

Flow through Porous Media
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Lecture – 26
Miscible Displacement (Movement of Concentration Pulse)

I welcome you to this module of Flow through Porous Media. What we are discussing was Miscible Displacement; that means, already a fluid is present in the porous medium and now, we are injecting fluid which can mix with this resilient fluid. So, in this case inside porous medium, how the mixing takes place that is something we want to understand.

So, before we get to this first we are trying to understand how the mixing takes place in a capillary and so, we were discussing about Taylor dispersion. That means, when the velocity profile itself is not uniform then that can contribute to the mixing so, how these sort of; how this mixing takes place.

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Diffusion in a moving front

Solution

$$c_1 = \frac{M/A}{\sqrt{4\pi Dt}} e^{-\frac{z^2}{4Dt}}$$

when the pulse moves through a capillary at uniform velocity \bar{u} ,

$$c_1 = \frac{M/A}{\sqrt{4\pi Dt}} e^{-\frac{(z-\bar{u}t)^2}{4Dt}}$$

When the velocity profile becomes non-uniform, the governing equation changes, along with BCs

So, let us see here we have what we discussed earlier is that in a static system this is the equation we had arrived at. For the for the static system this is the equation we arrived at c_1 is equal to c_1 is a function of z and t position and time, and this is given by M total amount of solute one places at z equal to 0 in terms of kg of solute and as a function of z and t .

And, what is this we have when we are talking about Taylor dispersion or everywhere you will find that we want to stick to this particular form of equation. So, that at any time we can compare see here we get a feel, what is the feel that I said that I have introduced we have introduced a Dirac function, and we expect that this would diffuse this would diffuse like this. So, with time we expect that it would take these this kind of shapes.

So, what we did here is suppose I say that the concentration at any location it will increase with time like this when z is greater than 0. So, now, if we change let us say diffusivity D , what would be the effect? If D is increased we always know that the broadening will be faster because it is diffusion coefficient, something has a higher diffusion coefficient; that means, it mixes faster right something has a very low diffusion coefficient.

So, though they are supposed to be mixable so, I have too fair to say one liquid and other liquid we put them side by side they are supposed to be mixable. So, they will mix if you leave it for a long time, but it will take long time to mix. So, diffusion coefficient is that characterizing characteristic parameter which tells you that how fast it will mix how fast the molecules will diffuse.

So, naturally if D is increased I can see the impact here. So, now, suppose Taylor dispersion with all the equations they have considered and everything if they arrive at a very similar equation. But not D instead of that some other parameter, some other clubbed parameter, lump I mean it is say let us say instead of D they are talking about I mean I am giving you an example let us say R naught square v naught square by 48 some lumped parameter I have. All other terms remaining same, I have a lump parameter here.

So, in that case we know that if we change R what would be the how the lump parameter will change and if the lump parameter increases we know that it will have a very similar; you will have a very similar experience here as when the diffusion coefficient increases. So, we the Taylor wanted to I mean in this entire analysis Taylor wanted to stick to this particular form. So, let us see what they have done.

Now, one thing we have also pointed out at this time is that if this moves this entire this Dirac function that is the pulse that we placed if this moves at a constant velocity u bar so, the center of this pulse, so, then we can still find out a concentration c 1 as a function of z and time; space and time, but only thing is there has to be a correction z minus u bar

t whole square that correction has to be there to take care of the moving front rather than a static front.

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Diffusion in a moving front ... Contd.

$\frac{\partial c(r,z,t)}{\partial t} = \frac{D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c}{\partial r} \right) - 2 \frac{v_z}{R_0} \left[1 - \left(\frac{r}{R_0} \right)^2 \right] \frac{\partial c}{\partial z}$

$t=0, \text{ all } z, c_1 = \frac{M}{\pi R_0^2} \delta(z)$

$t > 0, r=R_0, \frac{\partial c}{\partial r} = 0$ (no flow)

$t > 0, r=0, \frac{\partial c}{\partial r} = 0$ (symmetry)

Solution

$\bar{c}_1(z) = \frac{1}{\pi R_0^2} \int_0^{R_0} 2\pi r c_1(r,z) dr$

$\bar{c}_1 = \frac{M}{\sqrt{4\pi E_2 t}} \exp\left(-\frac{(z - \frac{1}{2}vt)^2}{4E_2 t}\right)$ (Taylor Dispersion)

Here, $E_2 = \frac{\text{Dispersion coefficient}}{48 D} = \frac{R_0^2 v^2}{48 D}$

So, now we had talked about the stellar dispersion we have already talked about this governing equation. So, we said that this governing equation includes a radial diffusion; that means, diffusion in this direction radial diffusion in r direction and convection in z direction; that means, flow is continuously taking places in this direction. So, some with the flow some mass, some molecules that are put into this washer shaped differential element some are leaving because of this flow.

