Flow through Porous Media
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Lecture 21
Flow Equations (Viscous Flow in Capillary)

I welcome you to this lecture, to this module of Flow through Porous Media. What we were discussing in the last class is what would be the Reynolds number for a porous media and how this Reynolds number can be used to find out the pressure drop. So, this is one alternative way, to express pressure drop as a function of flow rate other than the very common, very useful Darcy's law.

So, we are we are heading towards this set of equations that I mentioned before, this Kozeny Carmen black number equation and finally, Ergun equation. So, to arrive at those equations one must know what is the Reynolds number; and how Reynolds number is used to calculate the pressure drop.

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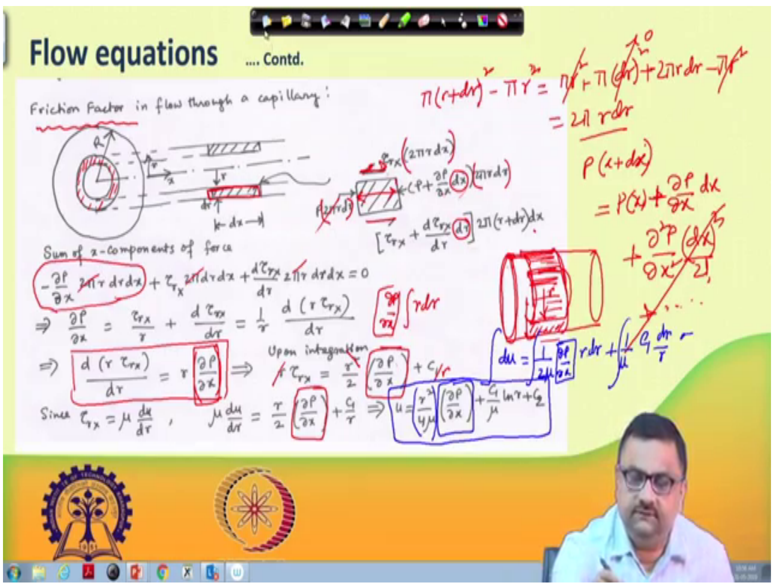

So, when it comes to this definition of friction factor; definition of friction factor so, we have to now think of this definition of friction factor here. So, here I, we try to arrive at this for a flow through a capillary. So, when there is a capillary and a flow takes place through a capillary, there are following there are these forces that are coming in that are existing. Here, we pick up a differential element, this is if I look at the side view this particular annular area we are referring to. So, this from the side view this is the annular area we are referring to. So, in this if I take the front view, after cutting this is the hatched portion that we have. So, this is at a radius R the thickness of this annulus is dr and the length of this annulus is let us say dx and r capital R is the radius of this capillary.

So, now if we look at the forces that are working on this particular differential element. If we pick up this particular differential element and if we try to find out what all forces are acting here, we can see what is the area of this annulus; annulus the area of the annulus would be pi into r plus dr whole square minus pi into r square; that is equal to; that is equal to pi r square plus pi dr whole square plus 2 pi rdr minus pi r square. So, these will cancel out and dr being small since this is a differential element, so this term is set to 0. So, you are ending up with the differential area is 2 pi r dr, that is the area of this annulus.

So, over this annular area pressure P is acting over this entire cross-sectional area here. So, over this annular area the force when you write; so, that would be pressure P into area. So, this area is 2 pi rdr which we found here; and from the other side once again the area is 2 pi r dr from the other end of this differential, this hatched element.

But here the pressure is changing; pressure is changing with x, the pressure has to change with x because we are expecting a pressure drop because of friction. So, this on this side the pressure is P plus del P del x dx. So, what essentially what are we doing here? We are taking a Taylor series expansion we are considering this length is; this length is dx right. So, we are taking at so, what if the pressure here is P and here pressure is pressure if we take a Taylor series expansion. So, P at x plus dx; dx would be the P at x plus if we take a Taylor series expansion it would be P at x plus del P del x dx plus del square P del x square dx whole square by 2 factorial plus there would be other higher order terms.

So, now what you do here in this case is you are assuming this term are not existing the higher order terms. Only this linear term existing why because this dx is small or in other words the pressure in x is changing it is definitely non-linear, but you choose to assign pressure to be linear, since this length dx is very small. So, this change is I can consider this change to be linear.

