Heat Transfer
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Lecture - 26
Heat and Momentum Transfer Analogy

So, we were discussing about the dimensionless form of the momentum equation, dimensionless form of the energy equation, the boundary conditions again in dimensionless forms; essentially for the case of fluid flow, the no slip velocity. And what would be the condition on what will be the condition of velocity at a point plot from the plate such that at that point the velocity is going to be equal to the local free stream velocity outside of the boundary layer. And similarly, we are also looking at the energy equation in what would the, what would be the form of the boundary conditions?

For example, what is going to T star that is the dimensionless temperature at any actual location; but on the plate itself? So, T star at x star comma 0; that means, y star equals to equal to 0 would be equal to 0 because of the way we have defined the dimensionless temperature T star. But T star were T star was simply T minus T s by T infinity minus T s. So, on the plate T is equal to T s; therefore, T star would be equal to 0. At a point plot from the plate, the temperature of the fluid would simply be equal to T infinity and the value of T star in that case would be equal to 1.

So, we were looking at 2 equations to governing equations for 2 processes; one for heat transfer and the other for momentum transfer. And we saw that the ones that separate, the combination of terms that separate these 2 equations that that differentiates between these 2 equations are the presence of similarity parameters. One is the Reynolds number for the case of momentum transfer and the second is Reynolds times Prandtl number for the case of heat transfer.

So, these are the only difference between Heat transfer and Momentum transfer. So, what we would like to do is we have we have proposed then that if we could keep the Reynolds number to be the same for heat transfer as well as or momentum transfer and if you can choose a hypothetical fluid with a Prandtl number to be equal to 1, then these two equations dimensionless form of these 2 transfer equations are identical.

And if in addition, we assume that the flow is taking place over a flat plate, then the governing equation the boundary conditions of the governing equations are also going to be identical. So, that is the case of dynamic similarity which they tells us that that a for a dynamic similar system, the expression of dependent variable for the case of momentum transfer which is u star can be replaced by the replaced by the dependent variable of the other equation which is T star.

And therefore, and analogy, a similarity and equableness between the momentum transfer and heat transfer can be established to obtain expressions, known expressions of one dependent variable from the known expression of another independent another dependent variable. So, will be look at that it will be very clear towards the end of this class, how it is done.

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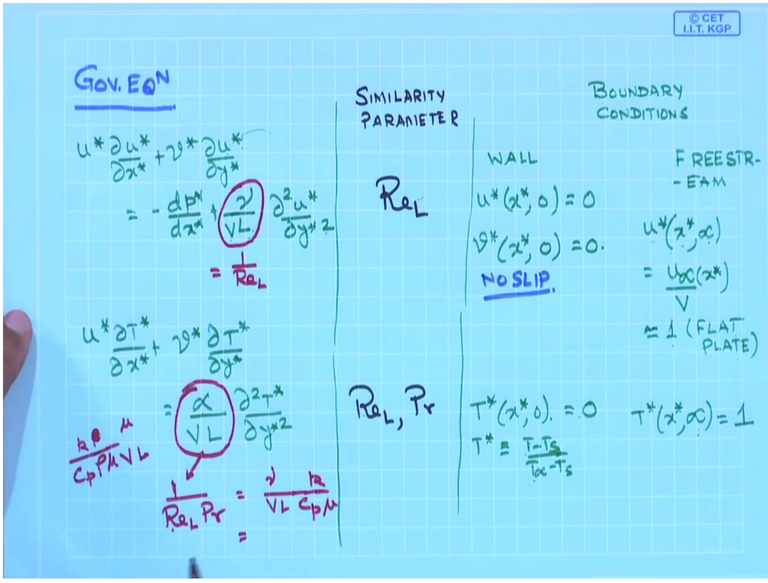

So, let us look at the this slide which was the last slide on the previous class, where I have identified the governing equations, the similarity parameters, Reynolds number and Reynolds and Prandtl number. This is for momentum; this is for energy and the boundary conditions using no slip and a point plot from the flat plate, what would be the velocity condition? The temperature at y equals to 0 and temperature at y equals to infinity.

