Optimization in Chemical Engineering
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Lecture – 48
Constrained Nonlinear Programming (Contd.)

Welcome to lecture 48. This is week 10 and you are talking about Non-linear Constrained Programming problems; so, Constrained Non-linear programming problems. In our previous two lectures we have talked about successive linear programming method and successive quadratic programming approximation method.

So, in those methods either we used a linear approximation of the problem general non-linear problem or we made use of quadratic approximation of the general non-linear programming problem; I am solved it.

We have seen that both linear programming problem and quadratic programming problem can be solved efficiently and the reliably. So, in case of successive linear programming problem and successive quadratic approximation problem the non-linear programming problem could have been solved efficiently.

So, in today's lecture and the following lecture we will see other class of methods known as transformation methods. So, today we will see penalty function method.

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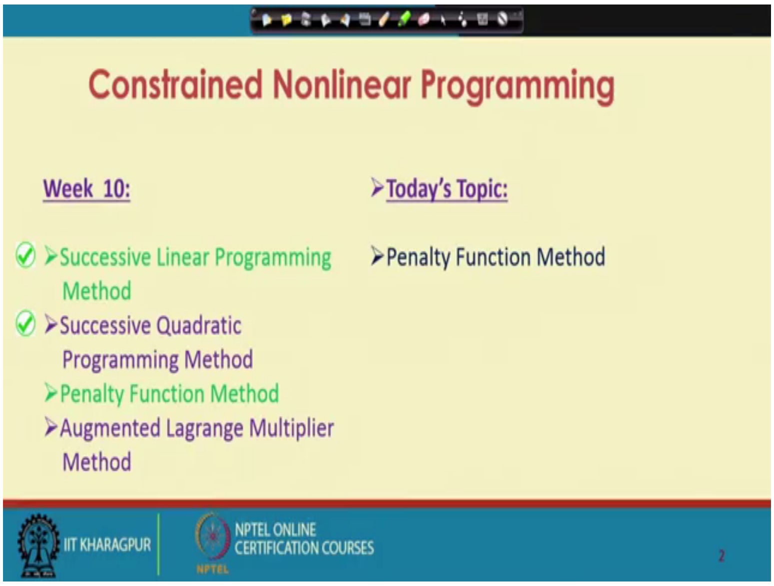

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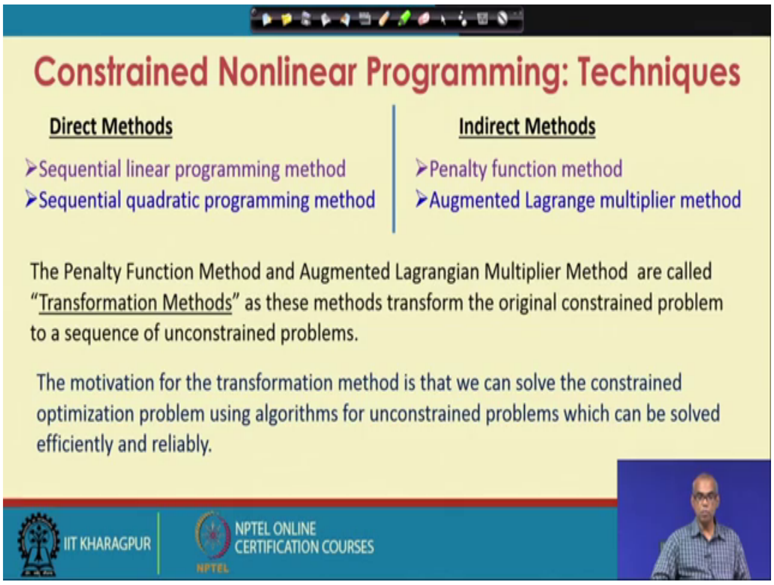

So, we will see penalty function method which is a transformation method. And in the following lecture we will learn about augmented Lagrange multiplier method which is another transformation method. These transformation methods convert the original constrained problem to a sequence of unconstrained problems. So, as the name suggests the transformation problem we will transform the original constrained problem to a sequence of unconstrained problem. And we have already seen that the unconstrained problems can also be solved very efficiently and reliably. The motivation for the transformation method is that we can solve the constrained optimization problem using algorithms for unconstrained problems which can be solved efficiently and reliably.

