# Flow through Porous Media

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## Lecture – 13
## Mass Continuity (Superposition of Elementary Flow)

I welcome you to this course on Flow through Porous Media, our the module that we were discussing earlier was a Mass Continuity where we initially in introduction section, we talked about various characterization or the couple of characterization parameters that is permeability and porosity and how we can work with I mean we can establish some flow equations using these two terms, that we discussed and then we tried to see the continuity as we see in fluid mechanics, how it can be extended to flow through porous media.

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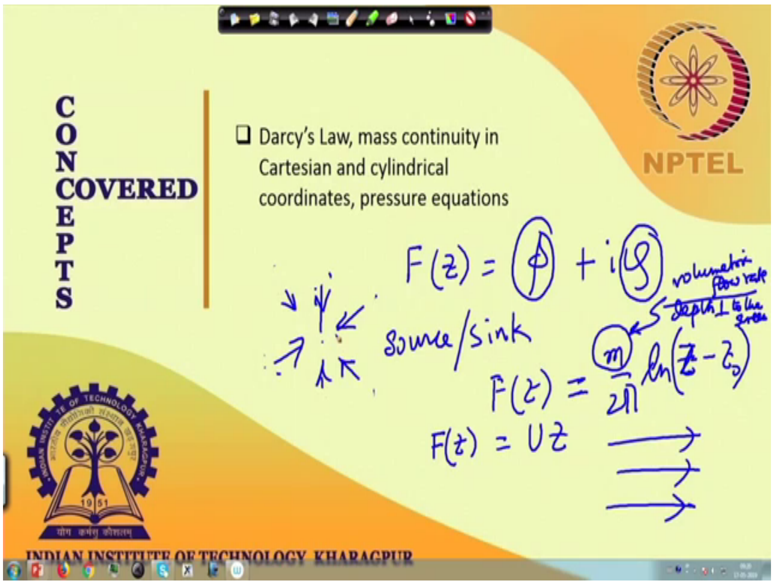


So, in this case we had talked about Darcy's law and we had talked about mass continuity in Cartesian cylindrical coordinate system to derive pressure equations and velocity profiles. Beyond this we have talked about the complex potential, that is we had we had discussed about this complex potential as F z which is equal to phi plus i phi plus i psi where this is referred as potential function and this is referred as stream function. And we have established F z for different elemental for flows per particularly which are of relevance for relevance to porous media application.

For example a source or a sink, where we have seen that F z would be equal to F z is equal to m by 2 pi l n z where or to be precise if we want to shift a coordinate, we can write this as z minus z 0 and this m is volumetric flow rate volumetric flow rate per unit depth perpendicular to this screen per unit depth perpendicular to the screen. So, this is this is how we had we had worked with and we have found out that this is the complex potential for source and sink.

Similarly, for complex potential for unidirectional flow, what we saw was that, you have F z is equal to u z this implies flow in x direction whereas, the source and sink means one point and then flow is either emanating if it is a source or if it is a sink then m will be minus m will be negative and in that case these would be simply sink means flow would be all inward. So, this is something which we discussed and then more importantly we said that if there are 2 unit flow the 3 unit flows or more than 3 unit flows if they are happening simultaneously in some place, then we can simply superpose them; that means, we can take F z of one elemental flow and take another the F z of another elemental flow and then sum them up and whatever the sum per potential functions sum of these potential functions and sum of the stream functions these can be used as if the super force flow, this will define that sum will define what is the potential function, what is the stream function, what is the velocity profile.

So, using this. So, I take this and I give you an example here the example is something like this.

