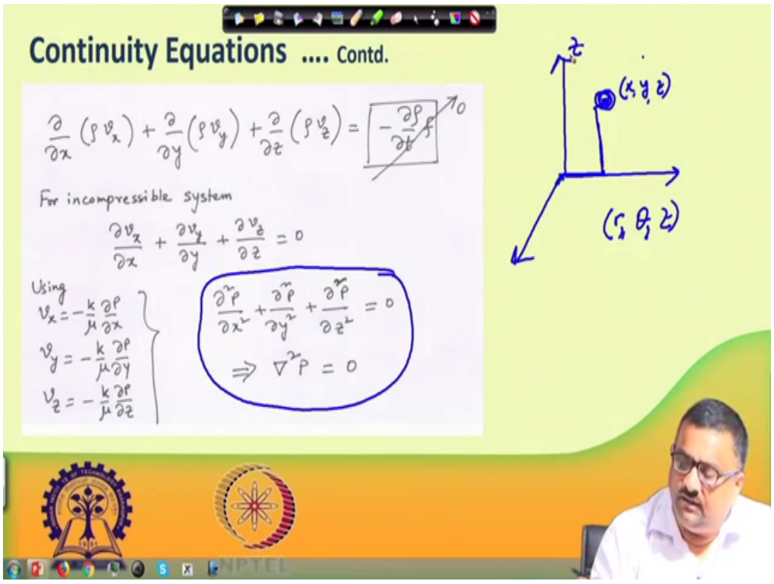
Flow Through Porous Media
Prof. Somenath Ganguly
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 06
Mass Continuity (Cylindrical Coordinates)

I welcome you all again to this course on Flow Through Porous Media, what we were discussing is mass continuity as it applies to porous media. So, what we discussed is Darcy's law how it how we extend is Darcy's law to mass continuity in Cartesian and cylindrical system. Cartesian we have briefly touched upon, now we get into the cylindrical coordinate system and how we get how we arrive at the pressure equation.

(Refer Slide Time: 00:55)



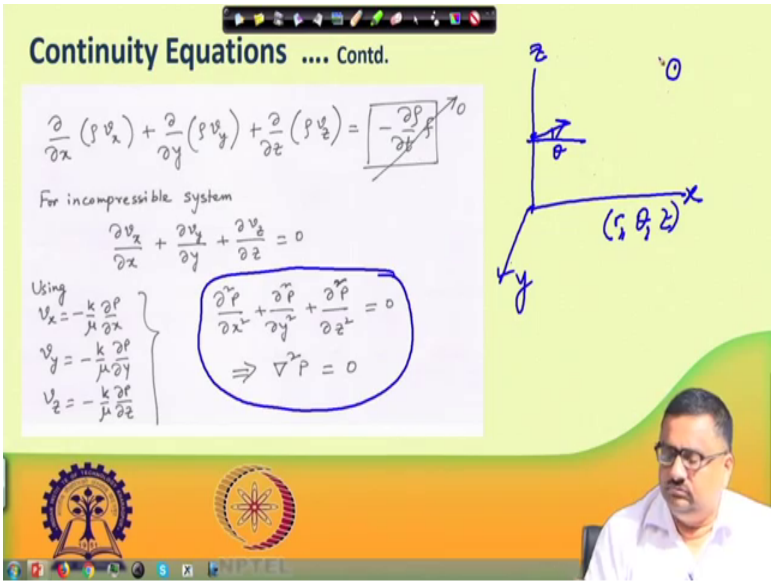
So, now this is the continuity equation that we are working with in the last lecture and we have seen that this equation; this equation is leading to the form here grad square p is equal to 0. Now, if we try to extend so, this is all in Cartesian system right; what we said here is that this is all I mean I have a x y and z and I can I have a point here. And then we are trying to find out how the pressure changes in x y and z. So, that we can if I have a porous medium and I set my origin and I set my xyz coordinates.

So, how pressure vary from point to point we see that the pressure has to follow certain rule and that rule is given here. So, this is for the Cartesian system, now many cases particularly when you work with a well be it in a be it reservoir where the some water is injected or oil is produced or some contaminant is injected into saline aquifer or. So, those of the there will not a Cartesian system of equations may not help you much if you are working with a 1 dimensional or 2 dimensional set up they are the Cartesian system is the only choice.

