Optimization in Chemical Engineering
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Lecture – 49
Constrained Nonlinear Programming (Contd.)

Welcome to lecture 49, in this week we are talking about Constrained Non-linear Programming. As of now we have discussed 3 different methods to solve a general non-linear programming problem with constraints. The first one you talked about was successive linear programming. So, what you did is given a non-linear programming problem. We successively used linearized version of that non-linear programming problem and then we could make use of linear programming technique to solve the problem.

Next we talked about successive quadratic approximation of a non-linear programming problem. So, there given a non-linear programming problem we successively approximated it by a quadratic programming problem where the objective function was a quadratic function and all the constraints were linear. And you could solve it efficiently using techniques of solution of quadratic programming problems.

Then we talked about penalty function method. Which is a transformation method and this transformation methods converts the original constrained non-linear programming problem to a sequence of unconstrained optimization problem. So, the unconstrained optimization problem can be efficiently solved by several powerful optimization algorithms that are available.

So, today we will talk about another transformation method which is known as augmented Lagrange multiplier method. So, this method also transforms the general constrained linear programming problem to an unconstrained function which can then be solved using unconstrained optimization techniques.

So, in this lecture we will see how we convert or transform a general non-linear programming problem to an unconstrained problem using augmented Lagrange lagrange multiplier method.

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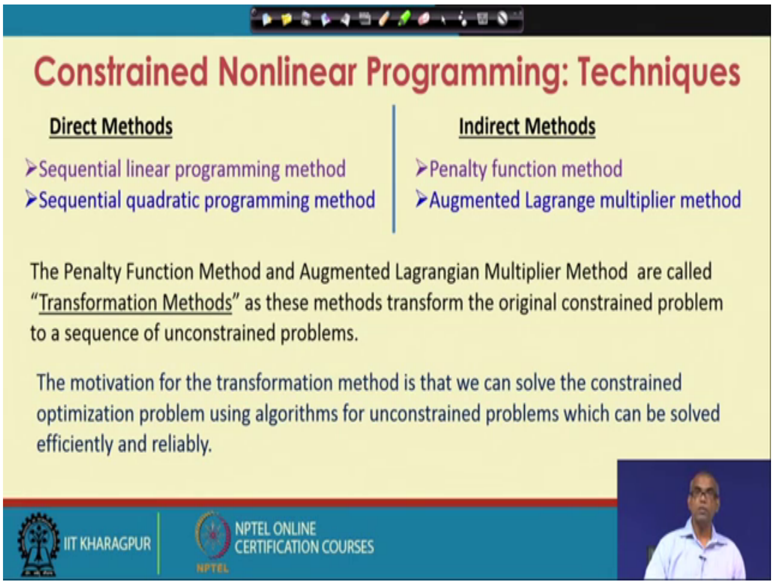

So, this is what we just talked about that penalty function method and augmented Lagrangian multiplier method are called transformation methods; as these methods transform the original constrained problem to a sequence of unconstrained problems. The motivation is that, we can solve the unconstrained optimization problem using algorithms for unconstrained problems which can be solved efficiently and reliably.

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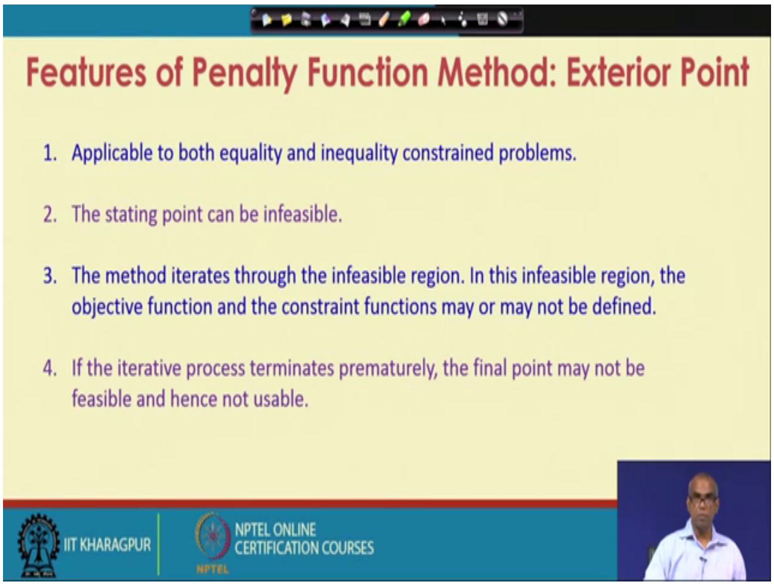

Now, we have talked about 2 different types of penalty function methods, exterior point method and interior point method.

So, let us talk about some features of exterior point method. The exterior point method is applicable to both equality and inequality constrained problems. The starting point can be infeasible. The exterior point method iterates through the infeasible region. In this infeasible region the objective function may not be defined and also the constrained functions in this in feasible region may not be defined.