So, with this knowledge when we by keeping the Prandtl number to be equal to 1 and keeping the Reynolds number to be the same and assuming that the flow is taking place over a flat plate; everything in this equation in the between these equations and the boundary conditions are identical, so we have a similar system, dynamically similar system.

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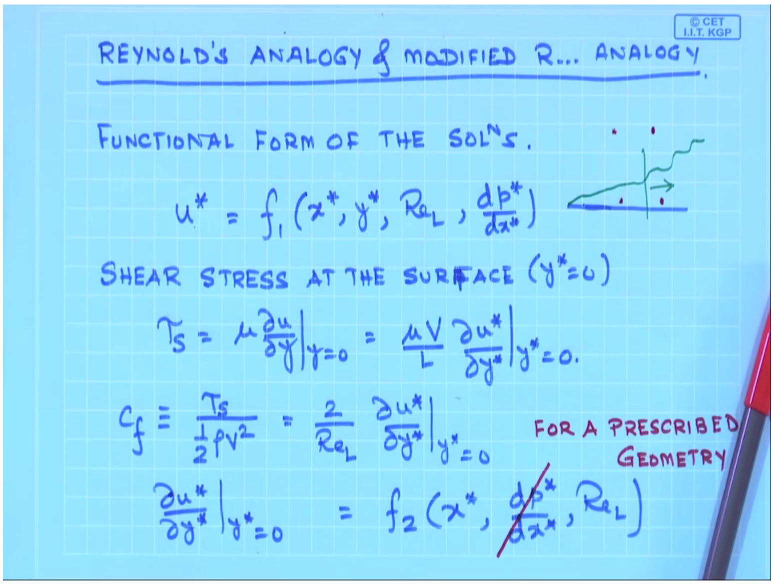

So, what I am going to what we going to do is find out what is Reynolds analogy and modified Reynolds analogy. So, for that I am going to look at the function what could be the functional form of u star. I do not know what would be the exact form of it; but I know that if I could write if I could write the u star if I could write the functional form of u star, it should contain the independent variable x star, independent variable y star, the similarity parameter Reynolds number and the pressure gradient present in the system which is d p star d x star.

So, my functional form of u star is going to be x star y star Reynolds number based on the entire length of the plate and d p star d x star. I do not know how u is going to be connected with x y or Reynolds number, but I know that a functional form like this would exist for the case of flow. Now in terms of engineering interest we would like to find out what is the shear stress at the surface? That means, at this by at by at the surface, I mean at y star to be equal to 0 that is at the surface.

So, which I let us call it as tau s, the shear stress which would be mu times del u del y at y equals to 0. So, this is the shear stress at the surface and if I if I non-dimensionalize it; it is simply going to be mu u by L del u star by del y star at y star equals to 0.

So, that would give me the expression for shear stress and shear stress coefficient, we understand that by definition its tau s by the dynamic pressure which is half rho V square; V is the approach velocity, rho is the density. So, that is a definition of C f. So, the definition of C f it can be written as Reynolds number for the entire length del u star by del y star at y star equals to 0. This was simply obtained by putting the value putting the expression of tau s over here and observing this half rho V square in it to obtain to by a real del u star by del y del u star by del y star at y star equal to 0.

So, this if I would write take write to find what is the, what is the del u star functional form of del u star by del y star at y star equals to 0? So, if you look at the expression the functional form, the hypothetical functional form of u star, I am trying to find out del u star del y star at y star equals to 0. Since, I am assigning a specific value of y star to be equal to 0; this must be a function of x star d p star d x star and Reynolds number based on length. Since, I have specified the value of y star to be equal to 0. So, therefore, the y star does not appear over here.

Now, this is the flow; this is a flat plate over which the flow is taking place and this site is the turbulent flow. Now, if the geometry is prescribed, then you would be able to obtain d p star d x star separately. So, this for a prescribed geometry, I will I will explain on it in a moment. Remember that what I have told you before is between in inside the boundary layer, the flow is viscous; outside of the boundary layer, the flow is in viscous. So, there is no effect of viscosity in here. Since, you have viscosity present in effect of viscosity present inside the boundary layer, you cannot use known equations which are available to give which are there to provide what is the pressure drop as a function of distance.