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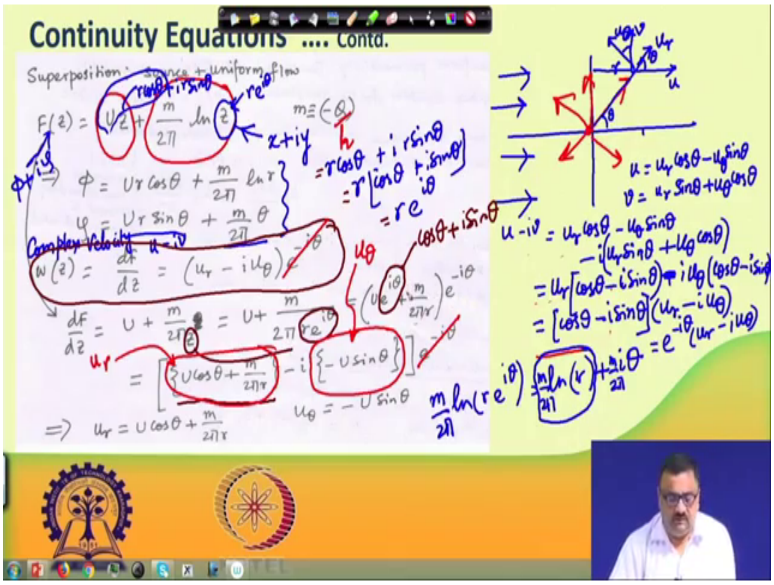


We are talking about superposition of source and uniform flow and both of them are at the at the at the origin; that means, I have let us say this is the origin. So, on one hand we have uniform flow u z, on the other hand we have a source means. So, if you have a flow in a porous medium you have this kind of a combination source and uniform flow, that in that case how would the streamline and potential line would be and what would be the velocity profiles in this case.

So, source would be let us say I have an injection well here. So, I am injecting. So, all are per unit depth perpendicular to the board. So, perpendicular to the screen so; that means, I am say in a three dimensional sense I have a uniform flow going on and simultaneously we have we have a well through which we are putting some we are we are disposing some water there and the water is flowing in this direction.

So, in this case what would be the what would be how the stream line will look? So, we have done this F z is equal to U z this is the elemental flow for uniform flow in x direction and this is m by 2 pi l n z this is the uniform this is the flow for the elemental flow for source ok. So, since Q in earlier context we defined the production from the well. So, it is it is basically m is equivalent to minus of Q, to be precise q per unit length because the earlier time we had worked with Q as a volumetric flow rate.

So, m is Q divided by h may be the depth of the depth of the porous medium. So, that also has to be accounted. So, anyway this is not that important m is here is a source which is opposite to production from the well. So, now, this is here the source is located at the origin. So, our x y system is taken so, that the source is located at the origin. So, there is no correction required here as z minus z 0. So, in this case if we sum them up we can see here what is the what is the first of all these z would be written as the z would be written as the z would be broken down into as we had done earlier in our earlier lecture x plus i y.

So, x plus i y means it is r cos theta right. So, we have x plus i y. So, this is r and this is x, this is theta. So, x is r cos theta y is r sin theta. So, r cos theta plus i r sin theta. So, we had taken r common and we had written it as cos theta plus i sin i sin theta, which by ols theorem we have used it as r e to the power i theta. So, we have put these l n z instead of z we put r e to the power i theta. So, r e to the power i theta when we put then we see here u z plus m by 2 pi l n.

Now, z this z also we can break it up into r cos theta plus i r sin theta. So, now, if we. So, these are the two elemental flows one is for the uniform flow another is for the source. Now if we try to add them and take up take only the real component of it. So, first of all l n of r e to the power i theta. So, l n of r e to the power i theta will break down into l n of a into b is l n a plus l n b and l n of e to the power i theta is basically i theta itself. So, l n r plus i theta ok.

So, that is why when we had written the real component this when we have m by 2 pi outside. So, then this would be equal to m by 2 pi and this would be equal to m by 2 pi. So, what is the real component arising from the source? It is m by 2 pi l n r and what is the real component arising from the uniform flow? It is u r cos theta because this is the imaginary component. So, u r cos theta is the real component so, that is why you can see potential function phi is defined as u r cos theta the real component of it and m by 2 pi l n r which is the real component of the source.

And similarly the since F z is written as phi plus i psi. So, the imaginary component will be equated with psi, basically this F z is equal to F z is equal to phi plus i psi. So, the real component has to be equated with the real component imaginary component has to be equated with the imaginary component. So, the real component is phi; this phi has to be equal to u r cos theta this is the real component arising from the uniform flow and m by 2 pi l n z is this one. So, this is the real component arising from the from the source.