So, that is what the assumption is, so you are based on this you write P plus del P del x dx into 2 pi r dr and here there would be a shear stress acting. So, here you have tau rx into 2 pi r dx. Mind it what is the area, annular area here in this case? it would be 2 pi r 2. So, this is if we look at this annular area it is basically this is r, so this is 2 pi r; this is 2 pi r multiplied by dx. So, 2 pi r dx is now the area over which the shear stress is acting.

The shear stress is acting; this is the annulus and this is the hatched area that we are working with. So, now, this as far as this part is concerned so, ok. So, it is r is the internal, internal radius. So, these particular areas, so we are looking at sorry, we this is not the right one. So, if this is the area out of this we are looking at only this part, this is the radius r. So, this particular area is 2 pi r into dx. So, basically this, so, this inner this, part is 2 pi r into dx the inner one and the outer one would be 2 pi r plus dr because this radius is r plus dr. So, outer one will also have an outer radius.

So this is 2 pi r plus dr into dx in into dx; so this 2 pi r plus dr into d x. So, this is the outer, outer layer you have. So, 2 pi r plus dr into dx is the area and this time it is tau rx is the shear stress at the inner wall and here it is tau rx plus d tau rx dr dr. Again, you are taking this Taylor series expansion of this tau and then ignoring the higher order terms assuming the tau is linear over this small distance dr. So, in this case this is dr and in this case this dx.

So, now one if one rise writes the sum of these all these x components of forces, then one would write this one arising from here, see P 2 pi r dr and P 2 pi rdr they will cancel out; so this is canceled out. So, this x component is positive and here this is negative. So, this minus this we are talking about. So, when we take this minus this; this this is gone with this P 2 pi r dr. So, this is left minus outside minus del P del x dx 2 pi r dr.

So, that is exactly what we are talking about here minus del P del x dx 2 pi r dr. So, this is the term arising from this minus this. Similarly, you have positive x we have this term and we have we have the shear term here and we have this term here. So, this is if we sum them up you we will end up with del P del x that is equal to; del P del x is equal to this these terms. So you can the other these are some of the terms are cancelling out for example, 2 pi will you can get rid of 2 pi everywhere.

So, it will be simplified to del P del x equal to this and this right-hand side tau rx by r plus d tau rx divide d tau rx dr; this can be combined as 1 by r d of r tau rx tau rx dr, so you can combine these two. So, essentially you will end up with these as the governing equation in this case.

I think this sign convention one should not use because both are positive in x. So, I think this should be taken as this only; because this both are both we are treating it as positive. So, this is the governing equation and upon integration if we do the integration here; that means, we are doing d of r tau rx that is equal to, now, this del P del x term we are not doing any del P del x we are treating it as if it is a pressure gradient.

So, del P del x later on we will simply equate it with delta P by l. So, del P del x though we are writing it in differential form this is basically a box and that box we are carrying in all down all the down the line. So, we are basically, when we do the integration it is d of r tau rx is equal to d r goes to the right-hand side. So, d of r tau rx is equal to some constant into rdr. Basically, that is how you are treating it, this is just a box we are carrying we are not getting into this.

So, r tau rx is equal to r square by 2 when you do the integration, it is basically integration of r. So, you have del P del x, this box is remaining outside into rdr on the right-hand side and on the left-hand side you have d of r tau rx. So, when you take the upon integration r tau rx equal to integration rd r is r square by 2 plus C 1 is the constant of integration.

Now this tau rx is can be written as mu du dr. So, from this you write this tau rx is mu du d r and this. So, one r was there; so this r is going to cancel this r square. So, it becomes only r and this becomes C 1 by r. So, you are dividing new both sides by r. So, here it is becoming only r instead of r square and this becomes C 1 by r.

So, then you have this; so, tau rx if you replace by mu du dr. So, you end up seeing here as mu du dr is equal to r by 2 del P del x del p del x we are keeping it as a box we are not touching anything there plus C 1 by r. So, we have if we do this integration once again now what we do is mu goes to the right-hand side du dr. So, you are doing this integration on u and this dr goes here. So, again you are doing rdr here this integration and C 1 by r dr.