So, objective function and constrained function may or may not be defined in this infeasible region. If the iterative process terminates pre maturely the final point may not be feasible and hence may not be usable.

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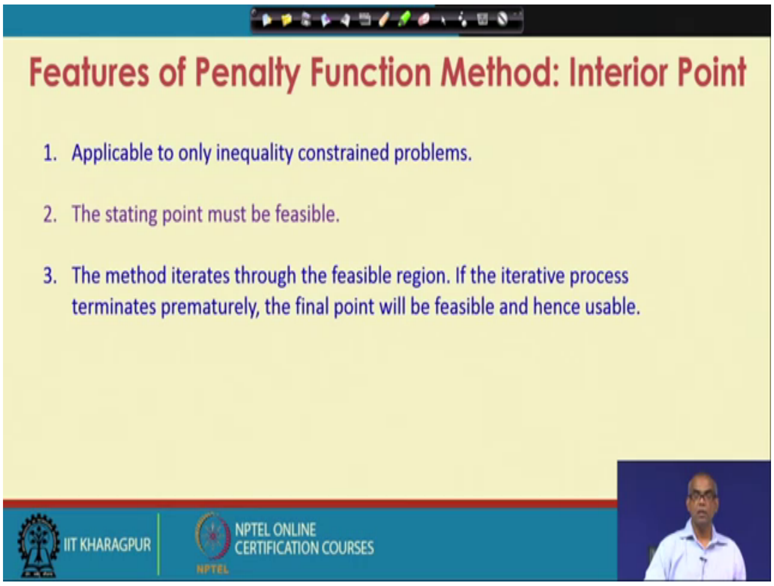

Now, let us look at some of the features that we have encountered with while talking about interior point method. Interior point methods only applicable to inequality constrained problems, the starting point must be feasible. Interior point method iterates through the feasible region. If the iterative process terminates prematurely the final point will be feasible and hence maybe useable.

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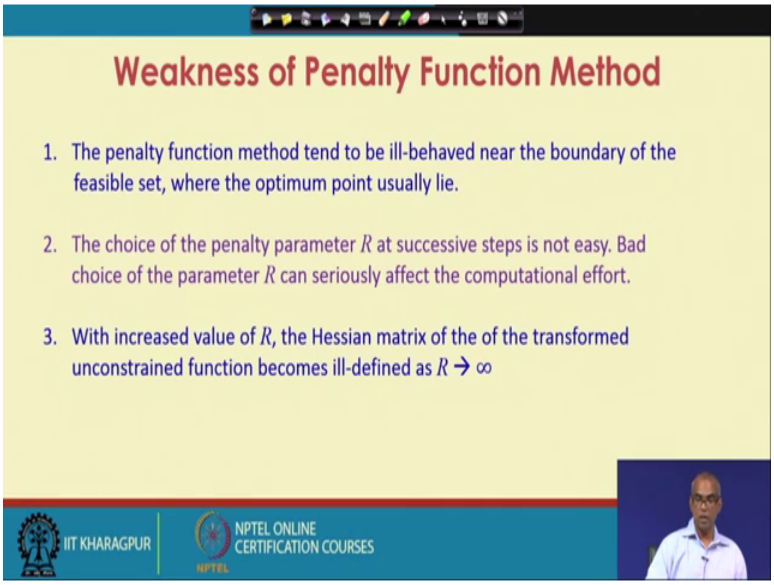

So, what are the weakness of the penalty function methods that we have noted? The penalty function method tend to be ill-behaved near the boundary of the feasible set and near the boundary of the feasible set the optimal point usually lie.

So, the penalty function method as it approaches optimum point the method tend to become ill behaved. The choice of the penalty parameter R at successive steps is not easy. Bad choice of parameter R can seriously affect the computational effort. With increased value of the penalty parameter R, the Hessian matrix of the transformed unconstrained function become ill defined as R tends to infinity. We have seen that for exterior method we start with a smaller value of R and go on increasing the value of R at each successive steps.

But, as we do that with increased value of R as the R goes to infinity the Hessian matrix of the un constrained function becomes ill-defined.

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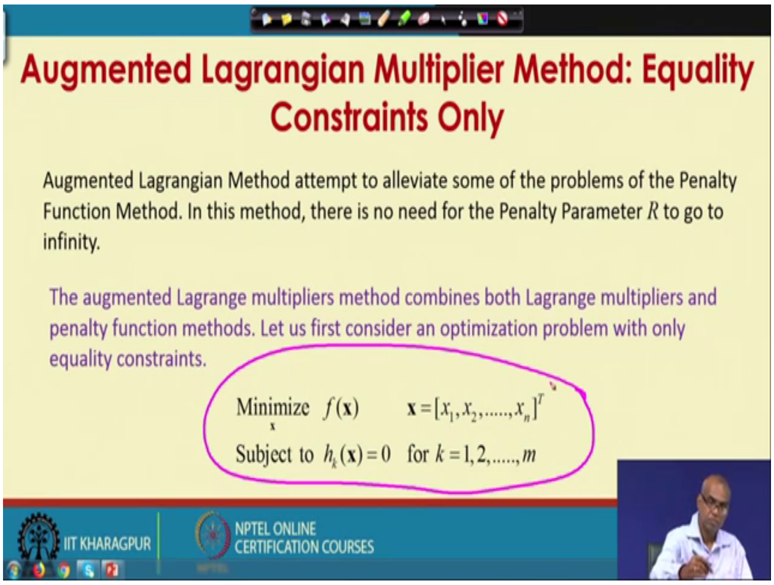