So, how much is the accumulation that is accounted in this ok. This was not important when you have a uniform velocity profile, but these are important when you have a non-uniform profile. So, with this and we have talked about the boundary initial and the boundary conditions. Now, at this time we must note one thing is that when it comes to a capillary, what we see at the outlet; when it comes to a capillary when we what we see at the outlet is not the concentration as a function of r, that is that is not possible for us to measure.

Think of this I have a capillary, if you want me to find out, if you want me to prove that this equation is valid, so, for that what we need to know is I have given you already a Dirac function I have just introduced a pulse of a salt or pulse of a dye ok. And, then at

the outlet I am measuring the concentration profile, but concentration profile with respect to r , see earlier we did not have a concentration profile with respect to r , mind it.

When you have a static system just simply traveling at an average velocity \bar{u} will uniform velocity profile; that means, a front is all uniform. So, I mean it is more like a plug flow. So, at that time concentration was only a function of position and time, but here we have an added problem that this concentration is a function of radial distance, axial distance z and r two axis as well as time because we have function of r , function of z and function of time.

So, concentration is a function of r , z and time, but frankly speaking we have no way to measure concentration as a function of r because these dimension we are talking about is to the order of 1 micrometer. So, within 1 micrometer it is next to impossible; finding out the concentration at various points I mean in our traditional way of doing things. So, rather what we do is we can collect slices of; so whatever is arriving at the outlet we can collect the slices of it and mix those slices.

So, I am collecting let us say this fluid and I am getting slices. So, this slices and if we get these slices they are all well mixed because I am collecting it in a cup. So, one cup for this, another cup for this, another cup for this, another cup for this and this cup whatever is collected they are all mixed up and that concentration we can measure by putting it in a spectrophotometer, UV spectrophotometer or measuring the refractive index or doing some other way of doing things. But, I have to collect it in a cup and there has to be some substantial volume to do this measurement.

So, obviously, what we are doing is; so, we said that we had stretched the pulse we have stretched the pulse like this. So, essentially so, at the outlet this stretched pulse will arrive maybe they will be more diffused, maybe it is more diffused here, maybe more diffused here because of the diffusion because already I said there will be a radial diffusion taking place. So, now, when we are collecting this I am collecting it in slices. So, this slice is one, next slice is one, next slice is one, next slice is one like this. So, this slice I am collecting it in a cup and I am measuring it measuring their concentration.

So, essentially we can collect we can get c_1 at z this is something which you can measure. So, basically this c_1 is this concentration is averaged over all r . So, how we can average it? We can write this c_1 at r, z is existing over this annular shaped area right;

over this annular shaped area this c_1 is existing. So, if we multiply this c_1 with $2\pi r dr$, $2\pi r dr$ is what? This, these cross sectional area these hatched area is basically you already know it is π of r plus dr whole square minus πr square and if you ignore π of dr whole square term because dr is small. So, you end up with this area as $2\pi r dr$.

So, what is the volume of this washer shaped element? This volume of the washer shaped element is $2\pi r dr$ multiplied by dz . So, $2\pi r dr dz$. So, $2\pi r dr dz$ so, what is this $2\pi r$ is in meter, dr in meter, dz in meters. This gives me meter cube. This is the volume of this washer shaped element this entire element. These entire element will have a concentration of c_1 , because c_1 is applicable, c_1 within bracket r and z ; that means, around r a differential area and around z a differential area so, this is the differential volume that you can construct around r and z . So, this is where c_1 is valid.