So, when you do this integration here C 1 by rdr you have to integrate and rdr you have to integrate here; because here essentially you are writing that du is equal to 1 by 2 mu;

this box del P del x rdr plus 1 by mu C 1 dr by r. So, this is what you end up with, if you break this up and when you do this integration, then you have to do integrate these, integrate these and there would be a constant of integration term C 2.

So, when you do these this part here this side it is u here it is again r square by 2 rdr integration is r square by 2 and then that 2 will multiplied with this 2. So, that is why we have r square by 4 mu del P del x remains a box as it is and when you do this here C 1 by mu comes out outside the integration and C 1 y mu comes out of the integration and dr by r is ln r.

So, that is exactly what you see here and C 2 is the next constant of integration for arising out of this integration there has to be a constant assigned which is C 2. So, this is the master equation for velocity profile for flow through a capillary.

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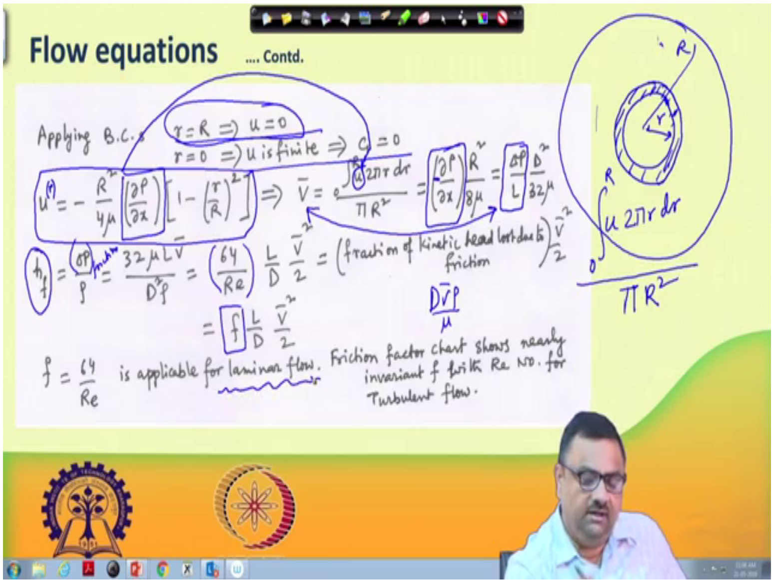

Now, if we proceed on these further, if we take ;if we work with that equation and if we apply the boundary condition that at r is equal to r, u is equal to 0; that means, at the inner wall of the capillary, the velocity has to be 0. That is basically no slip boundary condition, which is known as the fluid which is in contact with the wall which is static that fluid also has to be static; and at r is equal to 0 u is finite.

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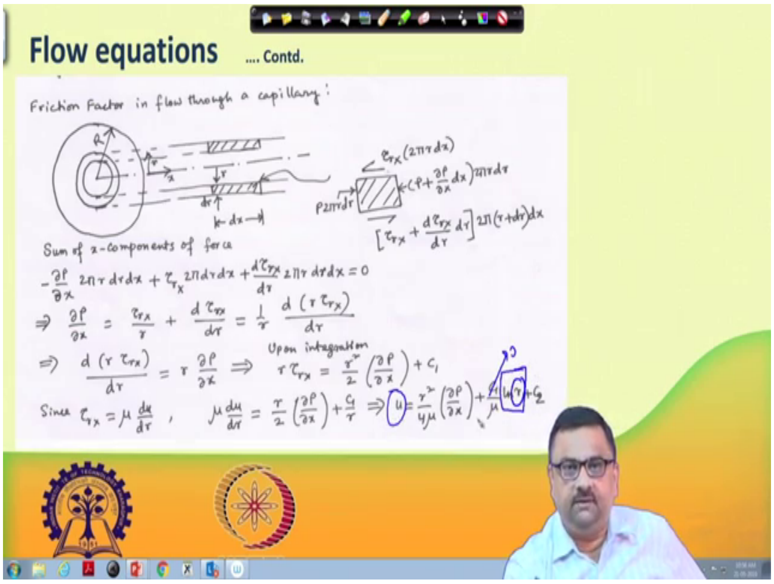

Why r is equal to 0 u is finite, what is the implication of this? If we go to this equation you can see here, that if r is equal to 0 at r is equal to 0 this term becomes ln 0. So, that will make this velocity infinite. So, if you want to hold this to be finite because at r is equal to 0 we know at that is the center of the capillary it cannot happen. So, naturally one can say that this C 1 has to be equal to 0. So, that this term is completely gone otherwise this cannot happen; you cannot otherwise this you would become infinite at r equal to 0 which is not the case we all know.