So, Augmented Lagrangian multiplier method was introduced to alleviate some of these problems. In this method there is no need for the penalty parameter R to go to infinity. So, the problem of R going to infinity and that causing ill-defined Hessian matrix of the constrained function can be even alleviated. The augmented Lagrange multipliers method combines both Lagrange multipliers and penalty function methods.

So, the augmented Lagrange multiplier method combines both Lagrange multipliers and penalty function method. So, we are familiar with both the matrix now. So, it will be easy for us to understand the augmented Lagrangian multiplier method.

So, first let us discuss augmented Lagrangian multiplier method as applicable to a constrained problem with only equality constraints. So, we first consider a non-linear programming problem with only equality constraints. So, we had minimizing an objective function f x, x is n vector and subject to m equality constraints.

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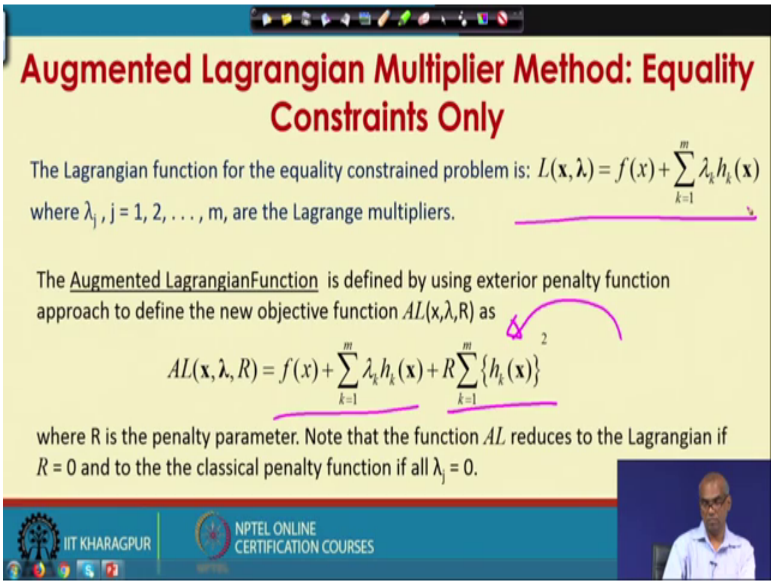

So, you can easily formulate the Lagrangian function for the equality constrained problem. All we do is we define Lagrange multiplier for each equality constraints. So, if you have m number of equality constraints we will take lambda 1, lambda 2, lambda 3 up to lambda m as Lagrange multipliers, then multiply each equality constraint with it is corresponding Lagrange multiplier and sum them up and add this sum to the objective function. So, that gives me the Lagrangian. So, you are familiar with this Lagrangian formulation.

So, now, the augmented Lagrangian function is defined by using exterior penalty function approach to define the new objective function. So, note that the Augmented Lagrangian function is nothing, but the Lagrangian function plus the penalty term. So, note that we have added the penalty term for the equality constraints. So, we have used the parabolic penalty term that we have discussed in previous class and this is an exterior penalty function.

So, the augmented Lagrangian function is defined by using exterior penalty function approach to define a new objective function. So, the Lagrangian function which is the function of the decision variables and the Lagrange multipliers lambda, we add the penalty term to this Lagrangian to obtain Augmented Lagrangian function.

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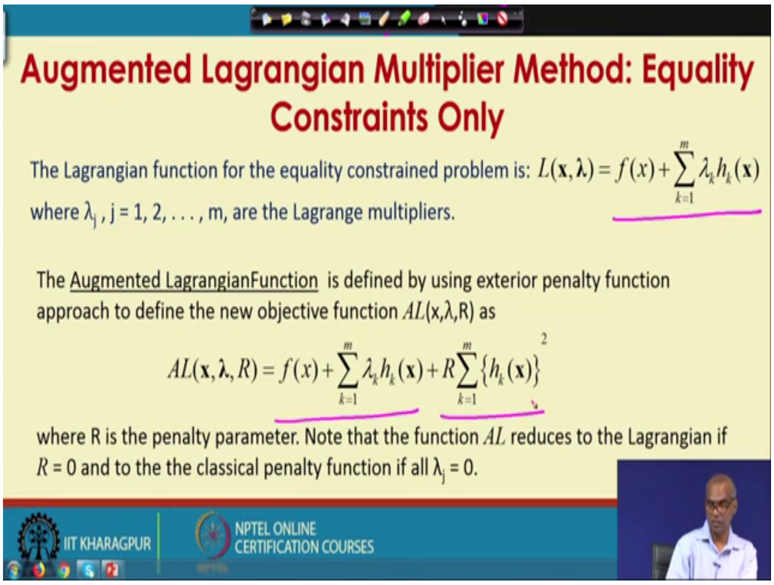