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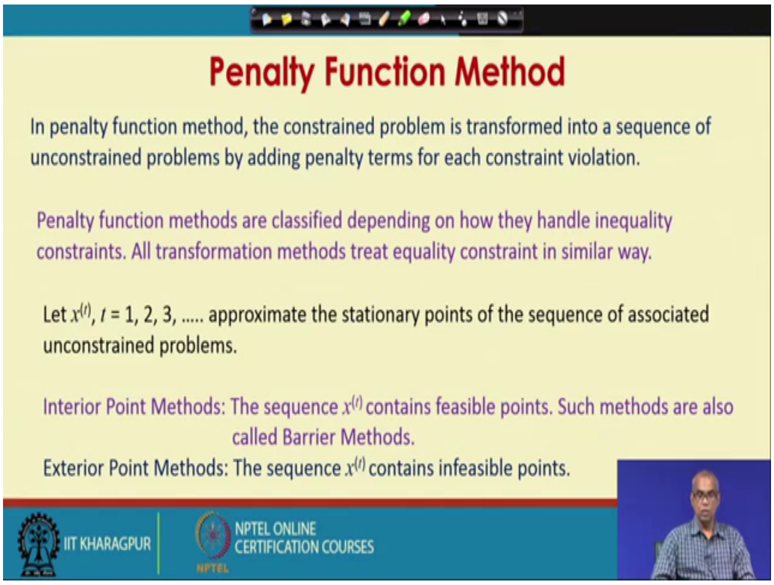

In penalty function method the constrained problem is transformed into a sequence of unconstrained problems by adding penalty terms for each constraint violation.

So, we at penalty terms for each constrained violation in case of penalty method. And that way we convert or transform a general non-linear programming problem into a sequence of unconstrained problems. Penalty function methods are classified depending on how they handle inequality constraints. All transformation methods treat equality constrained more or less in a similar way.

So, the way equality constraints are handled is similar in almost all methods that come under transformation methods or penalty methods. The penalty function methods are classified depending on how they handle inequality constraints, because the handling of inequality constraints are different for different penalty function method, but for equality constraints it is more or less similar.

Let us consider x t; t equal to 1 2 3 4 etcetera approximate. The stationary points of the sequence of associated unconstrained problems. We stated that a general non-linear programming problem will be converted to a sequence of unconstrained problem by use of penalty function approach.

So, we consider x 1 x 2 x 3 x 4 are the stationary points for the sequence of these unconstrained problem. Now if this sequence contains feasible points we call this method as interior point methods. Such methods are also known as barrier methods. If the sequence contains in feasible points we call the penalty function method as exterior point methods. So, there are broadly two types of methods interior point methods and exterior point methods. In case of interior point methods the stationary points of the sequence of the unconstrained problems will be feasible points. We also call this methods as barrier methods. In case of exterior methods the stationary points for the sequence of unconstrained problems will contain infeasible points.

So, interior points always contains interior point methods always contains feasible points as the stationary points for the unconstrained problems, but exterior point methods will contain infeasible points as stationary points for the associated unconstrained problems.

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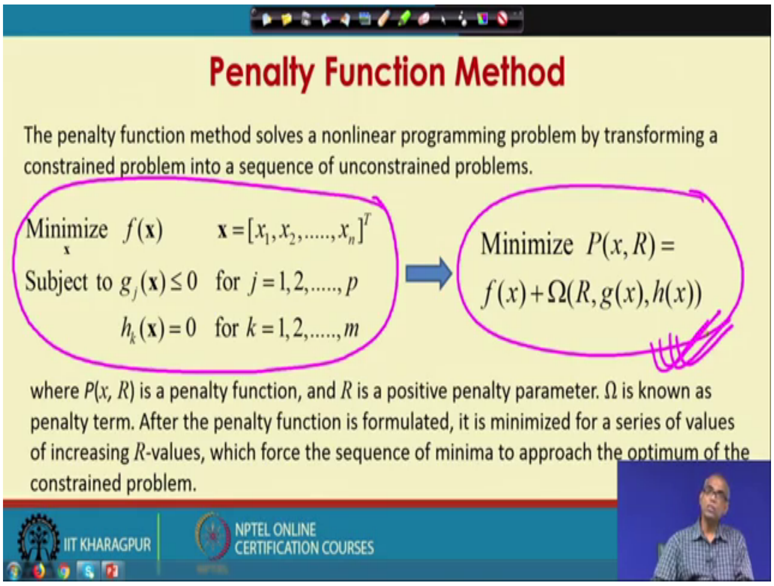

As you stated the penalty function method solves a non-linear programming problem by transforming a constrained problem into a sequence of unconstrained problem. Let us consider a general non-linear programming problem. So, you have minimization of function f x where x is n vector and subject to P number of inequality constant g x less or equal to 0 type and m number of equality constraints hk x equal to 0 k equal to 1 to m.