So, that is why the phi is written as U r cos theta plus m by 2 pi l n r and similarly in case of in case of the stream function psi, it would be just the imaginary component and the imaginary component here is the imaginary component here is in this case U i r sin theta. So, this part is the imaginary component because I have i here and for this part imaginary component would be m by 2 pi i theta because I have i here.

So, if the i into what we have here phi plus i psi. So, i into U r sin theta that is what i see here and m by 2 pi theta i is outside ok. So, m by 2 pi theta. So,. So, this is becoming the stream function and this is becoming the potential function if we simply equate the real component with the real component left side right side, and similarly imaginary component on the left side is equated with the imaginary component of the right side.

So, now if this is this is the potential function and this is the stream function. So, this is. So, if you if you if I say how the streamlines will look like, you must note here that streamlines will look like the streamlines would be lines along which U r sin theta plus m by 2 pi theta is constant that is what the information is. But still handling this information is not that straightforward let us let us dwell on this a little bit further, let us try to find out what are the what is the velocity in this context. Velocity in the sense we can we can work with Cartesian system or cylindrical system they are I mean we will be toggling between the two.

So, now w z which is the which is the complex velocity, this is the definition of this is the definition of complex velocity. So, complex velocity is w z that is equal to d F d z which is equal to u r minus i u theta e to the power minus i theta where did we get this? This is the vector identity we had worked on it earlier how did we get there? You remember u r this is the u r this is u r and this is u theta this is u theta and this is r u and this was the vertical was the v right.

So, this is u and this is v this is u r and this is u theta and what we did at that time is, we said this is angle is theta. So, u was at that time we had written u was u r cos theta minus u theta as a negative component negative h direction it has a component and since this angle is theta. So, this angle becomes this angle this angle became 90 degree minus theta. So, in the cos of 90 minus theta that is sin theta.

So, u r cos theta minus u theta sin theta and similarly v at that time we had written as u r this time it is 90 degree minus theta. So, v component would be u r the u r in the vertical component it would u r cos of 90 degree minus theta which is u r sin theta and this time this u theta the component in the vertical direction or u theta, this would be positive because both u r and u theta both are in positive v direction that component and since this angle is theta. So, it would be u theta cos theta.

So, these this was the identity we had at that time and then we said that d F d z was whereby your original definition d F d z was u minus i v u minus i v. So, we had written it here we had we had written it let me erase a bit. So, we had written it at that time that this u minus i v; u i v was equal to then we had u r cos theta minus u theta sin theta minus i u r sin theta plus u theta cos theta.

So, what we did at that time is, we had taken all u r terms together. So, it became cos theta minus i sin theta and u theta terms together. So, u theta terms together was when we took u theta terms together it would be i u theta into cos theta minus i sin theta. So, what do we have here then i plus i u theta cos theta, but do we have here minus i u theta cos theta. So, then this should be minus sign. So, this is a minus sign. So, then minus i u theta cos theta and minus plus and again my i square is minus.

So, then this becomes minus u theta sin theta. So, it makes perfect sense. So, this is this is a minus sign this is a minus sign. So, now, we had taken what we did is we had taken cos theta minus i sin theta common and we have on the other hand u r minus i u theta, and then this cos theta minus i sin theta is written as e to the power minus i theta. So, this was e to the power minus i theta into u r minus i u theta. So, that is exactly what we had written here d F d z is equal to u minus i v is equal to u r minus i u theta into e to the power minus i theta.

So, then in this case if we have d F d F d z that is equal to d F d z is equal to in this case we have u r minus i u theta e to the power minus i theta that is one thing. So, this is one thing we have this is one thing we have understood is that this is one thing we have understood that w z is u r minus i u theta e to the power minus i theta this is one thing another thing is if we take just d F d z if we just take d F d z of this. So, what is F? F is U z plus m by 2 pi l n z derivative of U z would be simply U and derivative of m by 2 pi l n z would be m by 2 pi and derivative of l n z with same by 2 pis constant outside and derivative l n z would be 1 by z.