So, this is unit of volume in meter cube this is in kg per meter cube. So, kg per meter cube kg per meter cube this gives me kg. So, all such annular shaped elements if you integrate from 0 to r , so, what you do is then one thing you can note here is that you do not need to work with this dz because this is the concentration of one for this differential volume and then you have the total volume of these system is basically πr naught square dz , that would be the volume of this whole so differential if you extend it so, for the entire cross section.

So, this dz you do not have to bother and this numerator has to be integrated from 0 to capital R , so, that is 0 to capital R 0 so, that is exactly this average concentration; that means, total mass contained within these washer shaped element divided by the corresponding volume. So, this is the average concentration. So, when you collect this in pieces you are collecting the average concentration.

Now, what Taylor has shown is Taylor has solved this equation, and Taylor has shown that the c_1 bar as a function of z that is written here is m divided by πr naught square divided by square root of $4\pi E z t e$ to the power this is exponential time e to the power minus z minus v naught t whole square divided by $4 E z t$. [FL] v naught is v naught is same as the earlier we had talked about u bar is same as the average velocity.

What is average velocity? Average velocity is; average velocity that we have studied in the context of flow through flow through a pipe or flow through a capillary when we define friction factor. There we have we have arrived at this particular expression and

there this is the that v naught that we considered here which is same as u bar which we considered in case of a uniform flow where the velocity is uniform. So, it is same as the average velocity. In case of a uniform velocity that is the single velocity, there is no other velocity; whereas, in case of parabolic velocity profile this is the average velocity.

So, this is how; so, now, this equation looks very familiar to us in the sense that this equation is basically M by a was πR naught square this is already we know square root of that time we had $4 \pi dt$ in case of static system. You recall it is $4 \pi dt$. Instead of D now we have a term which is $E z$ which is different, but that we look into, but $4 \pi dt$ e to the power there also we had the same thing minus z minus v naught t we can very well now relate. It was originally in static system it was e to the power minus z square by $4 dt$.

Now, then next what we did is we said ok, we have a moving front, but uniform velocity in that case we shift we were talked about a moving coordinate system and instead of z we had subtracted this v naught t . So, it becomes z minus v naught t whole square by $4 dt$ and now in this; but that time it was a z minus u bar t because there was only one velocity single uniform velocity. Here in this case in case of a Taylor dispersion we have the average velocity.

So, average velocity we can still relate. I mean average velocity means that the center of this pulse somehow is also moving at. So, that is basically a reference system, so, z minus v naught t whole square by $4 E z t$ so, this $E z$ is different. So, Taylor had been Taylor is successful in bringing this equation to the same form as we had seen in case of static system or the variant that we have seen in case of moving front with uniform velocity.

So, and now what is $E z$, Taylor found that this $E z$ is the dispersion coefficient this is called dispersion coefficient because now we are talking about dispersion it is not diffusion because we have a flow plus diffusion. Flow has already put a signature here by stretching the pulse itself, and this dispersion coefficient is R naught v naught whole square divided by $48 D$. So, what does this mean, this has some amount of implications here. What are the implications?

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Diffusion in a moving front ... Contd.

\frac{\partial c_1}{\partial t} = \frac{D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c_1}{\partial r} \right) - 2v_0 \left[1 - \left(\frac{r}{R_0} \right)^2 \right] \frac{\partial c_1}{\partial z}
$$t = 0, \text{ all } z, \quad c_1 = \left(\frac{M}{\pi R_0^2} \right) \delta(z)$$

$$t > 0, \quad r = R_0, \quad \frac{\partial c_1}{\partial r} = 0 \quad (\text{no flow})$$

$$t > 0, \quad r = 0, \quad \frac{\partial c_1}{\partial r} = 0 \quad (\text{symmetry})$$
 The solution is given as:

$$\bar{c}_1(z) = \frac{1}{\pi R_0^2} \int_0^{R_0} 2\pi r c_1(r, z, t) dr$$

$$\bar{c}_1 = \frac{M / (\pi R_0^2)}{\sqrt{4\pi E_z t}} \exp \left(-\frac{(z - v_0 t)^2}{4 E_z t} \right)$$
 This is labeled as 'Taylor Dispersion'. A note states: 'Here, $E_z = \text{Dispersion coefficient} = \left(\frac{R_0^2 v_0^2}{48 D} \right)$ '. To the right, a diagram shows a concentration profile that has spread over time, with a red arrow indicating the direction of flow and a blue arrow indicating the direction of diffusion. Handwritten notes in red ink define v_0 as 'average velocity', R_0 as 'radius of the capillary', and D as 'Diffusion Coefficient'."/>