Now, when you when if you if you if someone tells you that what is the equation that provides the pressure drop in a flow? The name that comes to your mind is Bernoulli’s equation because Bernoulli’s equation would relate the pressure gradient the pressure head, the velocity head and the gravity head. Now, if I assume the plate to a horizontal which is the case in this case. So, therefore, the it is going to be the summation of pressure head and velocity head to be constant. So, if I know this velocity or I can express the change in pressure in terms of change in velocity head that is what Bernoulli’s equation is all about. Now, there is catch though; the Bernoulli’s equation is strictly valid for in viscid flow for flow where the effect of viscosity is absent.

So, inside the boundary layer, technically I cannot use Bernoulli's equation as the flow is viscous there. So, this solution; but the observation is outside of the boundary layer the flow is in viscid. So, if the geometry is known to me then I would be able to use Bernoulli's equation in the flow domain outside of the boundary layer to obtain and expression for d p d x or d p star d x star independent of everything.

So, if someone gives me the geometric I should be able to obtain, what is d p star d x star outside of the boundary layer through the use of Bernoulli's equation and since the thickness of the boundary layer is very small, there is no change in pressure with y. That is an assumption which is, a valid assumption considering the small thickness of the boundary layer. So, I use Bernoulli's equation to find out what is d p star d x star. So, d p star d x star can be obtained and for a prescribed geometry d p star d x star is a constant; for that reason the from the expression of del u star by del y star at y star equal to 0 which had otherwise contained d p star d x star, I can drop that. Since for a given geometry this pressure gradient is known to me after every and is a constant.

So, in terms of functional form of the equation whatever I have written over here can simply be written once again as del u star del y star at y star equal to 0 is a function only of x star and Reynolds number I need not mention d p star d x star for a prescribed geometry.

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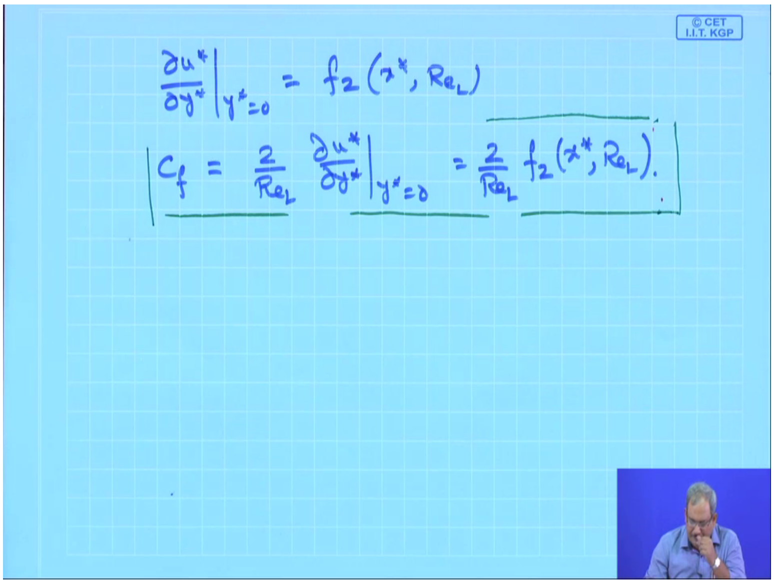

So, in other words my del u star by del y star at y star equal to 0 is simply going to be f 2 the function the unknown function x star times Reynolds number x star times Re L. Now, now if use C f which I have seen to be equals 2 by Re L del u star by del y star at y star equals to 0 which I have seen in here. This is the, this is where I have obtain the expression for C f. So, my C f is simply going to be 2 by Re L f 2 of x star and Re L.