Now this non-linear programming problem can be converted to an unconstrained problem by adding penalty. So, what do we do is the penalty function is defined as the objective function plus a penalty term; this is known as penalty function. So, penalty function is nothing, but the objective function of the original non-linear programming problem plus a penalty term.

So, the penalty term is a function of a penalty parameter R and all the constraints that the original non-linear programming problem will have. So, the penalty term is a function of a penalty term R inequality constrained g x equality constrained h x. So, penalty function becomes a function of the decision variables and the penalty parameter, because the objective function the inequality constraints as well as equality constraints are all functions of decision variables x.

So, the penalty function can be formulated as sum of objective function plus one penalty term. The penalty term is a function of the constraints as well as a penalty parameter. Note that the penalty function is now an unconstrained problem.

So, after the penalty function is formulated the unconstrained penalty function can be minimized for a series of values of the penalty term R. So, we can take various values of R and can solve a sequence of unconstrained problems. And you will get the sequence of minima to approach the minima of the original constraint problem.

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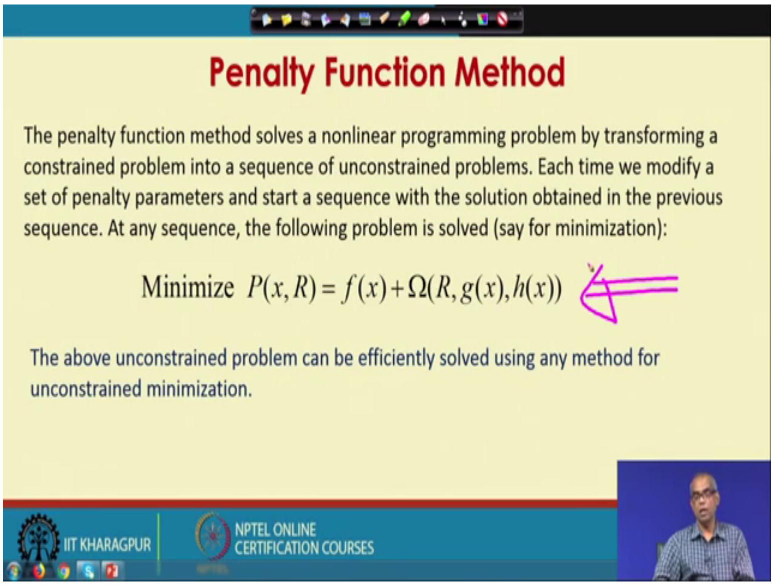

So, after you formulate this penalty function for the original non-linear programming problem. You solve this penalty function starting with one value of penalty term R. So, we minimize this unconstrained function using any standard numerical methods for minimization. Some of those methods we have discussed already.

Now, you again change the value of R and solve the next sequence of unconstrained problem. If we go on doing like this we will see the sequence of minima that we get we will approach to the optimum of the original constrained non-linear programming problem. The penalty function method solves a non-linear programming problem by transforming a constrained problem into sequence of unconstrained problems. Each time we modify a set of penalty parameters and start a sequence with the solution obtained in the previous sequence.

At any sequence we solve this unconstrained problem. Let us say we are solving for a minimization problem. This unconstrained minimization problem can be efficiently solved using any method for unconstrained minimization. Normally for practical problems analytical solution may not be applicable and you must make use of the numerical methods for solutions of this unconstrained problem. And there are host of powerful unconstrained minimization techniques which can solve this problem very efficiently and reliably.

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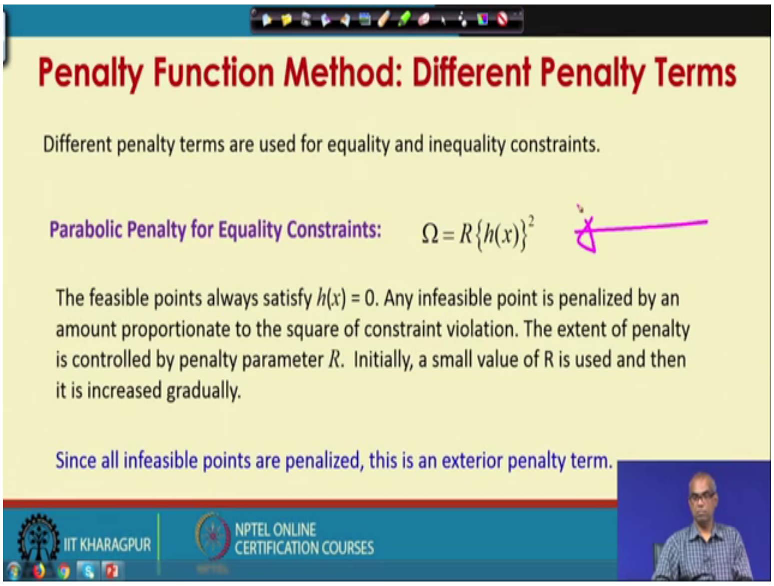

Now, let us talk about different types of penalty terms; different penalty terms are used for equality and inequality constraints. For equality constraints the penalty term that we use is known as Parabolic penalty.