But, if you have some kind of symmetry if you can work out in a cylindrical system and then it is better to work with a; work with a different coordinate system altogether. What would that coordinate system be? Let us say I have the same point, I have a point here that point is defined by x y and z. So, x it is going up this is my x axis I am going up to this, I am going up to y axis I am going up to this and then probably I am going in that direction and find out what is the corresponding coordinate. So, instead of that what we can possibly do is we can work with something called an r theta z system r theta z; r theta z means that I have first of all I have this is my z axis.

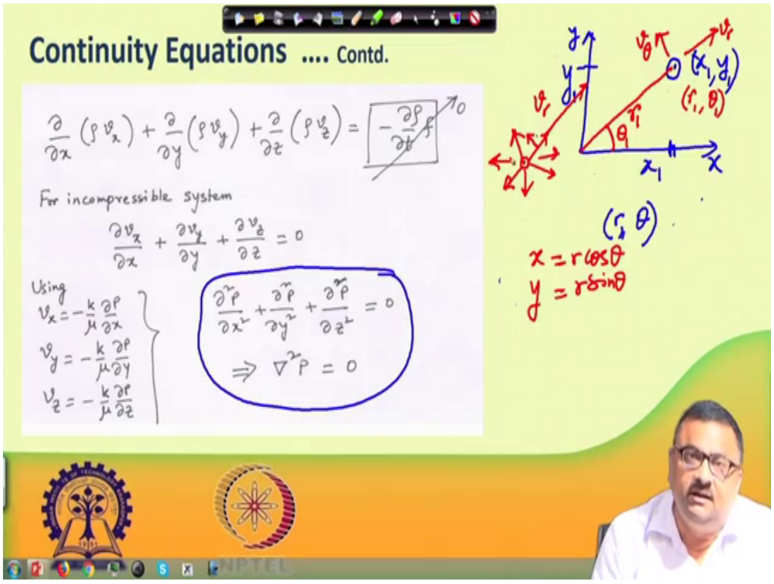
So, let us say we are working with alright I mean I can let us just the first focus on only on r theta system. It is the r theta system work with work in something like this, let us say I am defining a point here.

(Refer Slide Time: 03:45)



I am giving I am calling these as x, I am calling these as y, this as z and then I can think of these r it is making an angle theta in this direction. For a 2 dimensional setup let us say in xy system I have again let me change it just sorry about that. Let us first look at the 2 dimensional setup that would work out better. Let us say I have a point here and I want to know I can have by xy system I have my this coordinate given as x this is at x and this is at y.

(Refer Slide Time: 04:45)



So, I go this is this as a coordinate of x 1 y 1. So, I have this distance is x 1 this is my x axis, this distance is x 1 and this distance is y 1, this is the y axis. So now, I can define this point as x 1 and y 1, the other way of defining this point would be to go by these we call it as r and this as the angle theta. So, instead of calling it let us say this is r 1 and theta 1. So, instead of calling it x 1 and y 1 we can call this as r 1 theta 1 ok. So, in that case we do not have the xy axis anymore. So, in case of r theta system we can write automatically x is equal to r cos theta y is equal to r sin theta.

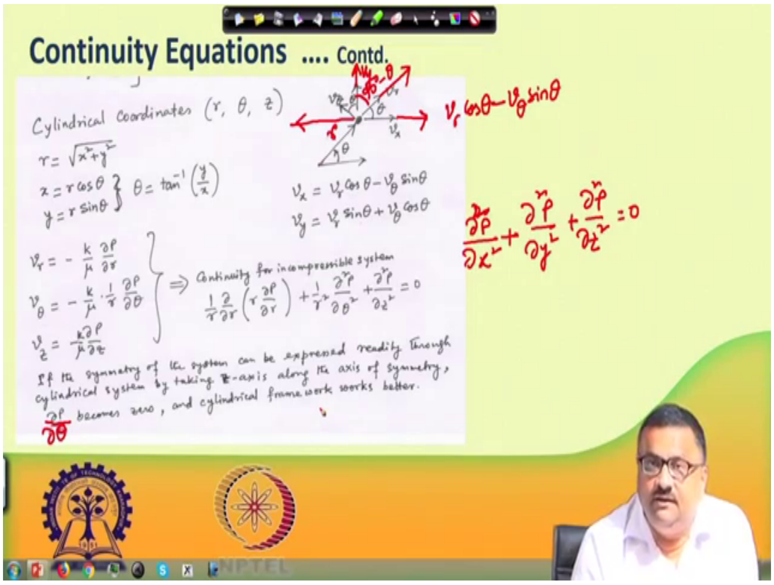
So, I do not need to rely on xy system at all, I can work completely on r theta system. So that means, the point is that I have to draw continuity not in terms of del del x or del del y instead I will draw the continuity on del del r and del del theta. So, that that could be a possibility and z would be simply this entire thing would be shifted upward because, if z is the third dimension. So, it will be shifted to the in that direction of third dimension. So, this is this way lot of times the I can it will be of benefit and in some cases where, if you have a similarity if you can; if you can; if you can remove one of these terms by considering a cylindrical coordinate this can very well happen.