So, my C f therefore, would be 2 by Re L f 2 the yet to be evaluated functional function of x star and Re L. So, these are the 2 equations that one needs to take a look at. First of all u is a function of all the independent variables, the operational parameter and the pressure gradient. From there I obtained the shares stress; from the shear stress, I obtained C f and for del u star by del y star at y star equal to 0, I obtained the functional form for this special case when the geometry is known to me. So, this should give me the expression for C f for flow momentum transport inside a boundary layer. Now, let us see what is going to happen to the temperature profile? So, if I look at the temperature in expression over here, over here that we have obtained.

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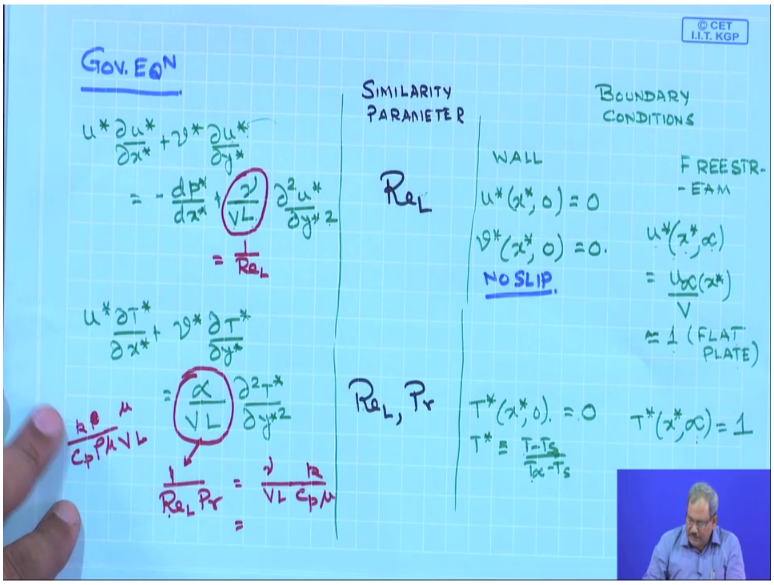

My temperature profile T star would be function of u star x star v star y star Reynolds number and Prandtl number; but this u star and v star are already a function are already known function of x star and y star and so on. For example, in this expression itself what we have seen is that u star is a function of once; you specify x y Reynolds and d p d x, u star is specified.

So, in the governing equation here you do not need to write T star is a function of u star because the moment you write T star is a function of x star y star and Reynolds number, you essentially specify u star. So, by incorporating u star once again in your a functional form that would be simply a repetition.

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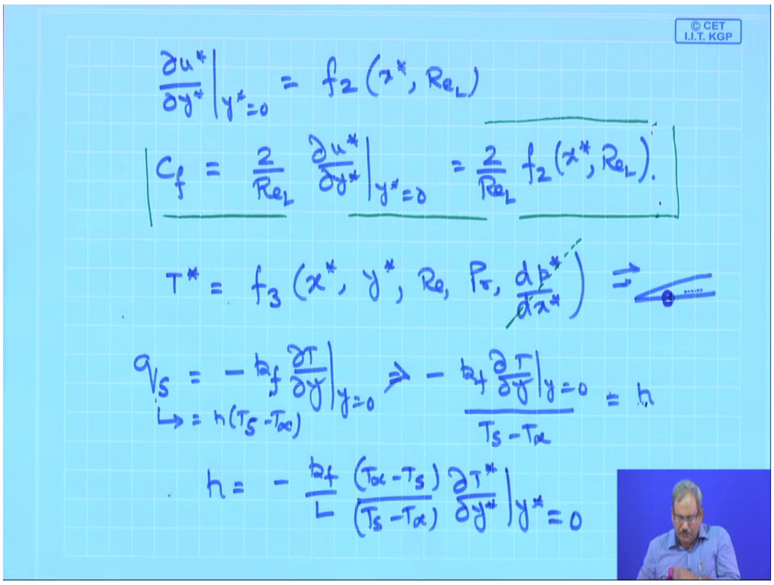

So, therefore, based on the knowledge of the of this governing equation, one should be able to write the functional form T star to be equal to a function f 3 and I put it as x star y star Reynolds number Prandtl number in d p star d x star. This d p star d x star I am keeping it just as just as to make it complete.