So, that is exactly what is written here d F d z is U plus m by 2 pi 1 by z. So, u plus m by 2 pi z ignore this part if it is the scratchy. So, u plus m by 2 pi z. So, now, instead of z we can write z as x plus i y and then z is x plus i y x plus i y is equal to r cos theta plus i r sin theta. So, r within bracket cos theta plus i sin theta which can be written as r e to the power i theta. So, if it is r e to the power i theta, then you can you can note here that it is instead of z instead of this z we have written it here as r e to the power i theta.

So, u plus m by 2 pi r e to the power i theta. So, now, these. So, what we did here is we multiplied e to the power i theta to the e to the power i theta to both the terms and outside we put e to the power minus i theta so, that it all they are leaving out ok. So, what we did is we multiplied u with e to the power i theta and m by 2 pi r e to the power i theta. So, e to the power i theta in that case cancelled out and then e to the power minus i theta remained at the outside. See you can you can you can automatically get this.

See u e to the power i theta into e to the power minus i theta. So, this would be e to the power 0 and which is 1 and then. So, you have only u left and m by 2 pi r e to the power minus i theta that is exactly what you have here m by 2 pi r e to the power i theta goes to the numerator as e to the power minus i theta. So, this is this is one and the same thing. Why we are doing this is because then if we write d to the put e to the power minus i theta outside, then we can essentially we are going to equate now this d F d z with this d F d z. So, when we equate this d F d z with this d F d z these e to the power minus i theta term will cancel out.

So, then we can write this u r minus i u theta we can equate it with this term these then we can equate real with real imaginary with imaginary and get the idea what is u r and what is the u theta that that is the idea. So, that is why we are doing this exercise. So, now, now here again e to the power minus i theta we can cancel out that that is not an issue. But this inside what we have u e to the power i theta plus m by 2 pi r m by 2 pi r is a real component we understand, but u e to the power i theta has both real and imaginary component. So, that is why we have to now break this e to the power i theta again u this e to the power i theta has to be broken down into cos theta plus i sin theta.

So, that I can because I have to compare real with real imaginary with imaginary. So, for that reason we had written here as the real component was u cos theta. So, so. So, what. So, we have this with the real component here is u cos theta and u i sin i u sin theta would be the imaginary component as far as this term is concerned and this term as far as this term is concerned these is all real there is no imaginary component. So, if we sum up all the real component in these, the real component would be u cos theta plus m by 2 pi r.

So, that is exactly what we have written here; the u cos theta plus m by 2 pi r this is the real component within this what we have within this bracket. Whereas imaginary component is only this one because m by 2 pi r does not have any imaginary component. So, minus this plus i sin theta is the only imaginary component and that is what we had written here. See we have multiplied outside we have u. So, this is u cos theta and this is i u sin theta. So, minus I had written and then minus u sin theta I have written so, so that this minus and this minus becomes plus ok.

I have intentionally written it as this minus i because I have to equate it with real minus imaginary something. So, that is why I put it as minus i within bracket minus u sin theta essentially it is plus i u sin theta which is arising from the star. So, this is this is what we have in the bracket and outside I have e to the power minus i theta. So, now, if we if we go by the now if we if we start cancelling, we can straight away cancel e to the power minus i theta is with e to the power minus i theta and we can write here that u r would be the real component here. So, this would be the real component which is going to be u r and this is u r minus i u theta and we are equating u r minus i u theta.

So, this is becoming the u theta right. So, this term becomes u theta. So, I can i. So, I can I could I could equate this whole thing I could in fact, I can I can clear up the space a bit. So, what we have written here is u r minus i u theta which is coming from this here d F d z.