For example, if it is a truly radial flow if it is a truly radial flow you can see here that I can consider only the v r which is the velocity in r direction. And I have to consider v theta, which is in which direction? Let us say if this is my r then theta would be in this direction right, theta it if this is the arc direction then theta would be in this direction. So, any velocity would be v r would be in this direction and v theta would be this direction.

So, instead of x y putting this you know source a point source as in x y terms we can or v x v y we can write it in terms of v r and v theta. And then if it is truly a point source then I do not see any reason why there should be a v theta term at all because, v theta we are looking at a swirl, we are looking at a vortex. So, when we have a vortex on the other hand if we have a true vortex, I do not see any reason why there should be a v r term at all ok.

So, it by this way so, by considering this r theta system we can simply consider for a source v theta is equal to 0 or for a true source or for a true vortex then you can consider the v r to be equal to 0. Then you can even just straight away you are getting rid of one term ok. But, if you are working with the x y system then you have to continue with all this terms and that will complicate the final expression. So, that is the benefit we are trying to get by considering the r theta system.

(Refer Slide Time: 08:37)



So, now if we look at this cylindrical coordinate system that is what we were discussing earlier here you can see this you have r theta system. So, you have v x in this direction; v x in this direction and v y is in this direction ok. So now, vx and v y vx and v y they are in this direction. So, now it can be replaced by v r and v theta ok, where r is this much this is basically the r and this angle is theta. So, then r is basically if you are working with x y system r is basically square root of x square plus y square and x is r cos theta y is r sin theta; theta is tan inverse y by x. This follows directly from geometry. And further if you write v x is equal to v x is equal to v r cos theta minus v theta sin theta you can see this here v r this is v r. So, corresponding component in the x direction would be v r cos theta.

So, here you have v r cos theta, then you can see here v theta is already moving this way. So, v theta will have also a component which is basically 90 degree minus theta in this direction right this angle is theta. So, if this angle is theta then this would be 90 degree minus theta. So, a cos of 90 degree minus theta would be sin theta and these would be in the negative direction of x. So, that is why it would be v r cos theta minus v theta sin theta basically, v theta cos of 90 degree minus theta and it is in the negative x direction. So, that is why you have a minus sign here. So, that is exactly what we have written here v x is equal to v r cos theta minus v theta sin theta.

So, if you do the same exercise with v y, v y would be equal to v r sin theta. So, you have this is the theta. So, this angle is 90 degree minus theta. So, v y direction. So, v y would be v r cos of 90 degree minus theta so; that means, v r of sin theta positive. And v theta v theta is also positive because v theta is in this direction. So, this will also have a positive component in y direction. So, that would be simply v theta cos theta and they would be additive. So, that is why we have now v y is equal to v r sin theta v r cos of 90 degree minus theta plus v theta cos theta. So, this is how v x and v y relate to v r and v theta and in in terms of vector component.

