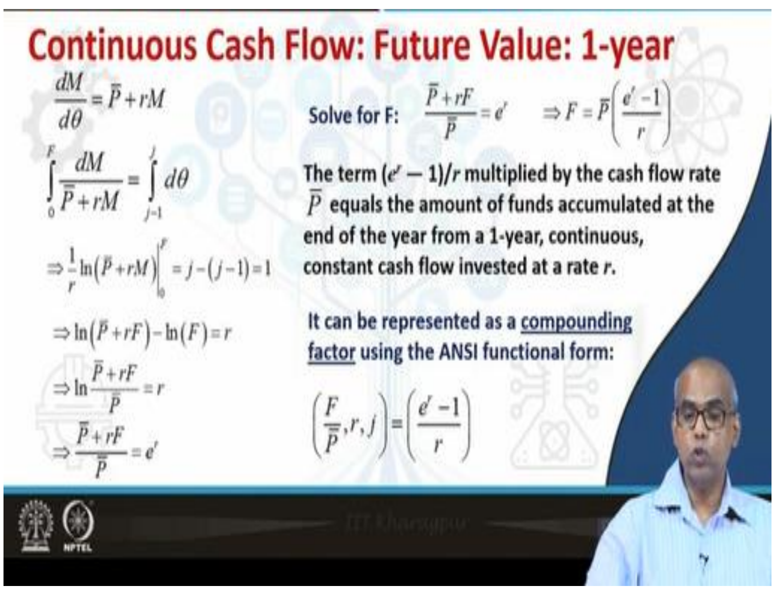

Now for one year let us consider they are j -1 to j, what will be the future value i can integrate it from 0 to F, d M by P bar + r F can be integrated from 0 to F and d theta from j -1 to j. So this gives a future value for one year when simplified, so after doing the integration what i did is P bar + rF By P bar equal to e to the power r. Now this can be solved for f and i get the expression for F which is the future value for one year.

F equal to P bar into e to the power r -1 by r. So the term e to the power r - 1 by r, multiplied by the cash flow rate P bar equals the amount of funds accommodated at the end of the year from a one year continuous constant cash flow invested at rate r, so what type of cash flow diagram we are talking about it is a continuous constant cash flow invested at rate r and occurring for one year.

So in terms of the functional form the compounding factor F by P or F by P bar here F by P, r, j equal to e to the power r - 1 by r.

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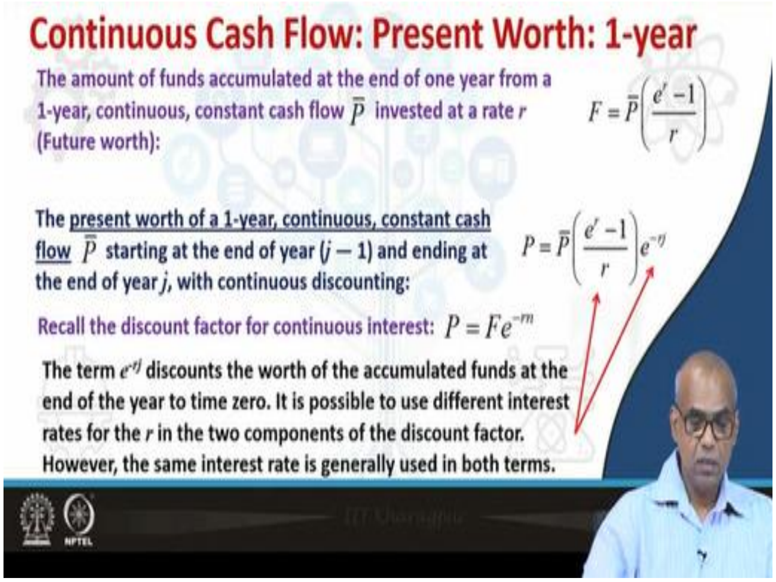

How to compute the present worth we know the expression for F, we have just computed that as F equal to P bar into e to the power r - 1 by r. The present worth of 1 year, continuous constant cash flow P bar starting at the end of year j - 1 and ending at the end of year j that means transfer 1 year, with continuous discounting is equal to P bar into e to the power r - 1 by r by discounting factor. Recall that discount factor for continuous interest P equal to F into e to the power - r n.