So, the parabolic penalty term is nothing, but the penalty parameter R into square of h; h is the equality constraint. So, the parabolic penalty term for equality constraint is R into h of x square. The feasible points always satisfy h x equal to 0 any in feasible point is penalized by an amount proportionate to the square of constraint violation. Note that any h x that is not equal to 0 is a violation of the constrained.

So, any finish any infeasible point is penalized by an amount that is proportional to the square of constraintviolation. The extent of penalty is controlled by the penalty parameter R by changing different values of R you can control the extent of the penalty. Initially a small value of the penalty parameter R is used and then it is increased gradually as iterations proceeds. Since all infeasible points are penalized by parabolic penalty for equality constraints. This penalty is an exterior penalty term.

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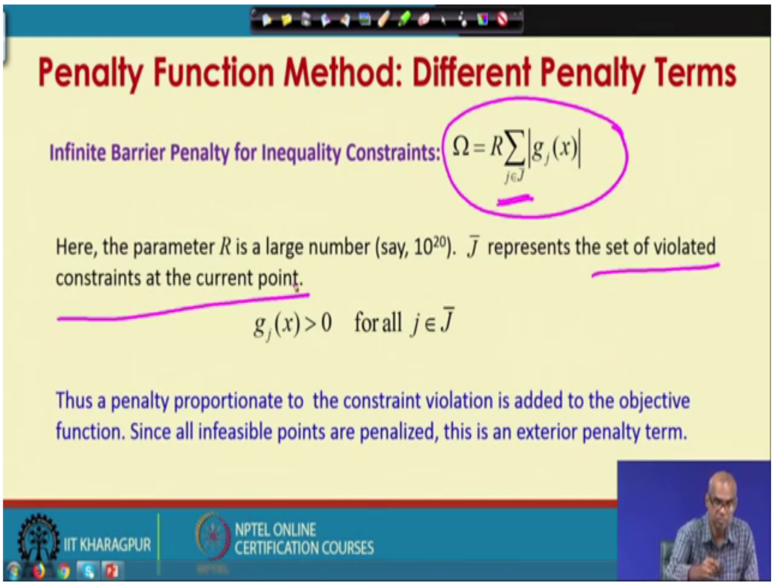

Next we talk about infinite barrier penalty; infinite barrier penalty is for inequality constraints. And this is defined as R into absolute value of the inequality constrained. If you have more than one inequality constrained you have to sum them up, but you will take or you will consider only for the violated constraints.

So, note that the infinite barrier penalty term is defined as R into sigma g j of x where j belongs to J bar which represents the set of violated constraints at the current point.

So, basically you adapt the constraint violations. Here the penalty parameter R is a very large numbers say 10 to the power 20 or. So, thus a penalty proportionate to the constraint violation is added to the objective function.

Since all in feasible points are penalized this is also an exterior penalty term.

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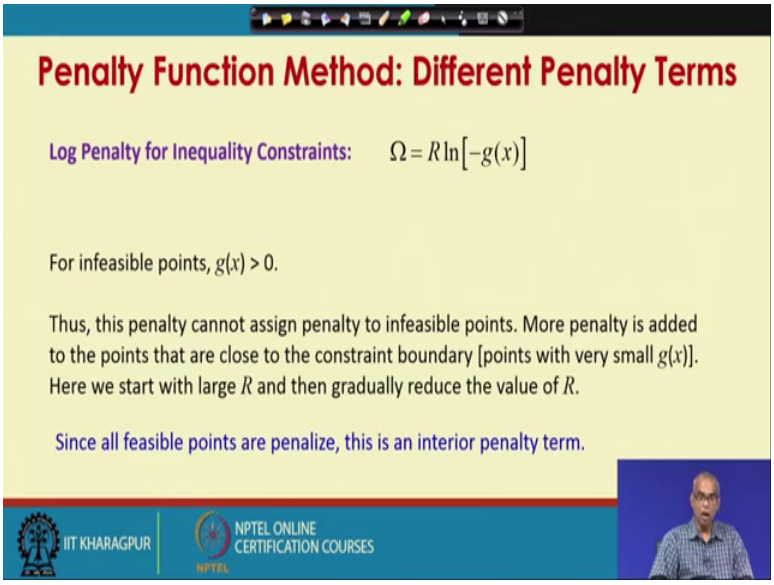

Next we will discuss is log penalty for inequality constraints. Log penalty for inequality constraint is defined as R ln minus g of x. note that for in feasible points g of x is greater than 0. Thus this penalty cannot assign a penalty to infeasible points more penalties added to the points that are close to the constrained boundary that means, points with very small g x. Here we start with large values of R and then gradually reduce the value of R. Since all feasible points are penalized this is an interior penalty term.