But we understand that for a prescribed geometry, I can drop this d p star d x star. So, the same way I have done it for the case of shares stress. I am going to write the same thing for the case of surface heat flux which I call it as q s. So, this is a solid plate, you have the profile and have flow taking place; I am trying to find out what is the surface heat flux at y star equal to 0. So, the surface heat flux is k thermal conductivity of the fluid times del T by del y at y equals to 0.

So, that is the Fourier law equivalent. That is a Fourier's law in which can be expressed as minus k f del T del y at y to be equal to 0 divided by T s minus T infinity and this is going to be equal to h. Because my q s, this is the equality of conduction and convection at this point, at the point where the liquid molecules by due to no slip are stuck to the solid.

So, the heat transfer from the immobile liquid molecules to the mobile liquid molecules, there you have the conduction and convection equality. So, this q s can be expressed in terms of Fourier's law; this q s can also be expressed in terms of Newton's law which is h times T s minus T infinity. So, h times T s minus T infinity is also equal to. So, these 2 are simultaneously valid at y equals to 0 and therefore, the expression for h can be obtained in this fashion.

So, when you express it in dimensionless form this becomes h is equal to minus k f by L T infinity minus T s by T s minus T infinity times del T star by del y star at y star equals to 0. So, this gives I am slowly moving towards the dimensionless form of the expression over here.

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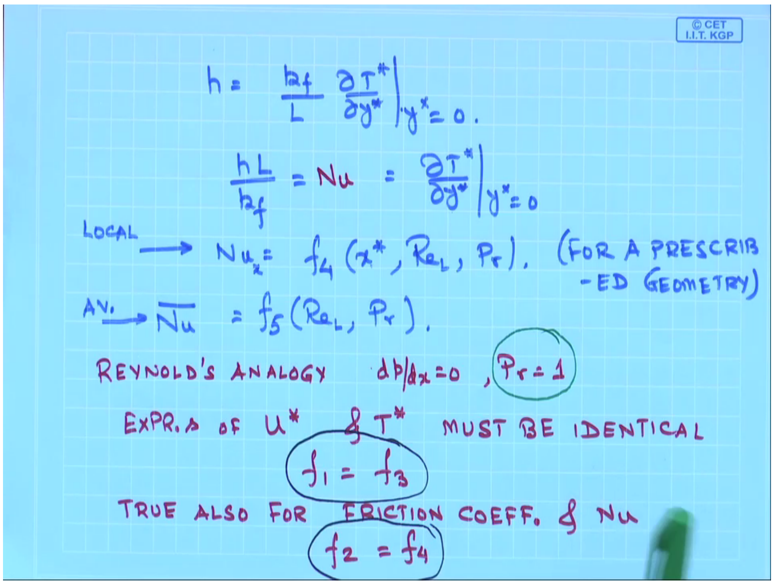

So, when you do that when you cancel that the numerator and the denominator what you get is h is equal to k f the thermal conductivity of the solid times del T star by del y star; y star to be equal to 0 or in other words, you can write h L by k f is equal to del T star by del y star at y star equals to 0.

So, what is h L by k this is nothing but the Nusselt number. So, we it in convection, we always try to find what is h or what is the expression for Nusselt number? So, now, I write a Nusselt number is used for f 1 f 2 and f 3 over here. So, Nusselt number is del T star del y star that that y star equal to 0. So, when I say its del T star del y star at y star equal to 0 that function should be a function of x star Reynolds number and Prandtl number provided the geometry is known to us.

So thus, this Nusselt number expression would be some function f 4; I do not know what this f 4 would be? But, some function of x 4 Reynolds number sorry x star Reynolds number and Prandtl number. So, this is obviously, for a prescribe geometry and if would like to find out what is the average value of Nusselt number, length average value of Nusselt number; the moment you do that, the length average value of Nusselt number; then, x star is obviously drop it should be another function f 5 Re L times P r.