So you multiply this quantity by e to the power - r j, I will discount F to time 0; that means present value or present worth. So F can be discounted to get P, so that will be e to the power - r j discounts the worth of the accumulated firms at the end of the year to time 0, it is possible to use different interest rates for the r in the 2 components of the discount 1 factors, look at the arrows there are 2 factors. So you can use 2 different r is that of interest. However, the same interest rate is generally used in both the terms.

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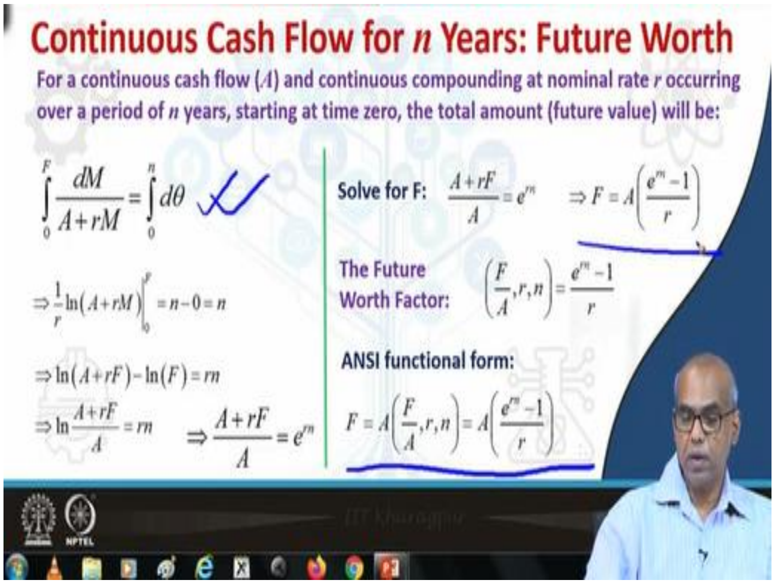

Now, we talked about what happens in 1 year? What happens in n years? what will be the future value? So instead of integrating j - 1 to j, let us integrate 0 to m; m for the d theta, so d m by A + r, m integral 0 to F equal to 0 to n integral 0 to n d theta. Here, A is the continuous cash flow. The amount is A which is being compounded continuously at the nominal rate R this cash flow is occurring for n years starting from time 0 and ending at time n.

So the future value will be you just simplify and solve for F, F equal to A into e to the power n - 1 by r. So the future worth factor will be e to the power, r n - 1 by r. We can also write in our usual notation of the ANSI functional form. So all we are doing here is for one year, we integrated this from j - 1 to j where integrating from 0 to n and there we are getting e to the power r - 1 by r here, we are getting e to the power n - 1 by r.

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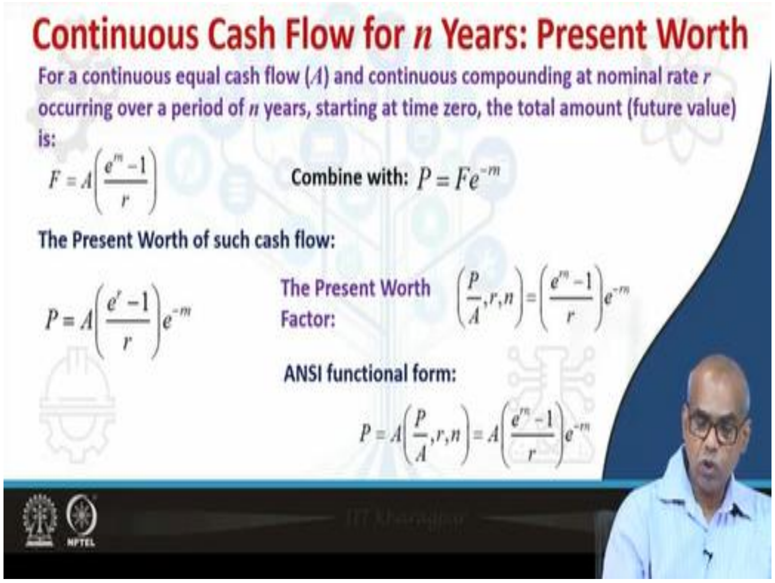

How to obtain the present worth, for continuous cash flow that is occurring for n years we know F equal to a into e to the power r n - 1 by r will combine it with P equal to F into e to the power - r n and then we obtained the present worth factor P equal to this not P equal to F into e to the power - r n, this F we obtained from here. So the present worth factor will be this term. We can write in terms of ANSI functional form P by A.

