**Flow through Porous Media**
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**Lecture - 12**
**Mass Continuity (Source / Sink)**

I welcome you once again to this class of Flow through Porous Media. We are discussing about these Mass Continuity. We were talking about the mass continuity in Cartesian and cylindrical coordinates and we are working with the pressure equations and of and also the velocities and in particular in the last lecture we were talking about the we are talking about the stream functions and potential functions and we combine them in terms of complex potential F z and we said that we in fact, we set up a few elemental flows of source, sink, vortex then we have unidirectional flow.

So, we there are more number of elemental flows this is not the complete list. You can have a flow like flow of a doublet flow of other elemental flows are possible, but essentially what I am trying to convey here as is how to play with this elemental flows to simulate a complex situation ok. So, simulate a situation where you have a well which is continuously ejecting water, you have another well which is continuously producing, you have some other type of elemental flows imposed in the porous medium and then the resultant of all these, resultant of all these means we if we superpose all these elemental flows, sum them up and then the summed up F z and if we express that summed up F z as phi plus i psi and then looking at this psi looking at this phi or take that summed up F z and take the derivative of it; dF dz is w z and w z is u minus iv.

So, what is u, what is v? So, we can get a fair bit of idea without you know scratching our head we can just simply taking the elemental flow and summing them up ok. So, we will continue this exercise we now we will start gradually into superposing different elemental flows. So, let us see what we have now here.

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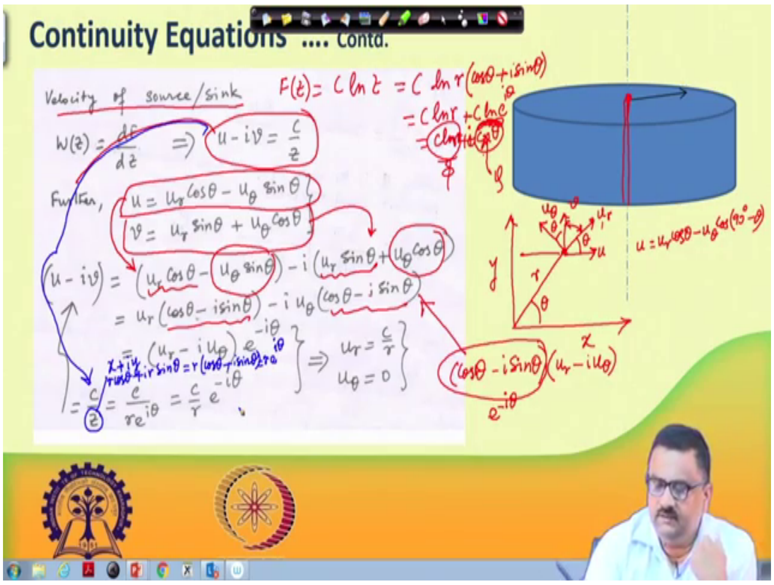


Here we are talking about now we had talked about F z for source or sink we said F z was C ln z. So, F z was C l n z. So, that is the expression for F z and we broke it up you remember F z as in the last lecture, what we did is we wrote C of ln and z we have written it as x plus iy and x is written as r cos theta and y was written as r sin theta then r cos theta plus.

So, r we have taken out we have taken r common. So, then it was ln of r cos theta plus i sin theta; and cos theta plus i sin theta we had written it as e to the power i theta. So, this became C ln r e to the power i theta. So, this we had written it as C of ln r plus C of ln e to the power i theta. So, these becomes equal to C ln r plus C i theta. So, then we said that this is the real component C ln r this is the this this c theta not in C i theta c theta leave out the i. So, this should be plus i c theta. So, C theta would be the imaginary component.

So, this is see if we write this as v plus i psi then this becomes psi and this becomes phi the potential function and we had we had worked with these phi n psi. Now if we if we try to find out what are the velocities in this case. So, how will we find out the velocity? Now the source and sink is particularly important for us because if we are working with cylindrical coordinate for example, we are looking at a well. So, for us to define this well, these this this this type of this type of this this source and sink type flow is very important.

