**Heat Transfer**
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**Lecture - 03**
**Heat Diffusion Equation**

So, we have gone through the fundamentals of Heat Transfer, the different modes of heat transfer conduction convection radiations their special characteristics and so on. The first part of this course will be devoted towards understanding conduction, deriving the relevant equations, and how those equations can be applied in space for specific practical applications.

One of the fundamental relations, which is quite common in conductive heat transfer, we have spoken about this Fourier's law of conduction, which defines the material property thermal conductivity. And, where we have seen that the heat flux, which is a vector, is proportional to the temperature gradient and the proportionality constant defines the thermo physical property k, but the use of this equation for a control volume will give rise to an equation, where the temperature can be expressed. As a function of special coordinates like what sea temperature at a specific xyz coordinate in a fixed coordinate systems as well as what is the time rate of variation of temperature at a specific location. So, this heat diffusion equation which we are going to derive today would allow us to find the temperature at any temperature as the function of time and xyz.

So, space and temporal variations of temperature can be obtained, if we can solve the equation the governing equation that that can be derived based on Fourier's law of conduction. In order to do that in order to derive the equation the first thing one has to do is define a control volume.

As we discussed before the control volume con has a fixed mass and it is defined by control surfaces the control surfaces do not have any mass of their own. So, the conservation equation, which is applicable for a control surface would simply be equal to in, is equal to out.

So, control surfaces do not have any mass of their own whereas, the control volume will have a specific mass. So, the conservation equation for a control volume would be rate of heat in minus rate of heat out of the control volume, plus the rate of heat generation inside the control volume is equal to the time rate of change of energy stored inside the control volume.

So, this generation of energy can be by nuclear means a nuclear reaction is taking place in the cont inside the control volume it can also be due to ohmic heat due to joule heating some amount of energy is gen generated inside the control volume. So, rate of energy in rate of heat in this case in minus rate of heat out plus rate of heat generation inside the control volume, the algebraic sum of these 3 must be equal to the rate at which energy is stored inside the control volume.

So, we will specify a control volume define a control volume and write this conservation equation, the physical description of the equation is going to be written in mathematical form which would ultimately give rise to the governing equation for heat conduction in a solid.

And, once we have this equation we would see what are the simplifications, that can be made to this equation and under those simplifications and with the help of appropriate boundary conditions, which will discuss next what is how what are the special form of equation for temperature distribution that can be obtained inside the control volume. Because the main aim of any conduction analysis is to obtain what is the temp what is the form of the temperature profile what is a temperature profile?

Because, once we have the temperature profile; for example, let us say t as a function of x, then we can find out by simply differentiating the profile what is del t or dt dx at a specific location? If you know what is dt dx, multiplying that with minus k k being the thermal conductivity we would be able to evaluate what is the heat flux at that location. So, it is of paramount importance to know; what is the temperature distribution inside let us say a solid at a given condition.

So, the derivation the understanding of the derivation of the heat diffusion equation is what we are going to try in today's class based on a or defined control volume of dimensions del x del y and del z where this del x, del y, and del z are small scales of length, which together define a control volume.

So, we will assume that it is cuboid which is defined by the length scale del x del y and del z and we would see how the heat comes in to the system, goes out of the system, what is the rate of heat generation? And the sum total must be equal to the time rate of storage time rate of change of energy storage inside the control volume. So, let us look at this picture which essentially tells us.