So, this part is the usual Lagrangian function and I have added the penalty term which is the parabolic exterior penalty term here to the Lagrangian to obtain the Augmented Lagrangian.

Now, look at this Augmented Lagrangian function, R is the penalty parameter. So, if the penalty parameter R is 0.

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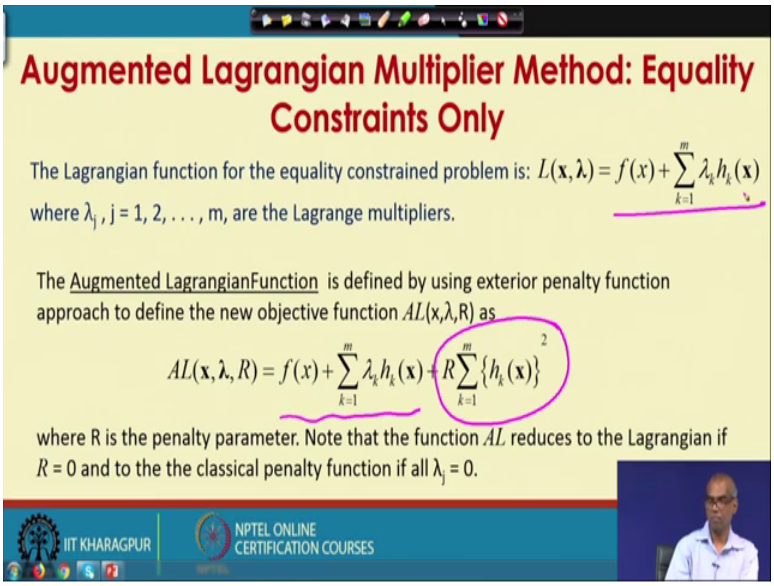

Then this term vanishes and what you obtain is nothing, but the Lagrangian.

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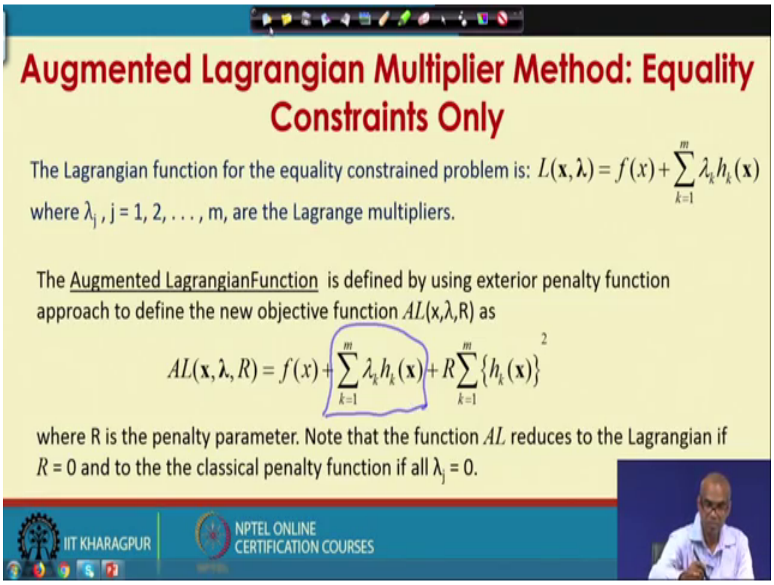

Similarly, if all the Eigen values sorry all the Lagrangian multipliers lambda 1, lambda 2 up to lambda m all are 0 then this part vanishes and what remains is nothing, but the classical penalty function that we have seen in our previous class.

So, you understand clearly that the Augmented Lagrangian function is Lagrangian function plus the penalty term. So, if the penalty term vanishes what remains is the Lagrangian and if the Lagrangian multipliers become 0 what remains is the classical penalty function.