So, now if we take a derivative of this that is w z is equal to dF dz basically we said that w z equal to dF d z and the when we take a derivative of a complex number, it should be u minus i v this we already we have seen in the previous lecture and since this is C ln z. So, when we take the derivative of C ln z this becomes c by z because derivative of ln z is 1 by z.

So, essentially this wz it leads to an expression u minus iv is equal to c by z that is that is on one hand dF d z is c y z when you take a derivative of this, on the other hand d F d z we have seen in the earlier lecture dF dz is u minus iv. Further we have studied these in one of the lectures earlier u can be written as u r cos theta minus u theta sin theta you may you may recall that I have you may recall that I have these x and y and at a particular location, I have in the x direction I have u and y direction I have v and on the other hand I have if we want to work with r theta system. So, this is the r and this angle is theta. So, we have this as u r and this as u theta and then we said that u is basically u is u this is theta. So, component of this u r. So, this is u is equal to u r cos theta and this this side you have u theta this angle is theta.

So, this angle is 90 degree minus theta. So, you have this here as minus because this this component of u theta would be there in the negative x direction. So, that is why I have a minus sign here u theta cos of 90 degree minus theta. So, that is what we see here as u is equal to u r cos theta minus u theta sin theta and similarly you can see v as u r sin theta this time because this angle is 90 degree minus theta.

So, this is u r cos of 90 degree minus theta that component and u theta will have a direct component of cos theta if because this angle is theta. So, this is the v. So, this is the relation last time we had for u and v and this is in fact, of sort of vector identity I mean whether what type of elemental flow we considered it does not matter, it will always be u is equal to u r cos theta minus u theta sin theta and v is equal to u r sin theta plus u theta cos theta. So, in this case if we try to write what is u minus iv using this expression of this.

So, instead of u we are writing u r cos theta minus u theta sin theta. So, u is written as u r cos theta minus u theta sin theta, this is this is what is going in here and this v is u minus iv. So, this is this is the v. So, this v is going here u r sin theta plus u theta cos theta. So, once you have this expression in place then you can take u r common. So, you have taken u r we have taken u r common.

So, this is one u r cos theta is there and minus i u r sin theta is there. So, take u r common. So, it becomes cos theta minus i sin theta and minus iu theta. So, now, we are looking at clubbing this u theta cos theta and u theta sin theta minded it u theta was written as i term. So, i u theta cos theta and minus i minus i square is minus 1.

So, that is why you have i square e this is minus minus plus and then i square again minus. So, finally, it is a minus u theta sin theta. So, if you if you if you break it up this is the expression which you end up with; u r cos theta minus u r into cos theta minus i sin theta minus iu theta cos theta minus i sin theta.

So, now you can take this cos theta minus i sin theta common. So, what we are doing here is, you are taking cos theta minus i sin theta common and your cos theta minus i sin theta common and then you are getting into u r minus i u theta. If you take this cos theta minus i sin theta and this cos theta minus i sin theta this if you take common you are left with u r minus iu theta. And then this cos theta minus i sin theta we have already seen that this can be written as e to the power minus i theta by Euler's theorem.

So, that is why you have written e to the power minus i theta multiplied by u r minus iu theta. So, this is this is exactly what is going in there. So, u r minus iu theta into e to the power minus i theta. So, in this case you have u r minus i theta into e to the power minus i theta on the other hand u minus iv is c by z. So, this is coming in here, u minus iv this is coming in here as c by z. So, u minus iv c by z whereas, u minus iv is this quantity already you have found out.

So, now c by z you can write these z again you break this z into x plus iy and these x plus iy you can write once again as r cos theta plus i r sin theta and then again you can write this as r of r into cos theta plus i sin theta then again you can write this cos theta plus i sin theta as e to the power i theta this is equal to re to the power i theta using Euler's theorem.

So, you can write c by z instead of that c divided by re to the power i theta because z is re to the power i theta. So, c by r e to the power i theta or in other words e to the power i theta can go to the numerator section. So, it would be c by r into e to the power minus i theta c by r into e to the power minus i theta. So, then you have you if you if you look at this so, essentially what do we get here? If we if we clear this entire thing up.