So, this is the local value of Nusselt number; this is n u x and this is the. So, this is local value of Nusselt number and this is the average value of Nusselt number and the bar over Nu simply denotes it is the average value which is the function of this in the f. For length average value, it would simply be a function of Reynolds number and Prandtl number.

Now when we when we use the Reynolds condition, Reynolds analogy; what is at d p d x is 0 and Prandtl number is equal to 1 and if that is the case, then the expression of u star and T star must be identical. This is what we were discussing so far. So, the expressions of u star and T star must be identical. So, what is expression of T star and u star? So, u star is f 1 and T star is f 3.So, if your if your Prandtl number is equal to 1. So, the equation becomes dynamically similar; d p d x is in is the dependence of d p d x is not there.

So, therefore, f 1 must a f 1 must be equal to f 1 and f 3; f 1 and f 3 are going to be identical ok. So, f 1 and f 3 are identical. It is also then true that the expression for the friction coefficient which is this f 2 must also be equal to the f 4 which is which is the relation for this case. So, expression for u star and T star must be identical would simply give you that f 1 is equal to f 3 ok.

In true also for friction coefficient and Nusselt number; so, if it is true for friction coefficient Nusselt number what you would get is f 2 is equal to f 4. So, these are collectively known as the Reynolds analogy. The important point here is the major problem that you one would face in the practical application of Reynolds analogy is the requirement that Prandtl number has to be equal to 1.

Where are you going to get a fluid who is Prandtl number is equal to 1 and if it is equal to 1, how are you going to use this analogy for other cases? So, f 3 f 2 is equal to f 4; how does how does that help us? f 4 is this, f 4 and f 2 if these 2 are identical; if f 2 and f 4 are identical, I will write these 2 equations once again to show how we can use them use them in this case.

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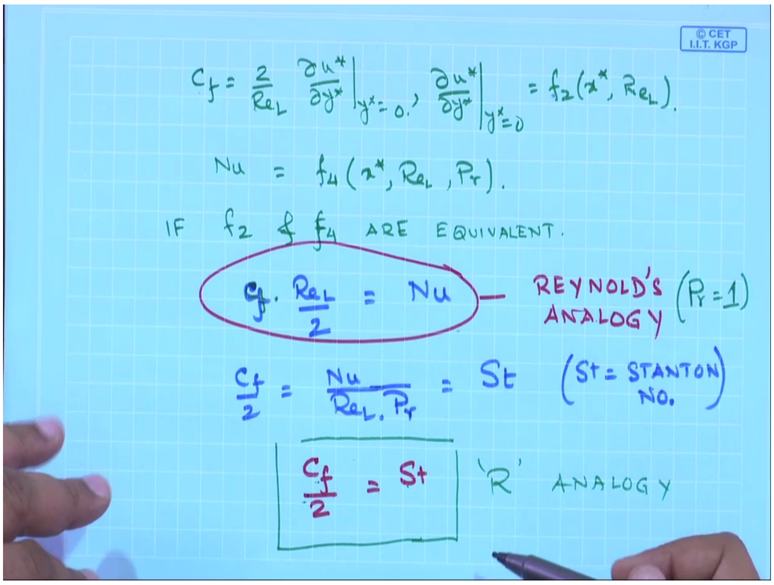

So, C f is 2 by Re L del u star by del y star at y star equal to 0 and we understand that del u star by del y star at y star to be equal to 0 is equals f 2 x star times Reynolds number based on length and for the case of Nusselt number, Nusselt number is simply equal to f 4 x star Re L times Prandtl number ok. So, if f 2 and f 4 are equivalent, then what we can say is that C f, C f from here times Re L by 2. So, C f times Re L by 2 is this which is f 2 must be equal to Nusselt number. So, if f 2 and f 4 are equivalent, then C f times Re L by 2 must be equal to Nusselt number.