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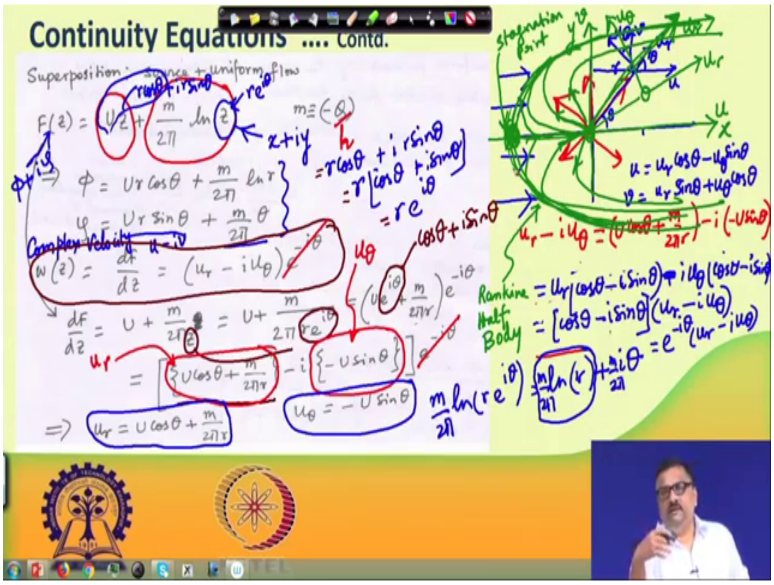


So, u r minus i u theta this is equal to; this is equal to on one hand we have u cos theta plus m by 2 pi r and minus i of minus u sin theta. So, this is this is what we have in this case e to the or minus i theta and e to the power minus i theta we have already cancelled.

So, if this is so, then we can say that u r is essentially u r is essentially this quantity. So, this is u r and this is essentially u theta. So, that is what I have written here these is becoming u r and this is becoming u theta. Now what does this signify I have first of all the I have what are what are these u r and u theta once again if i visit u r is. So, my origin was this right my origin was. So, this is the origin this is the origin ok. So, or from origin we had these as the x axis these as the y axis.

So, now we had we could we have we can write this in this velocity in this direction as u and the velocity in this direction as v. This we could have done, but instead of that we have chosen to write it in terms of u r and u theta. So, at any point at any point let us say I am I am taking some arbitrary point let us say I am taking arbitrarily at this point at this point I can have u and v or I can have u r and u theta in the r theta system ok. So, that is that is what it is and what we see here is that u r is this quantity and u theta is this quantity.

We want to know here first of all intuitively it makes sense that if I have a source; that means, flow is emanating out and I am having uniform flow. I mean intuitively what we see here is that the stream lines that are emanating here let us say this is emanating here this is emanating here, this is emanating this is a radial flow. So, this stream line would be then carried down like this, this stream line would be carried down like this because I have a uniform flow coming from the left. So, this stream line would be carried down like this.

So, you can see here at at some point at some point it would be this stream line that is arising from here and this stream line that is arising from here the velocity here would be 0. I mean that is that is where it is heading to unless I mean if we have to satisfy this, the velocity and I am creating almost an envelope like this and creating almost an envelope like this. So, this envelope and at this point I can see that the radial flow that is coming from the right and the uniform flow that is coming from the left they will these two streamlines they will hit each other and the fluid becomes static here.

So, this is referred as this point is referred as stagnation point. I mean intuitively I can see there would be a stagnation point and an envelope. So, all the radial flow that I am putting in here that would be enclosed in an envelope because of this uniform flow that is coming from the left ok. Just for as I am just i would like to inform you that this type of envelope this has a name to it generally this type of envelope in in the context of fluid mechanics is referred as Rankin half body Rankin half body.

So, our aim would be with this u r and u theta, our aim would be to look at this stagnation point where how far it would be on the left from the origin this stagnation point and what would be this envelope what would be the equation for this envelope let us target this I mean instead of you know these are still in general terms, but let us target these and find out what would be this equation for this envelope. So, still I mean I can I can make use of these to find out if I am if I am in in a reservoir in in a subsurface reservoir if some fluid is being fluid is sequestered and then it there is a uniform flow going on what to what extent this would this would travel in the lateral direction if we wants to know add infinity, if you if you try if it if it continues to travel further on the right.

So, these are the information we can extract out of it. So, I am going to close this module this section now, in the next lecture what I will do is I will I will revisit this equation particularly these envelope and the stagnation point and try to see what all information we could gather out of this superposition of flow

Thank you. Well, I will continue in the next session.