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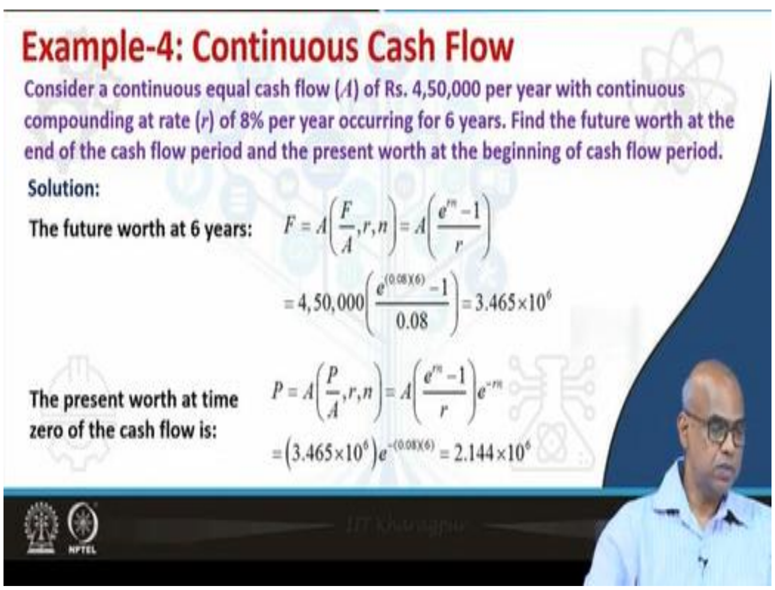

Let us take another example, consider a continuous equal cash flow our rupees 4,50,000 per year with continuous compounding at rate of 8% per year occurring for 6 years. Find the future worth at the end of the cash flow period and the present worth at beginning of cash flow period. The feature worth at 6 years can be computed by state forward application of the expression for F of future value.

Here, A is 4,50,000, R is 8%, so 0.8, so put these values in the expression you get as 3.465 into 10 to the power 6. The present worth at time 0 of the cash flow you use P equal to A into e to the power r n - 1 by r into e to the power - r n, we have already obtained this value as 3.465 into 10 to the power 6. So multiply that by e to the power - r n and you get 2.144 into 10 to the power 6.

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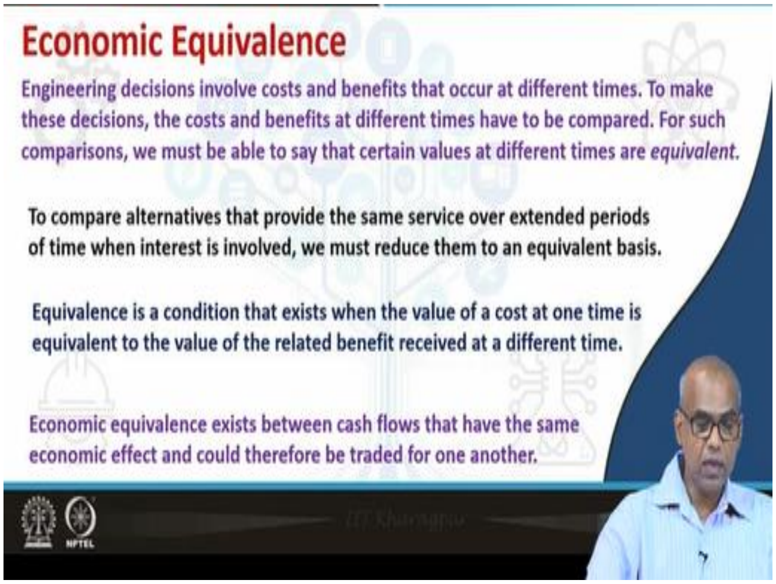

Now Engineering decisions involve cost and benefits that occur at different times to make this decision the cost and benefits are different times have to be compared for such comparisons, we must be able to say that certain values at different times are equivalent. To compare alternatives that provide the same service over extended periods of time when interest is involved we must reduce them to an equivalent basis.

Equivalence is a condition that exists when the value of a cost at one time is equivalent to the value of the related benefit received at a different time. Economic equivalence exists between cash flows that have the same economic effect and could therefore be traded for one another.