So, this is known as Reynolds Analogy. This is a some extremes in some cases, it is modified in a exactly different way; where it is written that C f by 2 is equal to Nusselt by Reynolds number based on the length. And since the value of Prandtl number is equal to 1, there is no harm in adding a Prandtl number in this case. I can do that since Prandtl number in Reynolds analogy is equal to 1. So, this Nusselt by Reynolds into Prandtl has a special name which is called Stanton Number. So, I can use Stanton number the I can introduce Stanton number. So, this is the value of Prandtl number is equal to 1.

So, the general form of Reynolds analogy is C f by 2 is equal to Stanton number. This is the commonly used form of Reynolds analogy. So, this connects the key engineering parameter of C f in fluid friction with h on Nusselt number in convective heat transfer. So, I would also like to draw your attention to the previous slide that I was showing Nusselt number is equal to del T star by del y star at y star equal to 0. This again reinforce reinforces my statement that the significance of Nusselt number is nothing but the dimensionless temperature gradient at the solid liquid interface.

So, that would be the definition of Nusselt number. The more important 1 is a Nusselt number contains h; this is an engineering parameter and here I connect Nusselt number with C f friction coefficient which also is an engineering parameter. So, through the use of this analogy, I connect the heat transfer with momentum transfer; but there is as I understand, there is a problem that is only valid for the case when Prandtl number is equal to 1. So, therefore, in order to extend the validity of Reynolds analogy 2 situations; 2 fluids whose Prandtl number may not be equal to 1; a correction factor is added to this analogy and then, it takes the is called the modified Reynolds analogy.

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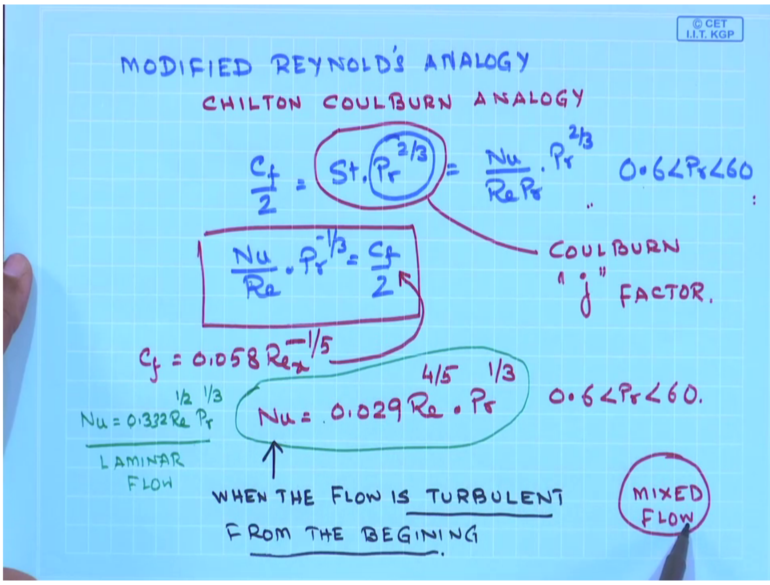

And is also known as the Chilton Coulburn Analogy to extend the, to extend the Reynolds analogy. A correction factor is added to this as Stanton into Prandtl to the power 2 by 3. So, this is the correction factor which is added in the Stanton Prandtl to the power 2 by 3 is Nusselt by Reynolds into Prandtl into Prandtl to the power 2 by 3 and this extends the Prandtl number to a large range of Prandtl number. So, what you get then is Nusselt by Reynolds in 2 Prandtl to the power minus 1 by 3 is equal to C f by 2 and this whole thing Stanton into Prandtl to the power 2 by 3 this is called the Coulburn "j" factor.

So, this is the expression for modified Chilton modified Reynolds analogy or Chilton Coulburn analogy and the validity of this is extended in most of the real systems real fluids, they have Prandtl number in the range; except for heavy oils which has Prandtl number more than 60 and the other extreme is liquid metals which as Prandtl number way below 0.6. So, for heavy metals sorry liquid metals and heavy oils, if we exclude these 2 special type of fluids most of the liquids the most of the fluids that you ordinarily use, ordinarily come across would be in this range. And therefore, the Chilton Coulburn analogy extends the extends this for a wide range of Prandtl number.