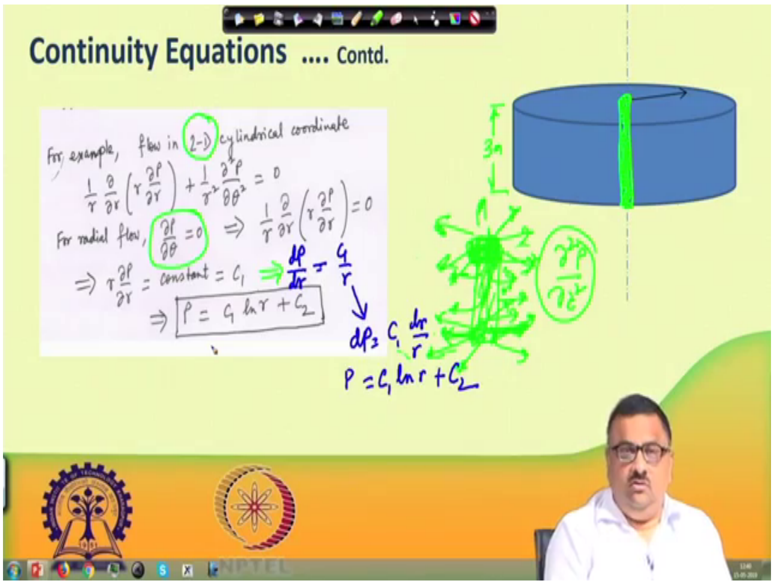
And further the definition of v r v theta and vz if you; if you; if you draw here you will find that v r is equal to if you; if you; if you simply use these equations and if you use this definition v r would be simply equal to minus k by mu del p del r ok. And v theta would be equal to v theta would be equal to minus k by mu 1 by r del p del theta this would be the definition of v theta now, in terms of del p del r and del p del theta. And v z would be as before minus k by mu del p del z ok; this would follow immediately if you take these and if you take these and put them together this would automatically follow. So now, if you bring in now if you try start writing the continuity equation for incompressible system.

So, what was the continuity equation for incompressible system in Cartesian coordinates? It was del square p del square p del x square plus del square p del y square plus del square p del z square that is equal to 0. On the other hand we have the continuity in r theta system. So, what is what are where are the differences you see here? The differences I see here is del square p del z square this term remains as it is ok, del square p del x square here we have 1 by r del del r of r del p del r 1 by r del del r of r del p del r. This would be the equivalent to del square p del x square term and here we have 1 by r square del square p del theta square.

So, this would be the continuity equation if somebody works with r theta coordinate system and the advantage I have already mentioned. I mean in some cases I mean; obviously, I can see we have added complications here. So, if it is a straight for if there is no advantage nobody would work with this kind of structure. But, if there are some advantages by which some of the terms can be completely struck off. So, then we can go for then there is a merit to this. So, where is where the merit is if the symmetry of the system can be expressed readily through cylindrical system by taking z axis along the axis of symmetry and del p del theta becomes 0.

So, when we do not have to consider this term then we and if you are working with a 2 dimensional system; then you are left with only this equation. And this is of advantage to me and instead of working with del square p delta because we have only one I mean the we can; we can; we can; we can make this an od instead of p d in that case. So, z axis is an x and del p del theta this is del p del theta del p del theta becomes 0; that means, you do not have the v theta component at all and cylindrical framework works better. So, this is when these are the conditions we would like to use this cylindrical system.

(Refer Slide Time: 15:09)



So, now let us look at the cylindrical system here where let us look at this cylindrical system here. So, what we did is we are working with a 2 D, 2 D cylindrical coordinate system. So, 2 D means a z is not there anymore. So, we have only 1 by r del del r of r del p del r plus 1 by r square del square p del theta square that is equal to 0. That other term with del p del z del square p del z square that term is not there we are working with a 2 dimensional setup.

Why do we work with a 2 dimensional set up, what is the advantage? The advantage is something like this suppose I have a well ok; so, what is what would be the how the well looks? Let us say this is a reservoir and I have a well from which we are drawing oil or we are injecting water.

So, this is the well alright; so, this is the well and these let us say this is the well and these well here it is all connected ok. So now, what is a what is that what is a depth of this? Let us say this depth is to the tune of let us say a few meters let us say 3 meters, let us say I have a reservoir I have a depth of let us say 3 meters.

So, when I am drawing something from here or I am injecting something let us say injecting would be easier because, then it is a source when I am drawing something then it becomes a sink. So, let us say this is acting as a source. So, in that case we would be we can say think of I have a conduit I have a conduit and then I have a porous medium.