Note that log penalty for inequality constraints is an interior penalty term because here all feasible points are penalized.

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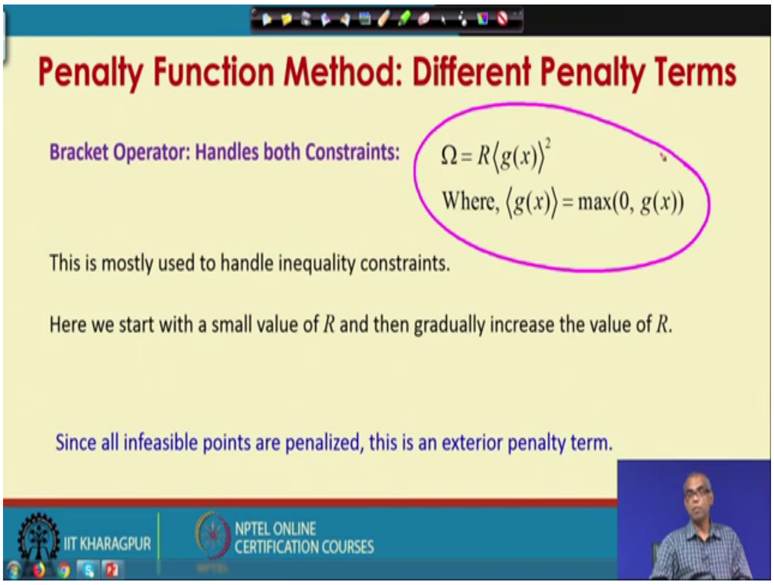

Bracket operators handles both constraints which is defined as max 0 g x. So, the maximum value of 0 or g x the vacant operator is mostly used to handle inequality constraints. Here we start with a small value of R and then gradually increase the value of R. Since all infeasible points are penalized this is also an exterior penalty term. This penalty term as defined max of 0 or g x is a very useful penalty term is mostly used to handle inequality constraints.

Here also you start with a small value of R and then gradually increase the value of R as iteration proceeds.

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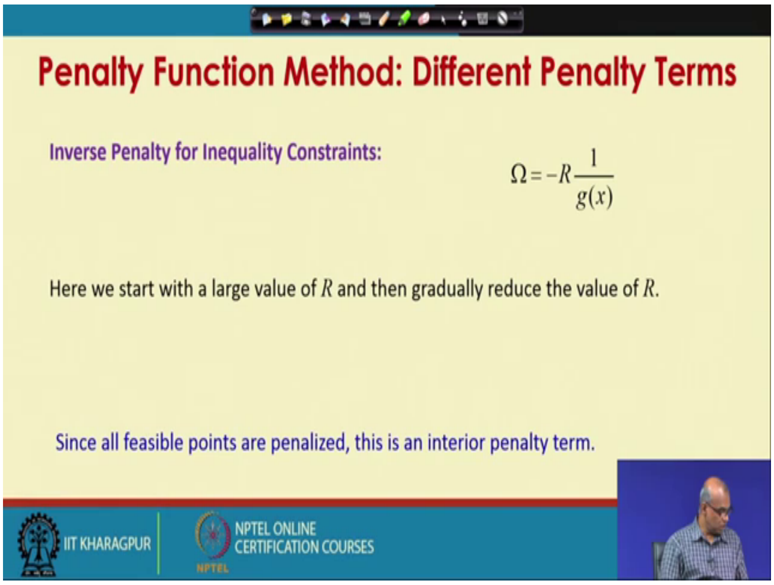

Inverse penalty for inequality constraints; which is defined as minus R by g of x. Here we start with a large value of R and then gradually reduce the value of R. Since all feasible points are penalized by this parameter by this penalty term this is known as interior penalty term.

So, inverse penalty for inequality constraints is an interior penalty term. So, these are the various penalty terms that you can use to form the penalty function. So, by adding these penalty terms for inequality and equality constraints you can formulate the penalty function. So, the penalty function will be formulated by adding these penalty terms to the objective function of the original non-linear programming problem.