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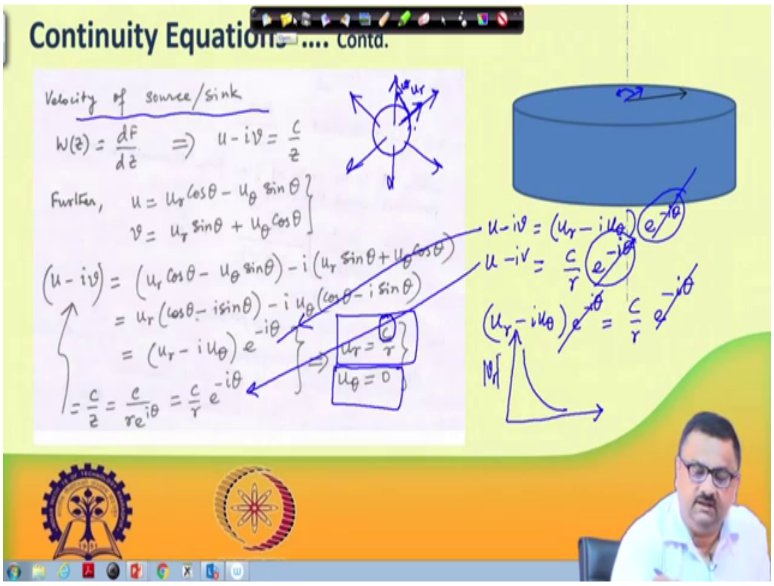


You are getting here on one hand u minus iv is equal to u r minus i u theta into e to the power minus i theta on the other hand you are getting. So, this is this you are getting from here. On the other hand your are getting u minus iv here as equal to c by r e to the power minus i theta. So, then if these are to be equated, then these e to the power minus i theta this e to the power minus i theta and this is e to the power minus i theta they cancel out if the we are equating these two.

So, you are getting here you basically you are getting u r minus iu theta into e to the power minus i theta that is equal to c by r e to the power minus i theta. So, you are cancelling e to the power minus i theta and e to the power minus i theta. So, what are you ending up with? c is a real constant. So, c by r is a real term and on the other hand left hand side you have u r minus iu theta. So, if we match real with real and imaginary with imaginary, there is no imaginary part of the right hand side.

So, you will end up with u r the real part on the left hand side has to be equal to the real part on the left hand side; that means, u r is equal to c by r that is there and u theta which is the imaginary part on the left hand side there is no imaginary part on the right hand side. So, nothing to match. So, u theta is equal to 0.

So, what you have here is u r is basically c by r and u theta is equal to 0 what is u r? u r is the velocity in r direction and that velocity is now we are saying it is c by r and u theta which is the velocity in theta direction. In fact, that is what that is what our original understanding was is not it? Our original understanding was that for a source or sink we have only radial flow if it is source it is going out and if it is sink they are all coming in the direction of the arrow changes.

So, in this case we are saying that if this is a stream line perpendicular to the stream line there cannot be any flow perpendicular stream line only thing that can happen is the potential line which is the circle here in this case. So, naturally u theta is. So, we have this as u r, this is our u r the velocity in radial direction and velocity in theta direction which is u theta u theta is 0 because there is no velocity a perpendicular to the r because it is a truly a radial system, it is a true source and sink in this in this cylindrical system.

So, we are noting here is that u r existing u r is existing there and u theta is u theta is 0, u theta cannot exist perpendicular to stream line there cannot be any flow. Further we note here you remember the last time we said in a in the previous lecture we said that u r is if c is positive it is a source. Now we can see if c is positive, then u r is u r is u r is u r is then positive whereas, when c is negative u r is negative; u r is negative means flow would be inward not radially outward flow would be moving inward.