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Solution

$$\bar{c}_1(z) = \frac{1}{\pi R_0^2} \int_0^{R_0} 2\pi r c_1(r, z, t) dr$$

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(Taylor Dispersion)

Here,
 $E_z = \text{Dispersion coefficient} = \left(\frac{R_0^2 v_0^2}{48 D} \right)$

v_0 = average velocity
 R_0 = radius of the capillary
 D = Diffusion Coefficient

E_z is equal to R naught v naught square by $48 D$ this you have to look at very carefully because we said that we have a velocity profile v naught is the average velocity. So, v naught is equal to average velocity, average velocity we have a parabolic velocity profile. So, we have maximum velocity at the center and then we have velocity at the wall is 0. So, we have some average velocity which is v naught.

R naught is the radius of the capillary and the D is equal to diffusion coefficient. These two terms are interchangeably used sometimes we call it diffusivity or we call it diffusion coefficient. So, it is one and the same thing as far as this lecture is concerned. So, earlier it was (D) ; earlier this term was D . Now, which we are calling it no it is not D , it is E_z .

So, whatever effect the D had, the same effect now these E_z is putting. So, that means, what was the effect of D ? I had this Dirac function and when D was higher I would have expected this to be stretching faster ok. So, it is spreading faster so, this was the profile at that time. So, now, E_z higher means this would be the same thing to this concentration profile right. So, if E_z is higher there would be more stretching.

So, what will make the E_z higher? E_z will be higher when the velocity is higher, when the velocity is higher the E_z will be higher. And when R naught is higher the E_z will be higher. So, that means, when we are working with a higher velocity we expect these dispersion coefficient to be higher; that means, the spreading would be higher. So, that

means, when you submit these Dirac function to a parabolic velocity profile, now we have enhanced, now we have more diffusion taking place because now this pulse has been stretched like these and so they, earlier it was just one single one and then there was some axial diffusion was taking place.

Now, we are not; you have ignored axial diffusion, but now we have talking about a radial diffusion. So, we can see here that now this diffusion is taking place over this whole slug and then we will be chopping, when we are collecting the slices; we will be collecting slices, this slice, then this slice, then this slice like this ok. So, when we put it in mixing cup this will be mixed anyway.

So, think of it that when v_{naught} is increased; v_{naught} is increased means you have this is also getting stretched more this one is getting stretched more. So, you are allowing this front to be even if the mixing is not that much, but it is front is anyway stretched. So, as long as we are collecting these samples we will continue to see the effect of mixing because this front has been stretched much right.

So, that is why I can see the effect of R_{naught} and v_{naught} coming in because R_{naught} and v_{naught} they are essentially stretching this particular profile here. And, diffusivity is now it is the inverse function. You can see it is this E_z is inversely dependent on diffusion coefficient. What that means, is that when D is higher, when the diffusion coefficient is higher the actual Taylor dispersion coefficient is lower.

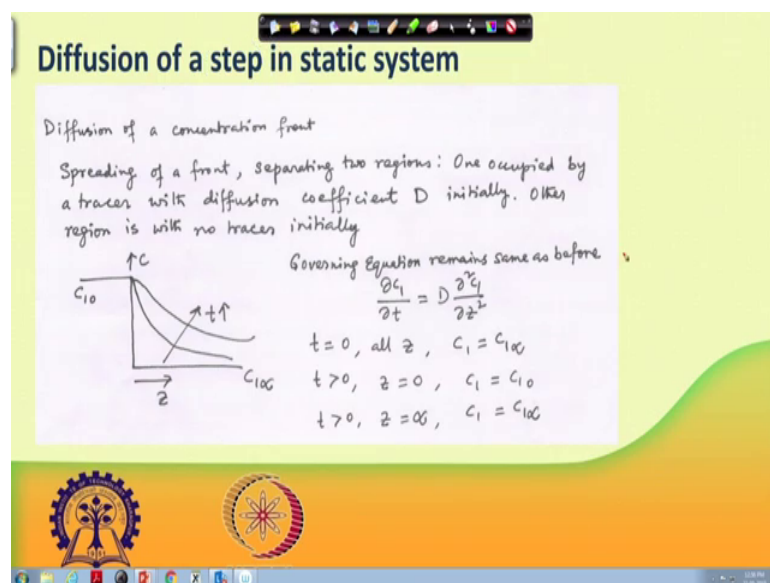
So, that means, when D is higher the; when diffusion coefficient is higher, the dispersion is not happening that much because if the diffusion coefficient is lower you are stretching it this way. So, you can let it you can continue to stretch and you can make it a this foot, that the footprint would be over a much larger length whereas, when the diffusion happens then this slug gets contained within a shorter distance. So, the footprint would be smaller.