So, that is why this u is finite implies C 1 is equal to 0, so this is one condition and at r equal to, r equal to 0. So, now, if you put these so, C 1 is already 0 and C 2 you can find out from applying this boundary condition. So, if you do that you will end up with a velocity profile which is given by this; u is equal to minus capital R square; R is the inner radius of the capillary, small r is any radial point ok, but capital R is the inner radius divided by 4 mu del P del x again we retain it as a box we are not touching anything there 1 minus r by r whole square small r is any radial point. So, this u is a function of r.

So, at any radial point, this will be the velocity profile and if one wants to find out what is the average velocity, we all understand that these u is applicable you when you talk about u as a function of r, we are talking about an annulus. See at a radius r we have an annulus let us say; let us say we have an annulus and this velocity of the fluid through this annulus is essentially u at r.

So; that means, this the area of this annulus we have already found out just before it is 2 pi rdr it is pi of r plus dr whole square minus pi of r square. So, the you noting this d r whole square term it is 2 pi rd r. So, this multiplied by u this gives me the volumetric flow rate, that is u is the velocity applicable to this annular are. If I am talking about if this is the overall capillary and if I talking about it here, here it this r is that some other r and the velocity around that annulus velocity of the annulus that annulus which you draw around that r that you are would be that you would be against that particular radial distance. At this radial distance this is the applicable u and the corresponding annular area is 2 pi r.

So, 2 pi rd r; so, this multiplied by this; this is velocity in meter per second into meter square this gives me meter cube per second. So, this is the volumetric flow rate. Now, if you integrate this for all are running from 0 to capital R; that means, you are doing this for all such annulus, all such annuli possible this for r running from 0 to capital R; which is the radius inner radius of the capillary. So, this is, the then if you integrate it then this is the volumetric flow rate over the annulus, when you integrate this gives me the volumetric flow rate over the entire cross-sectional area.

So, this is the volumetric flow rate and now if you divide by total cross-sectional area which is pi r square this gives me the average velocity through this cross-section and that is exactly what they have done in V bar which is the average velocity is integration 0 to capital R u 2 pi rdr by pi capital R square.

So, now, if you bring in this u here and do the integration it would be then you are basically doing u into 2 pi r dr. Now, u is a function of r here we have this functionality. So, when you bring in these and do the integration you will end up with this expression del P del x remains box as it is and this integration between 0 to capital R will take you to 0 to capital R will take you to R square by 8 mu. If one does this integration as per this put to; that means, putting these u in putting this u in here as this u; if you do this and now you do the integration, then you will be arriving at this.

So, now, if you instead of del P del x if you write these as delta P by L. In fact, there has to be a minus sign here which is which we are carrying here. So, this u when you bring in, so this is anyway this has to be minus del P del x and this del P del x will can be replaced by delta P by L. So, this delta P by L is the so, del P del x anyway we carried it as a box as if it is the pressure gradient and now we are saying it is the the finite delta P change over some length L; it is not a differential length dx.

So, this is the delta P by L pressure gradient; and this and since r square 2 r is equal to d diameter. So, instead of radius you can write it as d square by, so, this is to be 8 in to 4 in that case because when you square it, d is equal to 2 r that mean d square means d square is equal to 4 r square. So, this 4 will go here as 32. So, d square by 32 mu, so this is the average velocity.

Now when it comes to frictional loss generally in fluid mechanics they call it a frictional loss. Frictional loss is given by delta P which is the pressure drop due to friction divided by density rho; divided by density rho. So, what is the typical unit of this these, this what is the unit of this; this expression, I believe it is force per unit volume ok.

So, this is the amount of loss it that is that is happening. So, this delta P by rho; anyway, forget about this h f; if we look at delta P which is the frictional loss. So, this delta P is basically due to friction delta P by rho; rho is the density. So, it is (Refer Time: 23:20) customary divided by rho. So, that would be equal to if we delta P divided by rho would be; if we work with this from this expression V bar is equal to this quantity; V bar is equal to this quantity. So, if we equate the two, we get delta P by rho would be in that case all these terms will go to the left-hand side 32 mu L by d square will go to the left-hand side and moreover there is a rho in the denominator. So, there will be at rho in the denominator.