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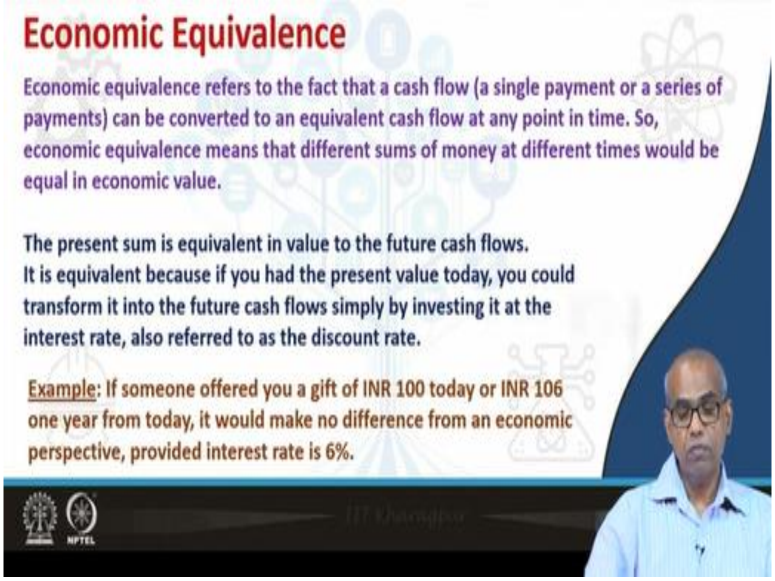

Equivalence refers to the fact that the cash flow a single payment or a series of payments can be converted to an equivalent cash flow at any point in time. So, economic equivalence means that different sums of money at different times would be equal in economic value. The present sum is equivalent in value to the future cash flows. It is equivalent because if you had the present value today, you could transform it into the future cash flows simply by investing it at the interest rate also referred to as the discount rate.

For example, if someone offered you a gift of INR 100 today or INR 106 one year from today, it would make no difference from an economy perspective provided the interest rate is 6% per year.

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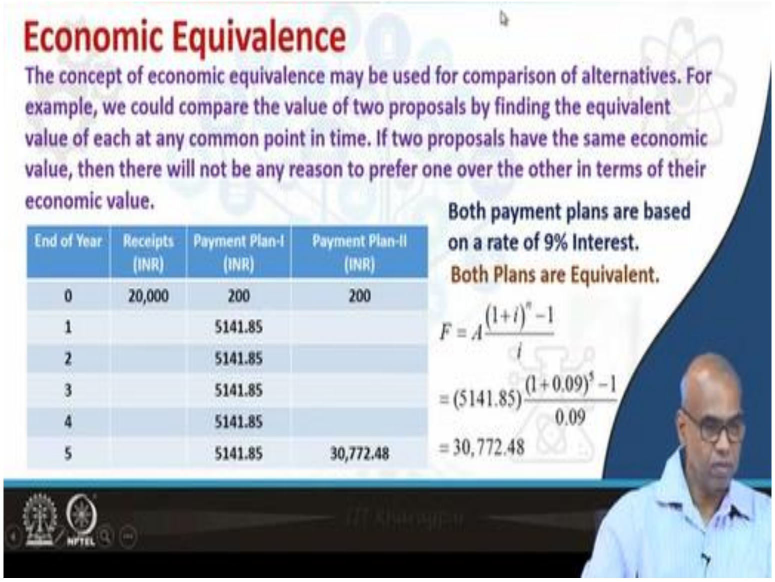

The concept of economic equivalence may be used for comparison of alternatives. For example, we could compare the value of two proposals by finding the equivalent value of each at any common point in time. If two proposals have the same economic value, then there will not be any reason to prefer one over the other in terms of their economic value. In other words, they are economically equivalent.

Look at the table you have taken loan of 20,000 rupees and the bank has given you two repayment, two payment plan one and plan two, both the payment plans are based on a rate of 9% interest. Now if you do the computation, you see that both the plans are equivalent, in case of in both the plans you make a down payment of rupees 200. So basically you are taken to 20,000 rupees, 20,000 – 200.

Now in payment two year finally you are making a payment of plan two enough you are making a payment of 30,772.48. Now, this is a uniform series, if you compute the future value you get the same value 30,772.48. So for the bank both are equivalent economically equivalent.