The advantage, what is the advantage? The advantage is as I mentioned C f expression is already known to us R e x triple minus 1 minus 1 by 5; put it in here and what you get is an expression for Nusselt number as 0.029 Reynolds to the power 4 by 5 into Prandtl to the power 1 by 3. The range of validity between Prandtl number 0.6 and 60. See the beauty of it. This is something which is really interesting. You have got an expression for Nusselt number, you have got an expression for h by simply using and analogy which has solid foundation. So, you the expression for C f is known to you; you are looking at the governing equations, non-dimensionalizing the governing equations; the similarity parameters clearly obtained out of this excises.

You look at the dimensionless boundary conditions; see under which condition these 2 equations governing equations become dynamically similar. The moment they become dynamic similar, the solution of one can be used as the solution of the other. So, del u star del y star at y star equal to 0 which is connected with C f can be substituted by del T star del y star at y star equal to 0 which is connected with Nusselt number.

So, the gradient of velocity or the gradient of temperature, all in dimensionless form; one related to C f, the other is related to Nusselt number. The momentum with them dynamical similar, these 2 gradients are identical and what you have then is an expression for C f and an expression for Nusselt number. The expression for C f is already known to you. Therefore, you obtain an expression for the Nusselt number in turbulent flow.

So, without getting into the complicated statistical analysis of AD formation, velocity distribution, unknown velocity distribution, the fluctuations in temperature in velocity; you have a tool now through the use of an analogy and an extended analogy by incorporating Prandtl number corrections, you now have the expression for convective heat transfer coefficient in turbulent flow. That is the beauty of this analysis or this analogy.

So, Reynolds analogy or modified Reynolds analogy which is also known as Chilton Coulburn analogy is a powerful tool which lets you find out the expression for h in highly turbulent flow. So, now, I have the complete picture in heat transfer; external heat transfer, flow the heat transfer in external flow simplest possible example flow over a flat plate. I have an expression for h in the early part where the flow is laminar up to a value of Reynolds number 5 into 10 to the power 5. And through the use of analogy, I have an expression for the Nusselt number beyond Reynolds number 5 into 10 to the power 5; that means, when the flow is turbulent.

So, together they give me a complete picture of what would be the heat transfer coefficient in laminar flow and what would be the heat transfer coefficient in turbulent flow? More importantly, this I would show you the next class a corollary of that is the flow is never fully turbulent and the flow can change from laminar to turbulent. So, in most cases any flow has a turbulent portion to sorry a laminar portion to begin with and then, it becomes turbulent.

So, those kind of flows are commonly encountered the known as Mixed flow. The early part its laminar later part it turns turbulent. So, how these relations can be modified to express the average heat transfer coefficient for the case of mixed flow. But that that is there is no new concepts and involved there. What is important is again, I would bring your attention to this equation which simply gives you the Nusselt number for the case of turbulent flow as a function of Reynolds number and as a function of Prandtl number.

I should mention as I was telling you this is when the flow is turbulent from the beginning. So, when the flow is turbulent from the beginning. This expression can be used to obtain to obtain the value of h and so on. But in most of the cases the flow is laminar to start with and then it turns turbulent those kind of flows are known as Mixed flow.

So, I will give you the expressions or mixed flow based on the expression of nusselt, Nusselt number in laminar flow and in turbulent flow in the next class. But however, I would once again write the Nusselt number for the case of laminar flow which here just to compare them is 0.3332 R e to the power half into Prandtl to the power one-third.

So, this is for laminar flow and this one is for turbulent flow. So, together if you if I combined this and this together what I get is mixed flow. But this is obtain almost completely analytically, this has some approximation built into it; but it gives us the analogy is give us a powerful tool to convert heat transfer data from the momentum transfer obtain an expression for heat transfer and vice versa.

So, will solve her quite a few problems on this to clarify the concepts and to show you how this analogy can be effectively employed in problem solving.