So, that way we will give that way you will get an unconstrained function which can be solved for a particular value of penalty parameter R. Then you can solve Duncannon problem again by taking different value of R and by doing so, you will slowly approach the optimum the original non-linear programming problem. You have seen that for exterior point methods we start with a small value of R and then gradually increase the value of the R, but in case of interior point method we start with a larger value of R and then slowly reduce with iterations.

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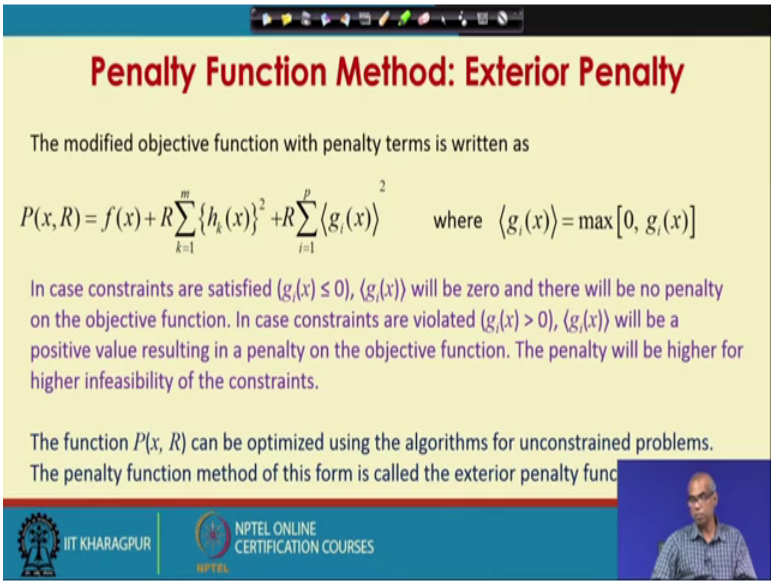

Let us see now how this exterior penalty works.

So, the penalty function will be formulated as shown. Note that I have added the parabolic penalty for the equality constraints and the max operator or the bracket operator for the inequality constraints. In case constraints are satisfied the g of x less or equal to 0 means the constraints are satisfied. So, we are solving a minimization problem with inequality constrained g of x less or equal to 0.

So, in case constraints are satisfied the contribution of this bracket operator will be 0 and there will be no penalty on the objective function in case constraints are violated that means, g x greater than 0 the bracket operator will be a positive value resulting in a penalty on the objective function. The penalty will be higher for higher in feasibility of the constraints that means, the penalty will be higher when the constraint violation is higher.

The penalty function P can be optimized using the algorithm for unconstrained problems. The penalty function method of this form is called the Exterior Penalty function method.

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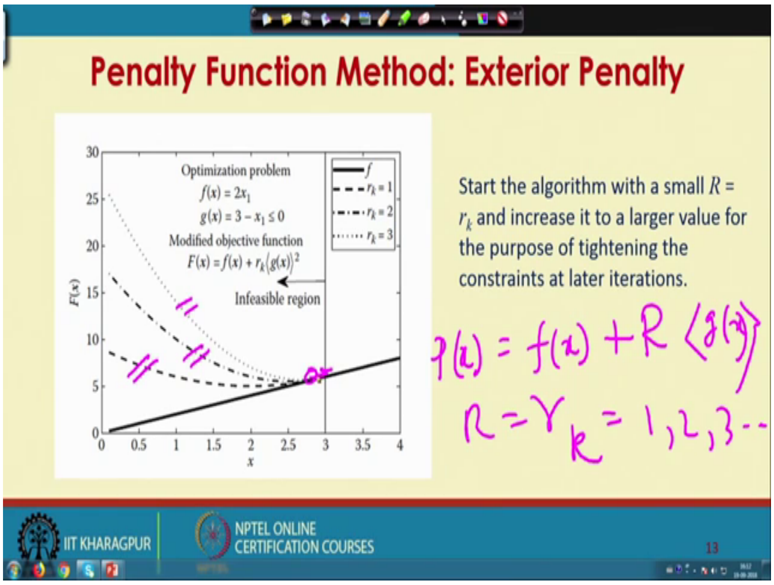

Now, here look at the graph the optimization problem that is being solved is f x equal to 2 x 1 subject to 3 minus x 1 less or equal to 0. So, the penalty function P x is the objective function plus the penalty term. So, for the penalty term let me use the bracket operator R g of x. So, let us take different values of these R equal to r k equal to 1 2 3 etcetera.