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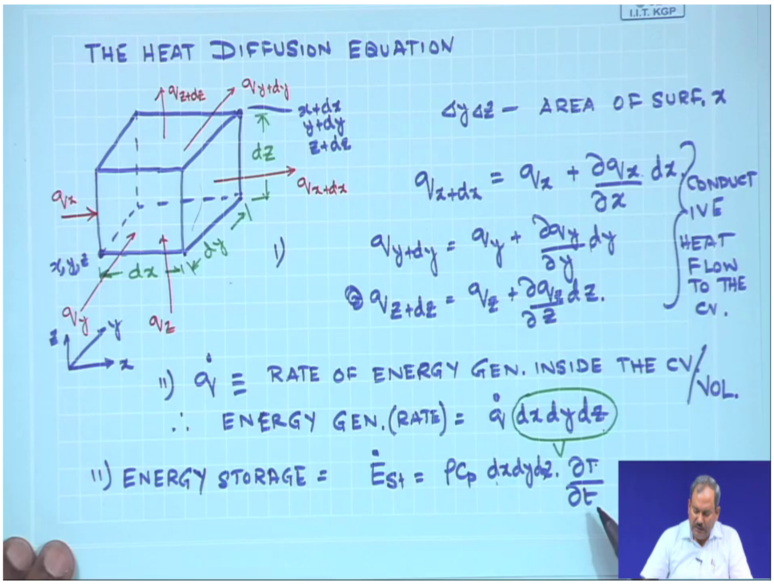


This is my cuboid, which has a length of dx dy and dz. So therefore, this is the x direction, this is the z direction, and this is the y direction. So, this is at this location is xyz and the top obviously, is going to be x plus dx y plus dy and z plus dz.

So, this point is x plus dx and so on. And, we would assume that q x is the amount of heat, which enters the control volume and it leaves from this phase as q x plus d x. So, q x denotes the amount of conductive heat flow to this control volume through the surface at x.

So, the surface at x has a has an area equal to del x del y del z is the area of surface, x. Similarly, the y phase which is perpendicular to the y direction has an area equal to dx times dz. And the z phase which is of area under at the bottom it will have an area of del x and that dx and dy. So, the by x phase or y phase or a z phase we mean that it is the area, which is perpendicular to the direction mentioned. So, the area x phase is perpendicular to the x direction and the amount of heat flow to the system would be denoted by q x ok.

So, we are going to we are going to write the energy equation with the conservation of energy for this control volume. So, we in order to do that first thing that we would do is we have to express q x plus d x, that is the amount of energy which leaves the x phase as the amount of energy, which comes in to the x phase plus the change of energy in the x directions, change of conductive heat flow in the x direction with x multiplied by dx.

So, this is a fundamental expression which is obtained by the teller series expansion of q x and neglecting all higher order terms. Now, if we see what is the, if we understand that dx is quite small there are Taylor series expansion of q x around x and neglecting the terms higher order neglecting the higher order terms is admissible it is accurate and therefore, q x plus dx can simply be expressed as q x plus the rate of change of q x with x multiplied by the length scale, which is d x over here. So, this analysis is only valid if dx is small which you have assumed to be small at the very beginning will be have defined our control volume.

So, this form of q x plus dx is therefore, definitely valid for the specific case once we write q x plus dx I should be also be able to write what is q y plus d y and q z plus d z exactly using the formulation that we have done for q x plus d x. So, let me write that and then I am going to write the conservation equation. So, your q y plus d y would simply be written as using the teller series expansion and q z plus d z would be equals q z plus del q z del z times d z ok. So, these this is the first thing which these are essentially conductive, heat flow to the control volume.

Secondly, let us assume that q dot is as the rate of energy generated inside the control volume. And as I have mentioned before inside the control as I have mentioned before this could be for various reasons from a electrical heating the nuclear heating and so on. So therefore, the total rate total amount of energy generation, this is per unit volume. So, energy generation or rate this must be equal to q dot times d x d y d z, because together this is going to be the volume.

So, therefore, if q dot is the rate of energy generated per unit volume, then the total amount of energy generated the rate of energy generation would simply be equal to equal to q dot times d x d y d z. And, the last thing which remains is energy storage and energy storage would simply be equal to E dot the dot denotes the time rate would be rho c p times d x d y d z multiplied by del t over time.