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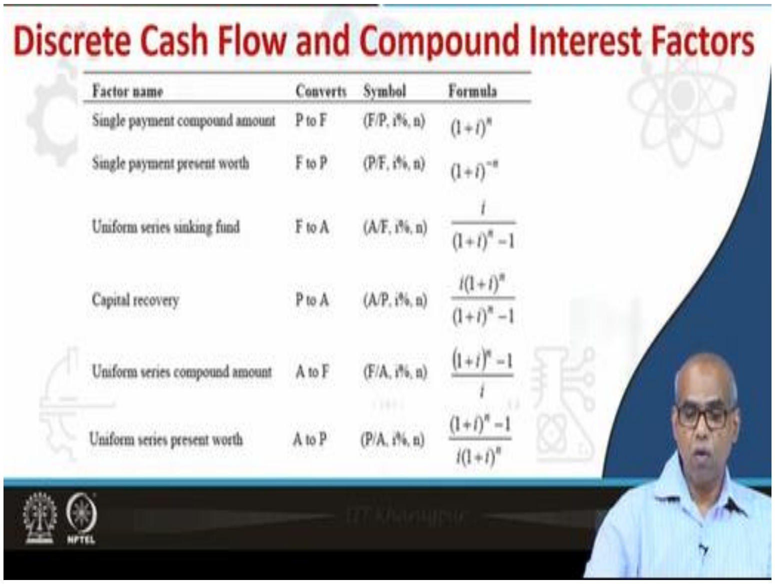

Now, here the several compounding factors, we talked about I have just summarized. Remember that the cash flow may be continuous or discrete, similarly the compounding may also be at the end of interest period or it can be continuous. So there may be four different types of options, so this table summarizes discrete cash flow and discrete compounding, then the compound factors under discrete cash flow and compounding. So this formulas some of these formulas we have derived.

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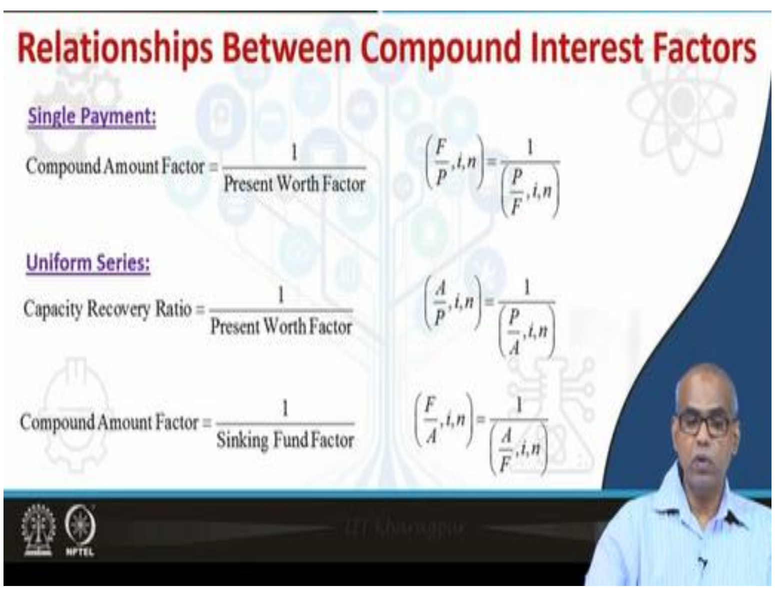

These are some relationship between components factors.

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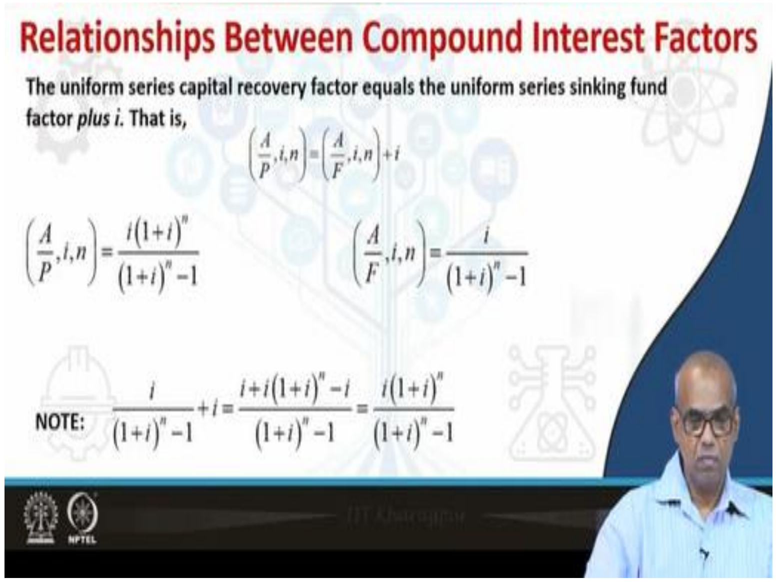

This is another relation between compound interest factors.