So, these are the sequence when you take r k or the penalty term value r equal to 1 2 and 3. And you see how you approach the optimum point for the original problem. So, this part is infeasible this is the feasible part and this is the infeasible region.

So, as we approach the optimum you always remain in the in feasible region we approach the optimum from within the infeasible region. So, you start the algorithm with a small penalty parameter R and increase it to a larger value for the purpose of tightening the constraints at later iterations.

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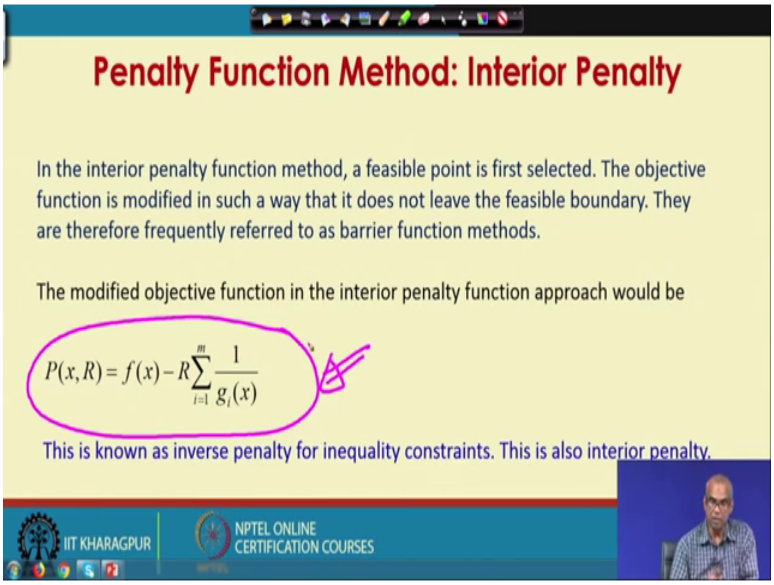

Now, let us look at the interior penalty. In the interior penalty function method a feasible point is first selected. The objective function is modified in such a way that it does not leave the feasible boundary they are therefore, frequently referred to as barrier function methods.

So, in case of interior penalty function method of feasible point is first selected the objective function is modified in such a way that it does not leave the feasible boundary it always remains within the feasible region. These methods are therefore, frequently referred to as barrier function methods.

So, the modified objective function in case of interior penalty function methods can be as shown. Note that I have used an inverse penalty term minus R by g by minus R by g x. So, if you have m number of constraints you have to add up 1 by g i x summed over all i equal to 1 to m.

So, this way you can formulate the penalty function method while using interior penalty term. Particularly here we have used inverse penalty term. Note that this is an interior penalty.

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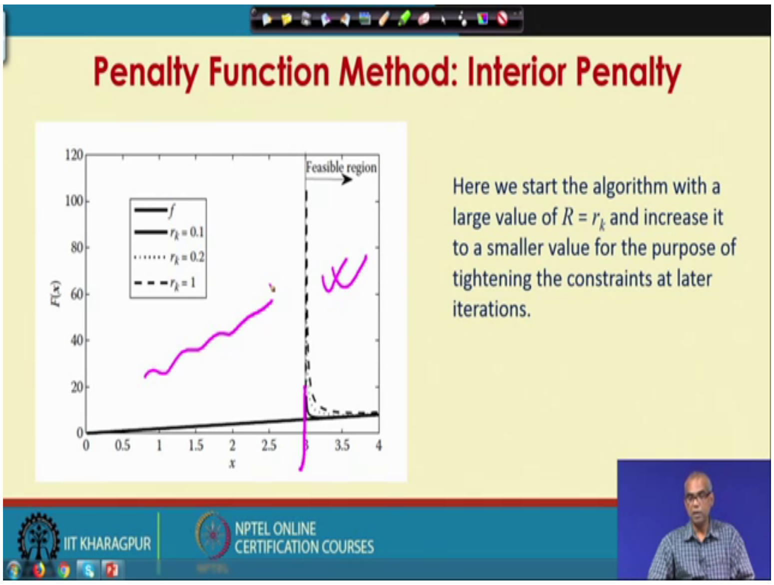

So, if you pictorially present this is the feasible region so, you start with a higher value of the penalty term R and then go on decreasing the value with later iterations. So, you see we always remain within the feasible region we do not leave the feasible region and approach the solutions from within the feasible region.

So, interior penalty function method approach the optimum while remaining within this feasible region method we started with higher value of penalty term and slowly reduce the value. In case of exterior penalty function method we approach the optimum while always remaining within the invisible region.

So, we approach the optimum from the in feasible region. And in case of exterior function method exterior penalty method we started with a smaller value of the penalty term R and increase the value of the penalty term in later iterations.