So, these are very unique dependence one can note that because by virtue of this one can see that the same system we are working with a uniform velocity profile in that case we see that it is simply the diffusion dominates. If diffusion coefficient is higher means the spreading would be more whereas, in case of Taylor dispersion; that means, when the flow is not uniform rather you are following a parabolic velocity profile.

Diffusion coefficient is more means stretching would be less sorry, diffusion coefficient is more means the dispersion coefficient would be less and dispersion coefficient less means the total effect on c_1 bar would be less. So, that means, you will see less stretching $E z$ is less. So, D is; D increases means $E z$ decreases and $E z$ decreases means c_1 bar decreases because $E z$ has the same effect as diffusion coefficient.

So, this is something which one must make note of while working with this scheme of things.

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Now, we must note here that the purpose of this entire exercise is; so, let us before moving. I mean in fact, my our next step would be; so, with this understanding of Taylor dispersion our next step would be to study the diffusion of a step in a static, diffusion of the step mind it. Next step would be diffusion of a step in a static system. So, diffusion of a step change; that means, earlier what did we do? Earlier what we did was diffusion of a pulse.

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Diffusion in a moving front ... Contd.

$$\frac{\partial c_1}{\partial t} = \frac{D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c_1}{\partial r} \right) - 2 \frac{v_0}{R_0} \left[1 - \left(\frac{r}{R_0} \right)^2 \right] \frac{\partial c_1}{\partial z}$$

$$t = 0, \text{ all } z, \quad c_1 = \frac{M}{\pi R_0^2} \delta(z)$$

$$t > 0, \quad r = R_0, \quad \frac{\partial c_1}{\partial r} = 0 \quad (\text{no flow})$$

$$t > 0, \quad r = 0, \quad \frac{\partial c_1}{\partial r} = 0 \quad (\text{symmetry})$$

$$\bar{c}_1(z) = \frac{1}{\pi R_0^2} \int_0^{R_0} 2\pi r c_1(r, z) dr$$

$$\bar{c}_1 = \frac{M/(\pi R_0^2)}{\sqrt{4\pi E_3 t}} e^{-\frac{(z - v_0 t)^2}{4 E_3 t}} \quad (\text{Taylor Dispersion})$$

Here, $E_3 = \text{Dispersion coefficient} = \frac{(R_0 v_0)^2}{48 D}$

Cumulative Flow = $\int c_1 v_0 dt = \int c_1 dz$

So, all along we have been talking about a pulse. So, initially this is the Dirac function and we will try to see that this Dirac function how it would feature. That means, it is something like you have concentration you have this capillary and then you have given a Dirac function. And, at the outlet so, at the outlet what do we get basically if we have a detector?

Let us say I have a detector which detects the concentration. So, what do we get at the outlet? At the outlet we will be getting concentration as a function of time; I mean if you go to any detector this is what you would get, with time if you have a continuous flow continuous flow is taking place and the detection is going on. So, with time you will get; so, with time when you start the concentration was all zero and concentration was all zero, because it take some time and then typically around the residence time, we have already talked about the residence time right.

So, typically around the residence time you would expect, little bit before residence time you see start building this concentration and maybe this is something which you are supposed to get. So, before residence time is reached you will get something at the outlet and after you get there; after the residence time also some and then it goes to zero. So, this is something which you get as concentration as a function of time.