So, porous medium has a large resistance for the flow to take place. So, if I look at the pressure profile intuitively if I look at the pressure, I would tend to see that the pressure will equilibrate within this even if there is a; so, so there would be flow taking place everywhere flow. Flow would be taking place out like this, but the pressure would be practically will not be much different over these 3 meter of height.

Pressure would be practically same, it would be; it would be; it would be flowing out. There would be some amount of pressure reduced because, there would be some extra flow that are that is traveling through this which is; which is; which is going. So, there has to be a pressure gradient; obviously, because you can see here the flow is going in and then the flow some material would be entering into the porous medium here.

So, these here there has to be some flow from this end from the top to bottom and so, there has to be some differential pressure. But, given if given the fact that this is this diameter is much larger compared to the pore space through which the fluid is pushed in I can say this pressure drop is not very significant.

So, in that case one can assume that one can work with a 2 dimensional framework and whatever they work with they write it is per meter depth perpendicular to this 2 dimensional plane. So, they are working with a 2 dimensional plane and everything that they work with say for example, the velocity, the flow rate etcetera they would be all per meter depth perpendicular to this 2 dimensional plane ok.

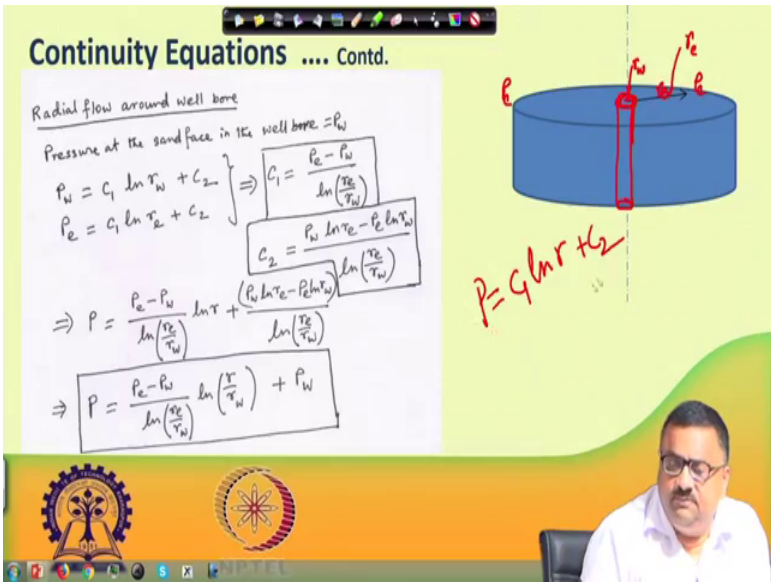
So, this is a; this is a common customs a custom and then you can and this has merit to it; these are this as definitely this has merit because this pressure drop is not much. So, what you are doing is you are ignoring these del square p del z square term ok. So, this term you are not considering instead you are treating as if these say these well is cylindrically the this this well there is; there is; there is a radial flow out from this well and whatever you see in the top of the well. And whatever you see in the bottom of the well there is not much of a difference. So, you can treat this all as same.

So, whatever you are doing you are working with a 2 dimensional framework only if this you are you are saying that this 2 dimensional work that I am doing now you would be valid over the next 1 meter depth perpendicular to this cross sectional area. So, over this 1 meter and that same thing is applicable over this entire 3 meter of depth that we are talking about ok. So, this is; this is; this is a very common way of doing things and it simplifies the matter quite a lot. So now, if you are working with for if you are working with a flow in 2 dimensional cylindrical coordinate then this is the equation you ignore del square p del z square term. And further if it is truly a radial flow; that means, flow is radially going outward then del p del theta would be equal to 0; that means, the velocity v theta is equal to 0.

So, you are left with only the first term only the first term is left. So, it is 1 by r del del r of r del p del r equal to 0. So, this is the only equation left. So now, if you do this integration you have first of all r del p del r would be equal to constant, if you take del del r of r del p del r equal to 0 or dd r of r dp d r equal to 0. So, r dp d r equal to constant and let us say that constant is c 1. So, then if this is c 1 then del p dp dr would be equal to c 1 by r and then d pd r d pd r. So, this would this would imply that dp d r would be equal to c 1 by r.