So, this means the energy storage the change in energy inside the control volume as a result of conduction would simply be equal to M C p delta t. And, if I take the time if I take find try to find out the rate of energy storage inside it, then it simply going to be rho C p multiplied the volume multiplied by del temperature over del time. So, these terms together, so this is going to give me the total amount of heat or conductive heat, which is coming to the control volume, this is going to be the total amount of energy generated inside the control volume.

And the last the third term will give us the time rate of change of energy storage inside the control volume. So, if we think of the think of the energy conservation equation the rate of energy into the system, which would be q x q y and q z and the rate of energy going out which would be q x plus d x q y plus d y and q z plus del z plus the amount of energy generated inside the system, which is e dot g which should be equal to q dot, which is the rate energy generation rate per unit volume. So, q dot multiplied by d x d y d z, the algebraic sum of all these must be equal to the time rate of change in energy storage inside the system, which is the last term that we have written.

So, this equation I am going to put together now, taking into account all the in and out terms and the conservations would give me the conservation equation with appropriate steps would give me the energy equation. So, let us see how this is done? So, with the terms which are already written before?

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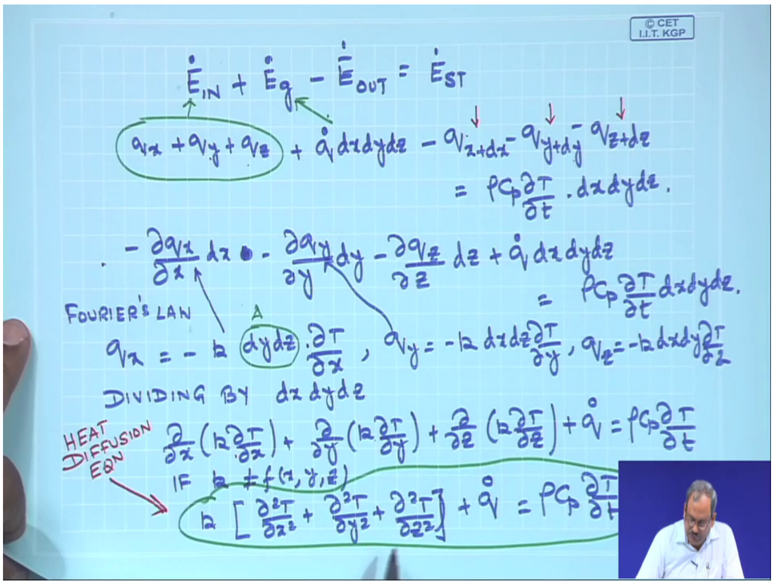

I am going to say that rate of energy in plus rate of energy generated minus rate of energy out is equal to the time rate of energy stored inside the control volume. So, it would be q x plus q y plus q z, which essentially is the in plus q dot dx dy dz minus the so, this is the generation term should be equal to energy stored, which is rho C p del temperature by del time times dx dy dz.

So, this is the equation and when I put in for q x plus dx or q y plus d y or q z plus d z if I put this term then the q x over here and the q x over here would cancel out and only this term will remain in between this and the and the and q x plus dx. Similarly between q y and q y plus d y, this would be cancelled and only this term will remain and same for q z.

So, once we do that simplification; that means, once I plug in the expressions of these 3 and combine with them, what I would get is the final equation as minus del q x by del x times dx minus del q y by del y times d y minus del q z del z d z plus q dot dx dy dz is rho C p del this term ok.

We also understand this on Fourier's law from Fourier's law I can write that q x is minus k times area which is d y d z times del T by del x. So, this is the area q is minus k A d T d x. And since we are talking about x phase it has area of d y d z similarly q y would be minus k times d x d z del T del y and q z would be minus k d x d y times del T del z. So, once these are put in here and you divide both sides. So, eh and this is put in here and you divide both sides what you would get is del del x of k del T del x plus del del y of k del T del y plus del del z of k del t del z plus q dot is rho C p del temperature by del time.

So, this is one of the forms one of the more general forms of the heat diffusion equation. And, if you look closely it gives you the variation of temperature with x y z and time. And, this is nothing, but the conservation equation, energy conservation eq equation where the only way only mode of energy transport is by conduction.