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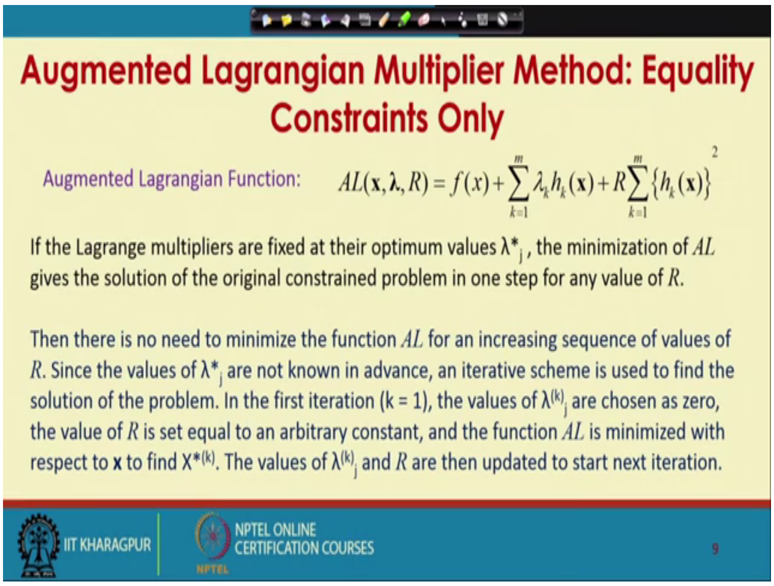

So, this augmented Lagrangian function needs to be solved and solution to this augmented Lagrangian function will lead to the solution of the original constrained non-linear programming problem. So, if the Lagrange multipliers are fixed at their optimum values, the minimization of the augmented Lagrangian function gives the solution of the original constrained problem in one step for any value of the Lagrangian multiplier R. Sorry, penalty parameter R.

So, I repeat once I have formulated the Lagrangian function, the optimization of the Lagrangian function will lead to the optimization of the original constraint non-linear programming problem. If the Lagrange multipliers are fixed at their optimum values the minimization of the augmented Lagrangian function gives the solution; that means, the minimum point for the regional constraint non-linear programming problem in one step for any value of the penalty parameter R.

In that case there is no need to minimize the function; that means the Lagrange Augmented Lagrangian function for an increasing sequence of values of the penalty parameter R. Since, the values of Lagrange multipliers are not known in advance and iterative scheme is used to find the solution of the problem.

In the first iteration the values of Lagrange multiplier such chosen as 0. The value of the penalty parameter R is set equal to an arbitrary constrained and the function; that means, the Augmented Lagrangian function is minimized with respect to decision variables x to find the optimum value of the decision variables. The values of the Lagrange multiplier and the penalty parameter have then updated to start the next iteration.

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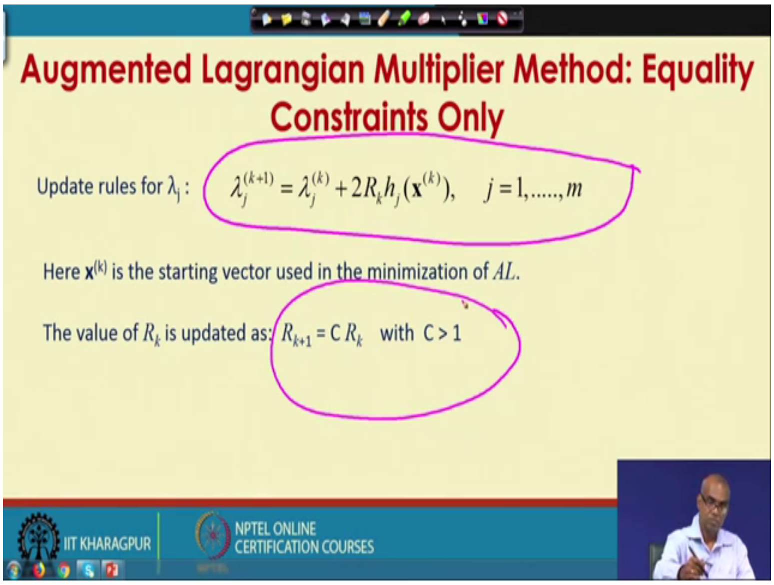

So, this is how we can update the Lagrange multipliers where k denotes the steps or the iterations. So, x k is the starting vector used in the minimization of Augmented Lagrangian function.

So, this expression gives me an update rule for the Lagrange multipliers lambda. Similarly, the penalty parameter R can be updated as R k plus 1 equal to C into R k. So, penalty parameter at k plus 1 step become a multiple of penalty parameter at k th step. So, C is that multiplier and C is greater than 1.

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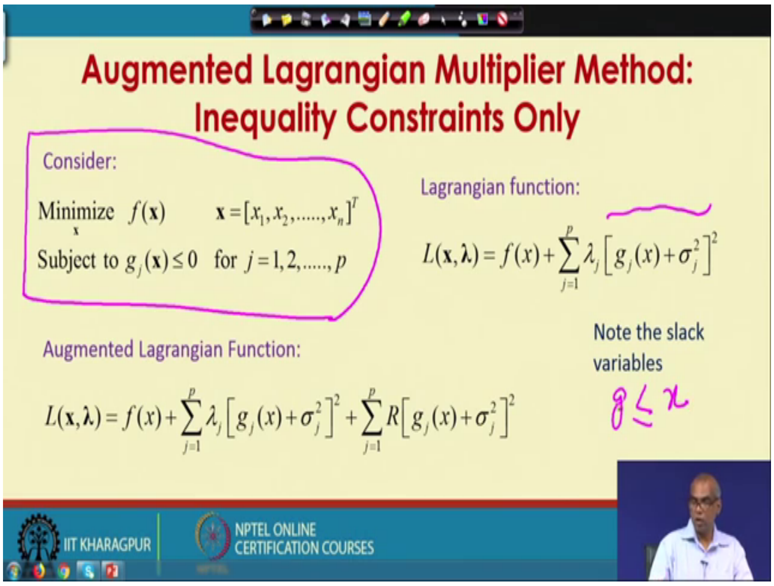