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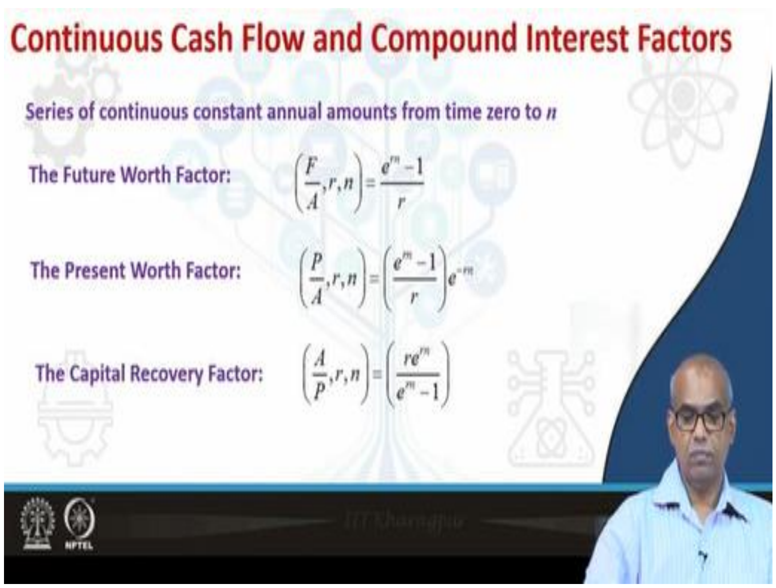

And, here I list again some of these factors for continuous cash flow and continuous compounding.

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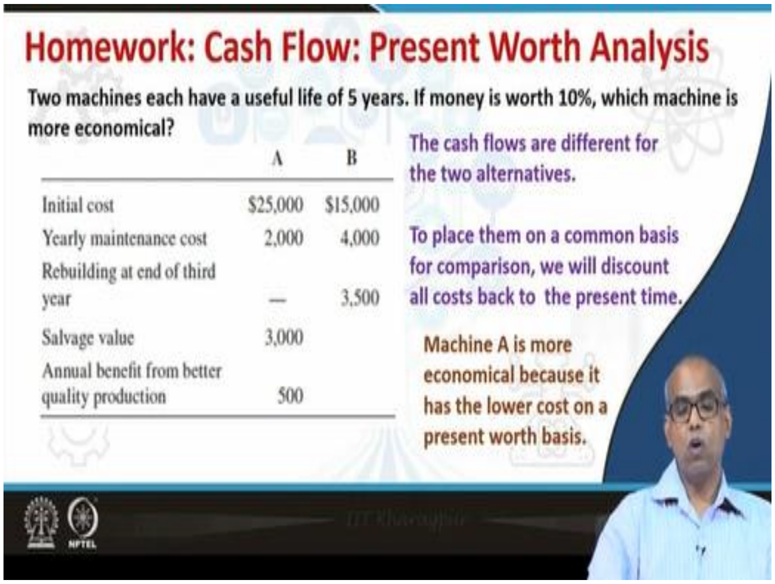

Now finally take this as homework problem; two machines each have a useful life of 5 years. If money is worth 10% which machine is more economical machine A and machine B. Initial cost are given, yearly maintenance cost, rebuilding at end of third year does not exist for A exist per B salvage value 3000 dollar for A, 0 for B, annual benefit from better quality production 500 dollar for A and 0 for B.

Now, we have to find out which machine is more economical, so basically economic equivalence you have to find out. The cash flows are different for the two alternatives, so first draw the cash flow diagrams for machine A and machine B to place them on a common basis for comparison will discount all cost back to the present time. So find out the present values or both the cash flow diagrams.

If you do that, you will see that the machine A is more economical because it has the lower cost on present worth basis. So complete this and take it as homework with this we stop our discussion on lecture, number 18.