So, that is that is the case when c is negative. This is in fact, making perfect sense because you remember u was when we when we were when we are trying solve this pressure this Laplace equation for pressure, in cylindrical system and only working with r ignoring theta and z we said that vr is proportional to 1 by r right vr was proportional to 1 by r mod of v r that has to be proportional to 1 by r.

So, that is that is exactly what we are heading to though we do not we do not understand fully what is this c that we do not have full understanding yet. But this much we understand that v r would be the u r equal to c by r is consistent with our earlier understanding, that vr is proportional to 1 by r which we have already established in earlier lectures. So, how to how to arrive at more meaningful c i mean what is this c? c is some constant we just put in there, but what is this c?

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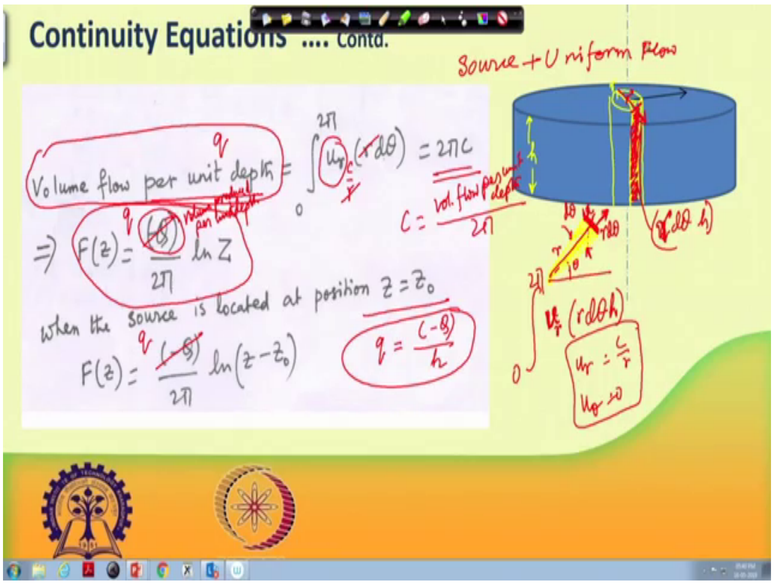


So, what we do here is, we try to see the meaning of c. In this case what is the volume flow per unit depth if we look at? Per unit depth we do not need to consider per unit depth as such, but let us let us see what was this let us say I am talking about a cylindrical, let us say I am looking at a cylindrical area with radius let us say r ok. So, let us say I have this height as h ok. So, we had discussed this earlier that this is here we are talking about a slice, here we are talking about a slice that slice has height h and all these vr that are that are that are coming out through these.

See we are we are looking at let us say this is the source, this is the source and we are looking at these source has let us say this is the vr this vr is operational let us say this this this makes an angle theta. So, vr is operational over this arc length vr d theta. This is this is the this is the area over which this this vr is operational ok. So, what we are essentially saying is that this is the vr this angle is d theta and vr d theta is the length of this arc and if you multiply this by the height. So, vr this is the area we are talking about this area is vr d theta into h at over sorry this is this this area is I am I am extremely sorry this is vr. So, this arc length is rd theta; this is this distance is r, this angle is d theta this is theta and this theta plus d theta.

So, difference is theta d theta. So, length of this arc is rd theta; so, rd theta. So, rd theta is the length of this arc multiplied by h. So, this gives you the area over which this vr is operational this is the area over which this vr is acting. So, the amount of flow that is taking place is vr is the velocity in meter per second, then r d theta into h this gives you the area in meter square. So, meter per second into meter square these gives you meter cube per second ok. So, then you have 1 as 2, but this is for the entire 360 degree one has to do this then only you can get the total flow. So, that is why you do this integration from 0 to 2 pi.

So, that is exactly what they have done your volume flow per unit depth and here the integration is from 0 to 2 pi that is fine vr, they are writing it as u r I mean in fact, my earlier terminology was u r; u r and u theta in when we started working with this complex potential. So, I think I should be writing it as u r not vr.