So, now, let us talk about the augmented Lagrangian multiplier method for inequality constraints. So; that means, we are now considering an optimization problem which has only inequality constraints.

So, I have to minimize the function f x which has x as n vector subject to p number of inequality constraints all inequality constraints are less or equal to time. We have seen how to convert a less or equal to type inequality constrained to an equality constrained by adding slack variables. So, we can convert a less or equal to type in equality constraint.

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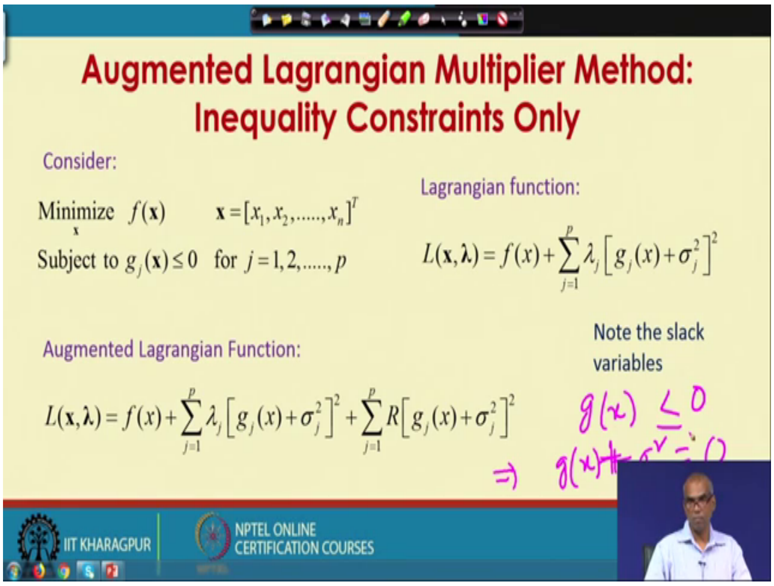

Such as g of x less or equal to 0 can be converted to equality constraint by adding a non negative slack variables sigma square. So, we can formulate the Lagrangian function.

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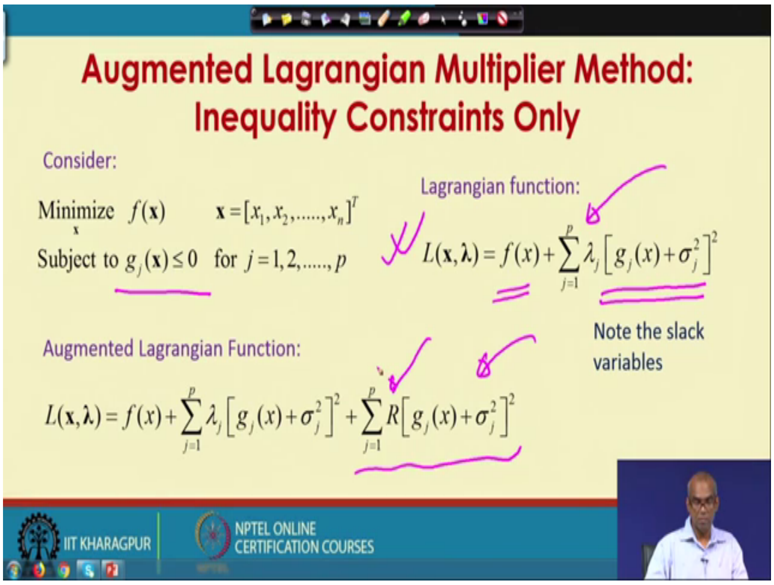

So, to formulate the Lagrangian function you first convert the inequality constraints which are all less or equal to type to equality constrained by adding a non negative slack variables sigma j square to each inequality constraints. And then for each converted equality constraints we define Lagrange multiplier and then take their product, sum them up and then add to the objective function f x. So, we have the Lagrangian function.

So, similarly, augmented Lagrangian function can also be now formed simply by adding the penalty term. So, we have now this converted equality constraints. So, you can make use of the parabolic penalty term.

So, you have the penalty parameter R and you have the parabolic penalty term.