So, if you now you can convert this concentration instead of time you can call this also you can plot this as a cumulative flow. Where cumulative flow is equal to; cumulative

flow is basically equal to Q multiplied by t , where Q is the volumetric flow rate multiplied by t . So, if the flow had taken place over say 10 minutes so, what 10 minutes whatever in meter cube per second you have so, if you take the product of 10 into 60 seconds and so, when you meter cube per seconds. So, meter cube per second into seconds. So, you get meter cube. So, cumulative flow means so many meter cube has flown through the system right. So, this is the cumulative flow; so, at this cumulative flow it will peak and then it will go down.

And, how do you get this cumulative flow? This cumulative flow these value we said this is recede when we plot this it is the same profile you get. Because the x axis we are simply multiplying by a constant Q right; earlier we plot it as t now we are plotting it as Q into t . So, it is just as Q ; so, Q is just a constant because we are having constant flow rate. So, the plot the trend will remain same.

So, now, this was that time it was the residence time when we talk about this cumulative flow these volume essentially if you think of it, these volume is the pore volume if you are working in a porous medium. And if you are working in a capillary this is not in that case the pore volume rather I would expect this to be the capillary volume we can call it, capillary volume.

So, this much; this is the volume this cumulative volume is basically represents the capillary volume. So, after you fill the cap after you inject one capillary volume then only you can say that you got the peak. So, in before itself you got this, there would be the treasure would be registered and the detector, and then are beyond this you will see that it will start going down and at one point it is zero; that means, the all the tracer has gone out of the system. So, this is something which we have this is something which we have noted here.

So, basically, so, what you are plotting here is; so, this concentration that we are getting these concentration is basically these average concentration c 1 bar. So, this concentration because we are not getting concentration at different radial points, but we are getting an average concentration for this slice as I said, we picked up slice by slice. So, this concentration is concentration of each slice ok. So, this is something which we can get at the outlet.

So, now what we are going to do next is we are going to now instead of going by the Dirac function, we will place a step change in concentration; step change in concentration instead of a Dirac function. So, we are not talking about Dirac function we will do a step change in concentration and in that case we will get at the outlet. At the outlet also you will have with cumulative flow you call it or with time you call it, we will get after some point we have supposed to get this step change. Because we have to provide this residence time or you have to overcome at least flow the pore volume, then only there will be a there would be the breakthrough.

And, this breakthrough earlier case as I said we had a Dirac function and we found at the outlet it has already started spreading. Similarly, this step change also will not be a firm step change. Ideally if there is no mixing we would have expected a step change.

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Diffusion in a moving front ... Contd.

$$\frac{\partial c_1}{\partial t} = D \frac{\partial}{\partial r} \left(r \frac{\partial c_1}{\partial r} \right) - 2v_0 \left[1 - \left(\frac{r}{R_0} \right)^2 \right] \frac{\partial c_1}{\partial z}$$

$$t = 0, \text{ all } z, \quad c_1 = \left(\frac{M}{\pi R_0^2} \right) \delta(z)$$

$$t > 0, \quad r = R_0, \quad \frac{\partial c_1}{\partial r} = 0 \quad (\text{no flow})$$

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$$\bar{c}_1(z) = \frac{1}{\pi R_0^2} \int_0^{R_0} 2\pi r c_1(r, z) dr$$

$$\bar{c}_1 = \frac{M / (\pi R_0^2)}{\sqrt{4\pi E_2 t}} e^{-\frac{(z - v_0 t)^2}{4 E_2 t}} \quad (\text{Taylor Dispersion})$$

Here, $E_2 = \text{Dispersion coefficient} = \left(R_0 \frac{v_0}{4} \right)^2 = \frac{R_0^2 v_0^2}{16 D}$

Cumulative Flow $\text{Flow} = \beta t$

But, if since there is mixing so, we would expect if this is the residence time or this is the pore volume we see a change which is something like this, step is now diffused ok; step is little distorted ok. So, this is something which you would observed if we do a step change in concentration at the inlet. We will work on this further and we will focus on if there are step changes how if there is a step change how what kind of concentration profile you get.

And, then we will; because you remember these, this expression is very important this is the Taylor this equation for Taylor dispersion and this dispersion coefficient because this information would be used in case of step change as well.

So, there at this point I want to close this lecture. And, in the next lecture I will continue with the theories or what predictions we can make for a step change in concentration at the inlet and extend this to porous media. So, that is all I have for this lecture.

Thank you very much.