So, u r into rd theta into h now since we are talking about per unit depth. So, h is equal to 1 in this case. So, that is why this h is not featuring here. So, now, if you do this integration here, u r is equal to what? u r is equal to we had written it as c by r u r is c by r that is what we found just in the previous slide right previous slide we found that u r is equal to c by r and u theta is equal to 0.

So, that is that is what our observation was in the previous slide. So, c by r into r d theta these r will cancel out with this r. So, u r left with c d theta; c d theta and d theta when you do the integration from 0 to 2 pi, you will get the integration gives you 2 pi. So, these integration gives you 2 pi c. So, volume flow per unit depth is 2 pi c. So, this now gives me a meaning of what is c. Because now instead of writing it as C ln z you can see now we are writing it as F z is equal to minus of Q divided by 2 pi ln Z why? Because volume flow per unit depth, volume flow per unit depth which is equal to 2 pi c. So, c is equal to volume flow per unit depth divided by 2 pi right.

So, now we can we can since the Q there is a sign issue because if you are producing the well vr becomes negative. So, vr and Q they do not they have the opposite sign. So, one has to keep this in mind, if the Q and the vr they are maintained as the same sign, then it would see it is basically volume flow per unit depth this is C is; C is equal to volume flow per unit depth per unit depth divided by 2 pi. So, that is the definition of C.

So, that is what we are done. So, this is basically volume produced per unit depth. So, this is this is how one can write the complex potential in case of source with; we in case of a in case of a source in case of in case of the elemental flow for a source. So, q by or volume produced per unit depth divided by 2 pi ln z. When the source is located at position z is equal to z 0.

See all these analysis that we have done so, far the source is located at origin right here also the source is located at the origin. So, we have done from the origin everything is happening. So, now, if the source is not located at origin rather source is located at some other position. This is very important because if you are working with three sources and two sinks distributed in the porous medium and you are trying to find out what is the resultant stream lines. So, in that case if you if you pick up one as the one source as the origin the other sources have to be then deviated from the origin.

So, one has to have a general equation for it and that general equation is ln of z minus z 0. I am not fully happy with this calling this a Q because last time the Q was flow rate not flow rate per unit depth. So, we can we can instead of minus Q we can still maintain this let us say volume flow rate we have to give some other name to it. Maybe we can give this name as small q that would be better. So, instead of capital Q we write this as small q and then then instead of capital Q we write this as small q and then we equate this that q is equal to q is equal to minus of capital Q divided by h.

So, that is because my then then that would be probably consistent with what capital Q we have defined in previous lectures. Basically it is a volume flow per unit depth and if v r is radially outward and it is positive then this small q is positive, but then if we are looking at the production from the well which is the reverse of this, then q has to be capital Q has to be negative. So, that has to be accounted.

So, anyway this is these are just the choice I mean when you solve problems you have to put these put these numbers. Best thing my sincere solution here is work with the units because we have we are talking about some case volumetric flow, some case volume flow per unit depth. Actually they have gone into volume flow per unit depth because you can see this is directly linked to the stream function right.

So, that that is. So, in one case we are talking about volume flow per unit depth, in another case we are talking about volume flow rate, another case we are talking about superficial velocity which is volumetric flux. So, best thing is whenever you work with these put the units unit. If you put the units, then you can see the units cancelling out and you can moment you see the left hand side is matching with the right hand side you you know you understand with you are you are under at the right place. So, that is very important for these this type of analysis. Anyway we will we will definitely if you solve problems in these areas it this this concepts will be all clear. This is all I have for this particular module of the lecture I will continue further probably my next attempt would be to superpose a source superpose a source and a uniform flow.

So, if we if we have a source and uniform flow clubbed together, how this type of flow play out what would be the what would be the resultant stream lines. So, gradually. So, source we have already understood by I mean we made good enough of an understanding with the source, we have we already understand uniform flow let us say u z if we are talking about only the flow in x direction. So, now, if we superpose the two then what would be the resultant stream lines? What would be the resultant velocity profiles in Cartesian system or cylindrical system? So, these are some of the things that I will get into in the next lecture. So, this is all I have as far as today's lecture is concerned.

Thank you very much.