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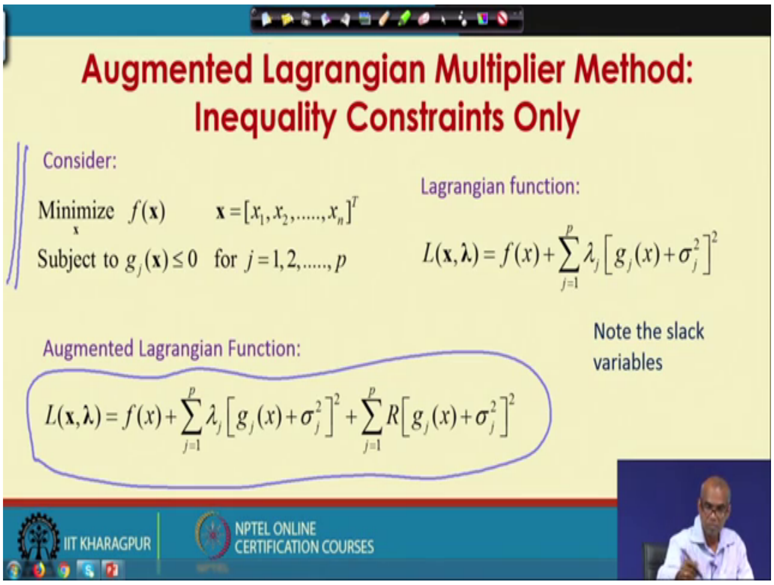

So, this gives me the augmented Lagrangian function. So, you have to minimize this augmented Lagrangian function to find out the minimum of the original constrained non-linear programming problem.

So, there are various unconstrained minimization problem and you can note that the augmented Lagrangian function is nothing, but an unconstrained function which can be minimized using a unconstrained optimization algorithm. And that will give me the solution of the original non-linear programming problem.

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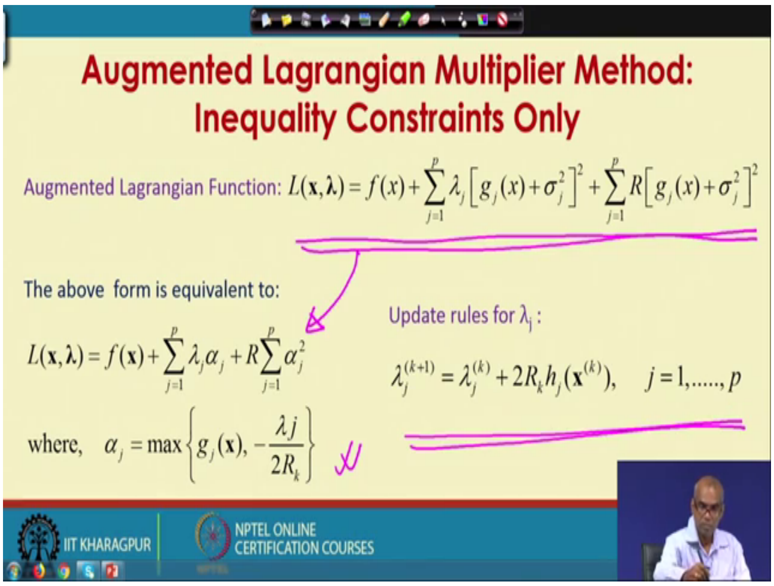

Now, this form for the augmented Lagrangian function is basically equivalent to a simpler looking form as shown, where we are defining a new term alpha which as the definition as shown. So, again you follow the similar update rules for the Lagrange multiplier at each steps.

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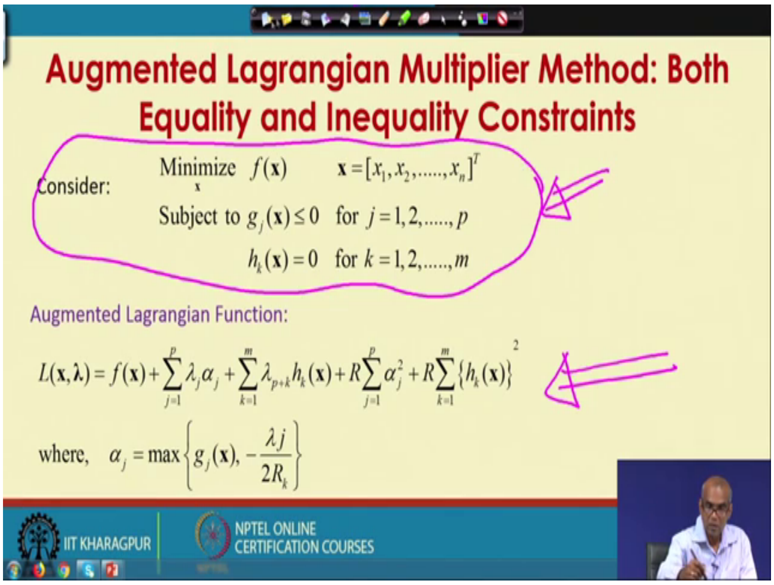

So, now if you combine this to strategies for handling inequality and equality constraints I will be able to handle any constrained non-linear programming problem with both equality and inequality constraints.