So; that means, when you do this integration it would be dp is equal to c 1 d r by r. So, then when you do this integration again. So, integration of dp would be equal to p that is c 1 d r by r that would be equal to c 1 ln r plus there would be a constant of integration and that constant of integration is written as c 2. So, any pressure profile here would take a form c 1 ln r plus c 2, if it is a truly radial flow. Now this so, make note of this expression p is equal to c 1 ln r plus c 2.

(Refer Slide Time: 22:17)



So, now if we continue this radial flow around a well bore. So, well bore means I have a well bore here I have a well bore here. So, this is the well bore; so, the wellbore has a radius. So, what is that radius? That radius is r w so, this radius is r w the well bore radius, well bore radius is r w and the pressure at the sand face in the well bore; that means, at the wall of these well and the sand face of this well bore let us say that is P w. So, this pressure here is P w and far away from the well bore let us say at a location r e, r e the pressure is P e. So, all in this entire perimeter it is the pressure is P e on that perimeter at radius r e the pressure is p e.

So, we are looking at these this radius is r e and the corresponding pressure is P e and P e is everywhere; P e is here also and this radius of the well is r w and on the sand face the pressure is P w. So now, we can apply this P is equal to c 1 ln r plus c 2 that we learned just now from the earlier section; what we learned just now is P is equal to c 1 ln r plus c 2. So, in this case we can apply the P w and P e; that means, it would it would apply to the point on the sand face; that means, the edge of the well here.

So, it is so, this is P w is equal to c 1 ln r w plus c 2 and this will also apply at the outer perimeter that we are talking about at radius r e the pressure would be equal to P e. So, it would be P e is equal to c 1 ln r e plus c 2. So, P is equal to c 1 ln r plus c 2. So, that would be that we can apply both at the sand face here at the well sand face as well as some point r is equal to r e.

So, these are the two equations we get, if we solve these two equations we get c 1 is equal to P e minus P w divided by ln of r e by r w and c 2 is equal to P w ln r e minus P e ln r w divided by ln r e by r w. So, these are the c 1 and c 2 that we end up with. So, now if you put this c 1 and c 2 that is, your original equation was what? Original equation was P e is equal to c 1 ln r plus c 2.

So, you put this c 1 and c 2, first you solve these two equations and then whatever c 1 and c 2 you get you put it here. So, what I get here in this case is P is equal to c 1 ln r plus c 2 and we can simplify this a little bit because, here we have ln of r, here I have a ln of r e by r w. So, sorry here we have a ln of r here we have a ln of r e and ln of r w. So, we have clubbed this together and we get P is equal to this quantity ok. So, this becomes the pressure profile, this gives you the pressure profile. So; that means, I have what all is given here? At r is equal to r e the pressure is P e is known, at r is equal to r w; that means, sand face of this well it is known.

So, if these two pressures are known and if this corresponding r e and r w at which radial point this is happening if these two are known and pressure the well pressure these are known; then pressure at any intermediate point can be obtained following this expression. So, this is; this is; this is this expression can be used to find out the pressure at any intermediate location I here.

So, this is the pressure, now we using this pressure we can take the take because, the v what is v r? We do not have any v theta in this case, we have only v r and v r is equal to minus k by mu del p del r. So, we can take the derivative of this with respect to r and find out what is the radial velocity.

So, that also we can do and we can do this exercise further. So, we can so, what I am going to close this lecture now. What I am going to do next is just remember that, this is the final expression we arrived at and this is the master equation. And if we when we applied the boundaries we arrived at this equation. So, we will work with this further in our next lecture to arrive at the velocity at different points and we will try to plot them; how pressure changes with radius, how velocity changes with radius and proceed on this further.

Our basic is the first step was our continuity, we have talked about Cartesian system; then we said that in some cases we can make use of cylindrical framework. And here this is one example where simply v theta term we considered to be 0 and v z is we are not considering any v z we are treating these as a 2 dimensional problem and we have sufficient justification for it.

And with those assumptions we are arriving at a pressure profile which we can take a further to find out what is the velocity profile. So, that I am going to continue in the next lecture. So, this is all I have for today's lecture. So, take home messages you remember this this P is equal to c 1 lnr plus c 2 and this expression that we arrived at, that is all I have for today.

Thank you.