So, that is why we have here 32 mu L V bar by d square rho and this can be written further as if you write these Reynolds number as DV; DV bar rho by mu. So, then in that case one can write this delta P by rho as 64 by Reynolds number multiplied by L by D it is again a dimensionless term for that pipe and V bar square by 2.

So, essentially this is you can say, this is fraction of kinetic head loss due to friction into V bar square by 2. So, this term this 64 by Re is referred as friction factor; this is referred as a friction factor and this this also the pressure drop due to friction by rho is written as friction factor f multiplied by L by D into V bar square by 2.

So, this is this friction factor; so now, we can see this friction factor is equal to 64 by Re, but this is applicable for laminar flow. When it comes to friction factor for turbulent flow, generally what people do in this case is they follow something called a friction factor chart. There is friction factor chart available for to compute friction factor as a function of Reynolds number.

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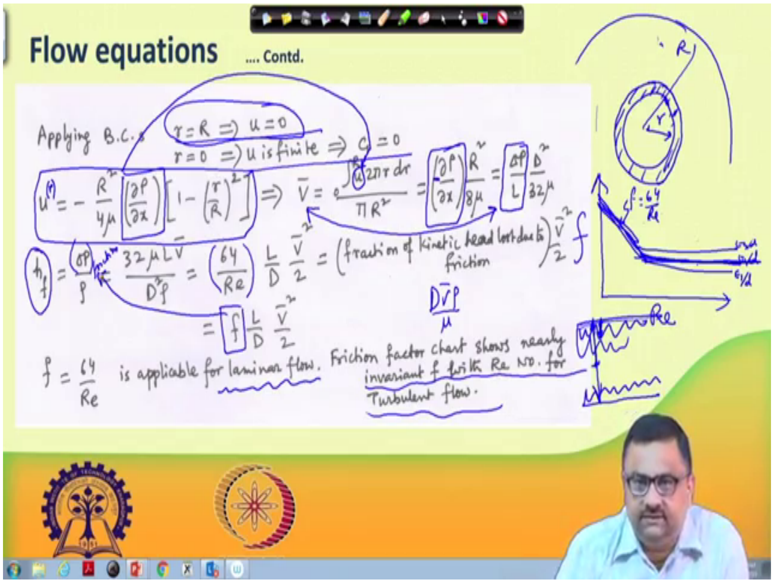

So, friction factor versus Reynolds number you can see initially it comes as 64 by Re. So, f is inversely proportional to Reynolds number. So, this slope of this it will be decided accordingly, so this part f is equal to 64 by Re. And then once the turbulent sets in, typically this plateau this, that is the, the change is there, but change is not that significant.

So, this part is primarily the turbulent region and this part generally the friction factor depends on the roughness. Various values of epsa by d for a pipe you can see there are different lines drawn and these lines are basically for different roughness's epsa 1 by d epsa 2 by d this a this epsa is not porosity by the way. This epsa is roughness in the sense that pipe inner wall is basically it is not smooth. This is there would be roughness involved and this roughness some pipe will have. So, how much would be this height? So, this is epsa divided by d is the diameter of the d is the diameter of the pipe.

So, with reference to the diameter of the pipe what is the height of these roughness? So, if some wall can be very rough, some wall can be very rough, some wall can be very smooth if it is very smooth; obviously, the pressure drop frictional pressure drop would be less if it is very rough frictional pressure drop would be more. So, that is what it in indicates. So, there would be different lines, but take-home messages this is a straight line following f is equal 64 by Re and then it is plateauing to some line.

So, this, so, friction factor chart shows nearly invariant f with Reynolds number for turbulent flow. So, this is what we understand for flow through a capillary. Now what we have to do is we have to extend this concept of friction factor to the porous medium. We have already found out how to calculate the Reynolds number for porous medium, but now we need to extend this idea friction factor and how this friction factor gives me this delta P. So, if the same route can be followed for flow through porous medium. So, that is somewhere that is something I am going to do in my next lecture. So, this is all I have as far as this session of the lecture of flow through porous media is concerned I.

Thank you very much.