This q dot is the energy generated per unit volume and this is simply temperature rho is the density and C p is the heat capacity of the system that we are talking that we are dealing with. If k is a constant; that means, it is not a function of x y and z; that means, the system is isotropic. And by isotropic I mean that the system properties do not change with direction, which is true for many of the common systems that we deal with it, but for some special crystals and so on and special when the system size is reduced this k can be a function of this k can be a function of the direction.

So, what do we assume that it is an isotropic system, which many of the systems that we would deal in our practical studies on heat transfer they are? So, therefore, this k can be taken out is not a function of x y and z. And therefore, the k can be taken out of the differentiation sign and what we get is the most common form of heat dequi diffusion equation, which would simply be k times del 2 T del x square plus del 2 T del y square plus plus del 2 T del z square plus q dot is rho C p del temperature by del time.

So, this is probably the more common form of heat diffusion equation that you would encounter. So, this simply says that it is an isotropic medium and therefore, you do not have any variation of k with either x y and z.

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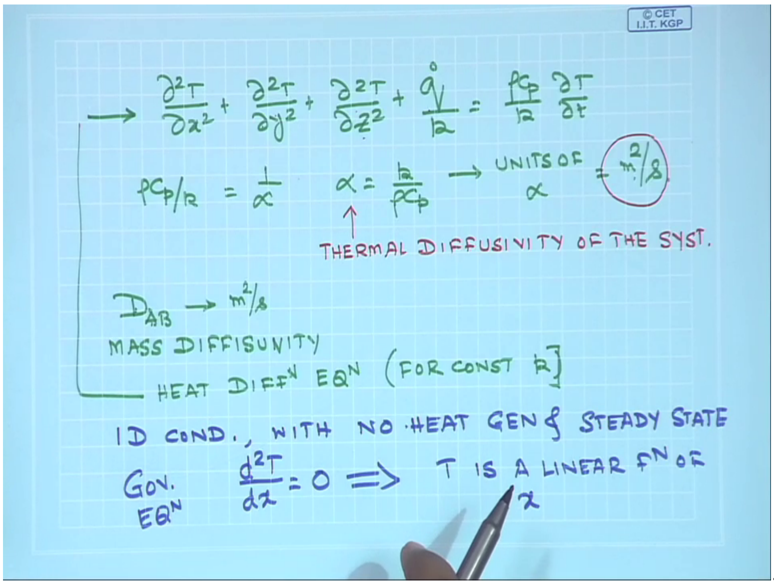


A slight rearrangement gives an form of this equation, which is more so, which is they are in the textbook delta T by del x square plus this term rho C p by k or rather the inverse of this term is called alpha. So, alpha is simply k by rho C p and if you put the units of k rho and C p in here you would see the units of alpha are simply equals meter square per second.

So, what this tells me is that this k this alpha is known as the thermal diffusivity of the system. If, you probably read later on is for the mass diffusivity, which is denoted by D A B which is the diffusion coefficient of A and in B has also units meter square per second.

So, this D A B which is more commonly we call as only diffusivity this is simply the mass diffusivity, but here we have a quantity which also has meter square per second and that is why it is known as thermal diffusivity of this system. And the same way diffusivity tells us how fast a species molecule will travel from 1 point to another. So, higher the value of diffusivity the faster it would travel to a given location given location and similarly the thermal diffusivity we will see, that thermal diffusivity would also tell us how fast a temperature front will propagate through the solid to reach another point far from the original point.

So, conceptually the thermal diffusivity and the mass diffusivity are going to be the same. And, you probably recollect that in fluid mechanics, you have studied you have seen that mu the viscosity divided by rho, which is which is the density this mu by rho also has units of meter square per second. And this mu by rho is known as the momentum diffusivity.