So, suppose, now we have this non-linear programming problem where I have to minimize an objective function f x which has n decision variables. There are p inequality constraints and there are m equality constraints. All the objective function and constraints maybe non-linear function of the decision variables.

So, first, by combining the Augmented Lagrangian function for inequality constraints and equality constraints I formulate the Augmented Lagrangian function for this general problem. And then this Augmented Lagrangian function has to be solved using an un constrained optimization problem, un constrained optimization algorithm to obtain the optimal solution here minimum point for this general non-linear programming problem.

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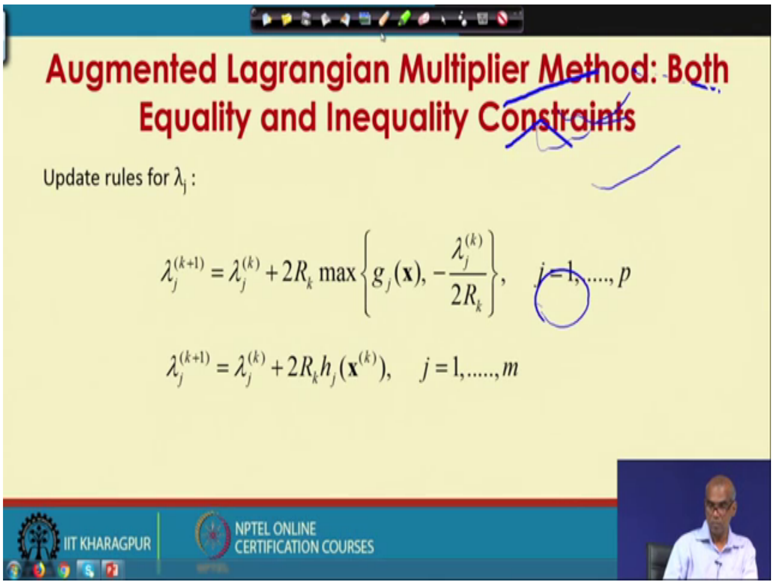

Here the update rules for the Lagrange multipliers are as shown. Note that there are set of Lagrange multipliers associated with the original equality constraints and there are another set of Lagrange multipliers which are associated with the inequality constraints. So, we have these update rules for the Lagrange multipliers as shown.

So, what we saw is the augmented Lagrangian function for problems with only equality constraints, problems with only inequality constraints and problems were both equality and inequality constraints are present. Once you can formulate the augmented Lagrangian function we can make use of any unconstrained optimization algorithm to find the minimum of the augmented Lagrangian function.

So, the minimum of the augmented Lagrangian function will give me the minimum of the original constraint problems. So, with this we will stop our lecture 49 here.

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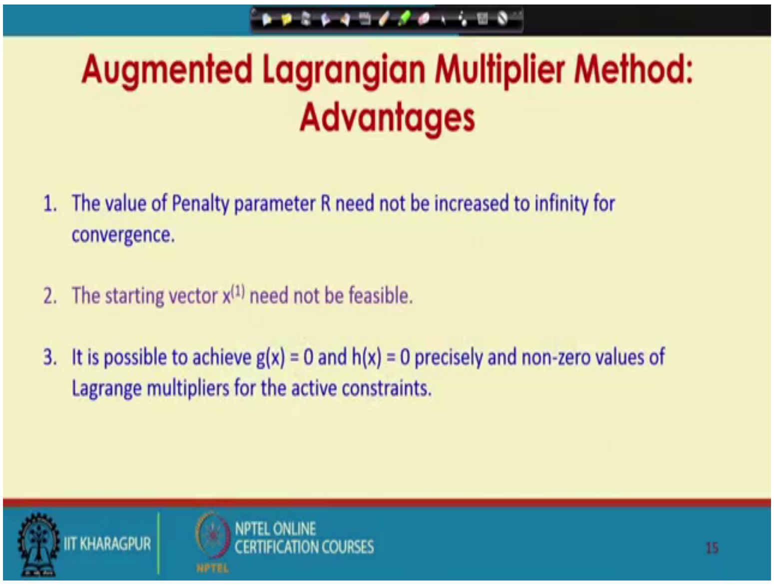

But, before that we will talk about some advantages for the augmented Lagrangian multiplier method. The value of the penalty parameter R need not be increased infinity for convergence. The starting vector need not be feasible, it is possible to achieve g x equal to 0 and h x equal to 0 precisely and non-zero values of the Lagrange multipliers for the active constraints.

So that means, the exact constrained satisfaction is possible. So, the main advantages are that the penalty parameter R need not be increased in to infinity for convergence, the starting point need not be feasible and exact constrained satisfaction is possible. So, with this we would like to stop lecture 49 here.