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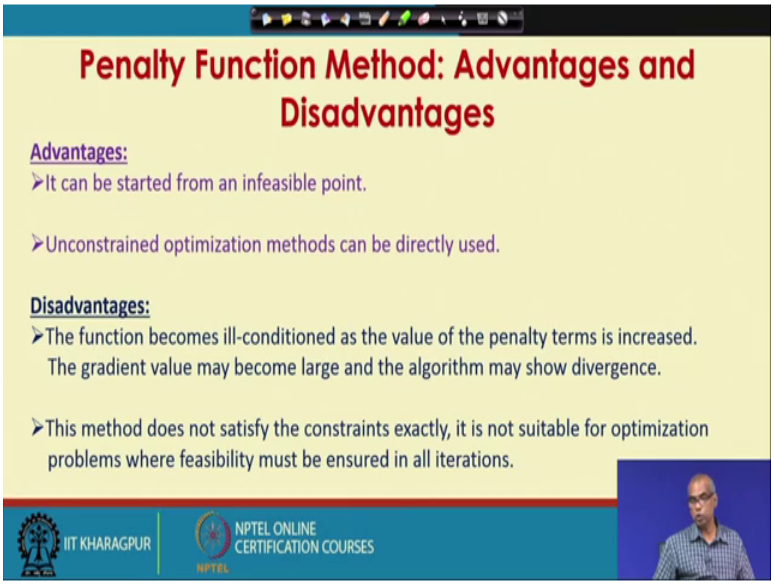

What are the advantages and disadvantages of penalty function method? It can be started from an infeasible point for exterior penalty function method. The unconstrained optimization methods can be directly used. As I told you there a host of powerful robust unconstrained optimization methods which can be used directly to solve the non-linear programming problem after you have convert it into an unconstrained problem using penalty function method.

So, conceptually the penalty function approach is extremely simple in terms of implementation it is also simple and since the unconstrained optimization techniques can be used it has an advantage. There are some disadvantages also the function becomes yield condition as the value of the penalty term is increased the gradient value may become large and the algorithm may show divergence. And other disadvantage is that this method does not satisfy the constraints exactly. It is not suitable for optimization problems were feasibility must be ensured in all iterations.

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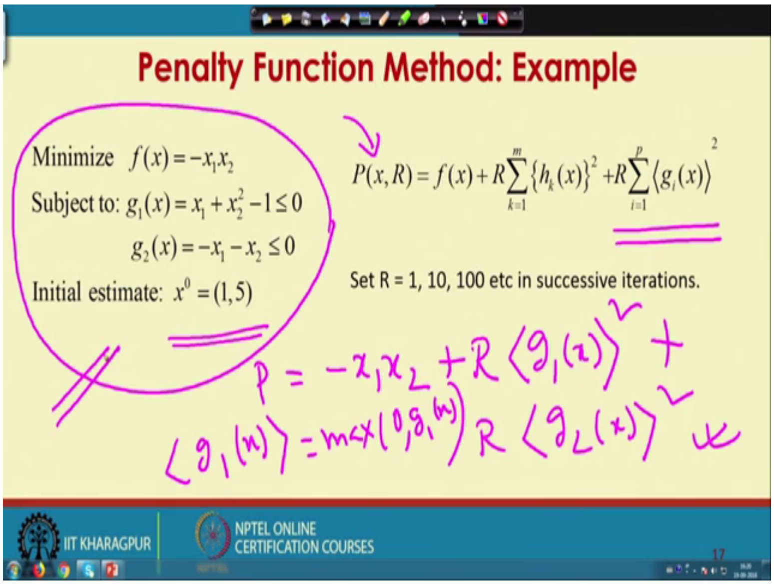

Let us consider this non-linear programming problem. You have to minimize the function f x equal to minus x 1 x 2 subject to 2 in quality constraints initial estimate is x 1 equal to 1 and x 2 equal to 5. So, how do you go about it see formulate the penalty function.

So, penalty function will be minus x 1 x 2 as the objective function plus these terms that means, R into g 1 x whole square plus R into g 2 x whole square and you know g 1 x is nothing, but max of 0 or g 1 x similarly for g 2. So, this is an unconstrained function.

Now, you set R equal to 1 and solve it. So, you get so, x 1 and x 2 as solution for that unconstrained problem next you take R equal to 10 again solve you get solution for as x 1 and x 2. You again consider R equal to 100 and again solve using numerical methods. And then you see that you are slowly approaching the optimum solution for this original constrained non-linear programming problem. With this we stop lecture 48 here.