So, what you would get 3 diffusivity is which you would see in one for the case of momentum transfer, which is mu by rho the other is the case of thermal the energy transport, that is k by rho C p and the third one is the more common more the one which you commonly come across is the mass diffusivity which is denoted by DMA. The units of all 3 quantities would be meter square per second, and the same units for these 3 quantities would lead to the unified treatment of heat mass and momentum transfer, that you would come across in transport phenomena course in your later years, but the fact that all 3 have the same unit would possibly have led people to think, that it is possible to use the same type of analysis for heat mass and momentum transfer.

So, the fundamental equation which we now then have for conduction is this. So, this is known as the heat diffusion equation. And this specific form is for constant thermal conductivity this is mu. Now many simplification to this equation are possible for example, if you assume that it is the heat conduction is taking place in a system in which you do not have any heat generation.

So, this q dot term would simply be equal to 0. If, you assume that this system is a represents a system at steady state and by steady state I mean that the temperature is a could be a function of x y and z, but it is not a function of T. So, at steady state this temperature change with time would simply be equal to 0 and you will be left with only the 4 terms on the left hand side. What is the simplest possible situation that you can think of it describes the heat diffusion equation, let us say it is being applied to a case where you have only 1 dimensional conduction with no heat generation and it is at steady state.

So, that is the simplest possible thing which you can think of which would which would give you that T is not a function of y, T is not a function of z, there is no q dot present in the system and T is not a function of time. So, there is the case time then I will only have this term present in it and I do not have to write the partial sign since t is a function only of position.

So, the governing equation for such a case would simply be d 2 T d x square equal to 0. So, this would be this if you solve this equation then T is a linear function of x. So, if in a solid if in a solid let us say T is a function of x only and it is at steady state no heat is generated in the system, then for that special case you are going to have a linear distribution of linear distribution of temperature inside the system. The same type of equation can also be also be derived for the case of cylindrical systems and spherical systems.

So, in cylindrical systems the temperature can be function of radial location r theta or axial location z. So, T is going to be a function of r theta and z and in the case of spherical systems the T can be a function of r theta and phi. So, these 2 equations in spherical and in cylindrical coordinates are available in your texts.

Since, the geometry is slightly more complicated for the case of spherical and cylindrical systems the heat diffusion equation that you see for spherical and cylindrical systems do look more complicated, but fundamentally there is no difference. They only use conservation of energy to explain, the physics of the system then only conductive heat transfer is present in the system.

So, what we do next is whenever you have a governing equation like the way we have seen for the case of 1 dimensional steady state conduction with no heat generation I need to my aim is to obtain as I have said before to obtain t as a function of x y z and time. So, wherever you are integrating whenever you are solving the governing equation, you are going to get integration constants. And the integration constants can on only be evaluated if you know what is the boundary condition.

So, we would see in the next class; what are some of the pertinent boundary conditions, which are commonly encountered in conductive heat flow and how with the use of appropriate boundary conditions we would be able to obtain the temperature profile inside a solid undergoing conduction. So, in this class from first principles from energy generation energy conservation equation we derived, what is what is known as the heat diffusion equation? And we have done it for Cartesian coordinate systems; you can also be done for cylindrical and spherical coordinate systems.

We have also seen that simplification of the equation can lead to several conditions, several situations, were in some cases it would be possible to solve the equation with boundary conditions, that we would see in the next class. And this also shows us that this k by rho C p you this defines a new property of the system, incorporating the rho the density this specificate and the thermal conductivity.

So, the progression of temperature front inside a solid depends not only on the thermal conductivity, it also depends on a specific heat and density. So, together they define a thermal diffusivity, which would tell us how fast or how slow a temperature front will move into a solid object undergoing conduction.

So, that way it is similar to mass diffusivity and we have seen in fluid mechanics that we have something called momentum diffusing as well, which would be the which would be an start of an unifying concept between heat mass and momentum transfer.

So, we will stop here today. And, in the next class we will start with the relevant boundary conditions and the solution of heat diffusion equations.