Optimization in Chemical Engineering
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Lecture – 45
Linear Programming: The Simplex Method (Contd.)

Welcome to lecture 45. So, this is the last lecture of week 9. In this week, we have talked about Simplex Method. As of now, we have started with the assumption that an initial basic feasible solution is readily available. So, today we learn what happens if an initial basic feasible solution is not readily available. So, how do I start the simplex procedures, when readily an initial basic feasible solution is not obtain.

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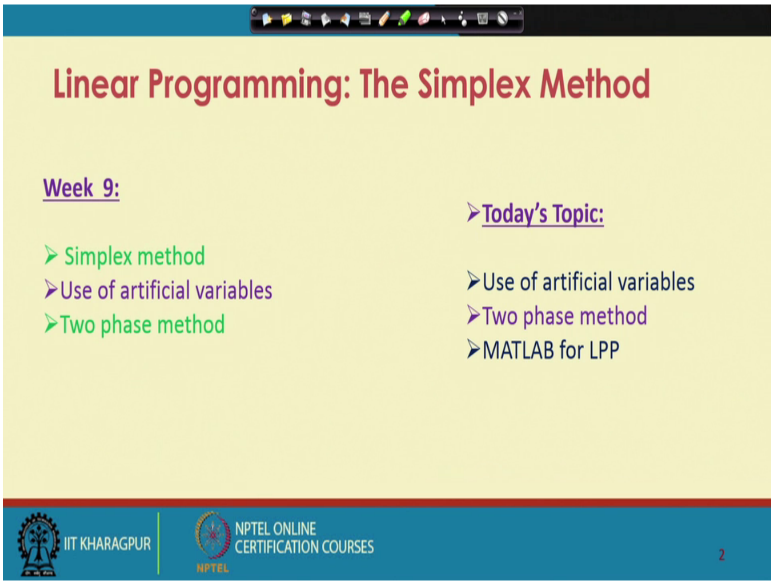

So, in this context, we will talk about interruption of artificial variables. So, we will talk about use of artificial variables. We will talk about two phase method. And finally, we will conclude with a brief discussion on MATLAB for linear programming problems.

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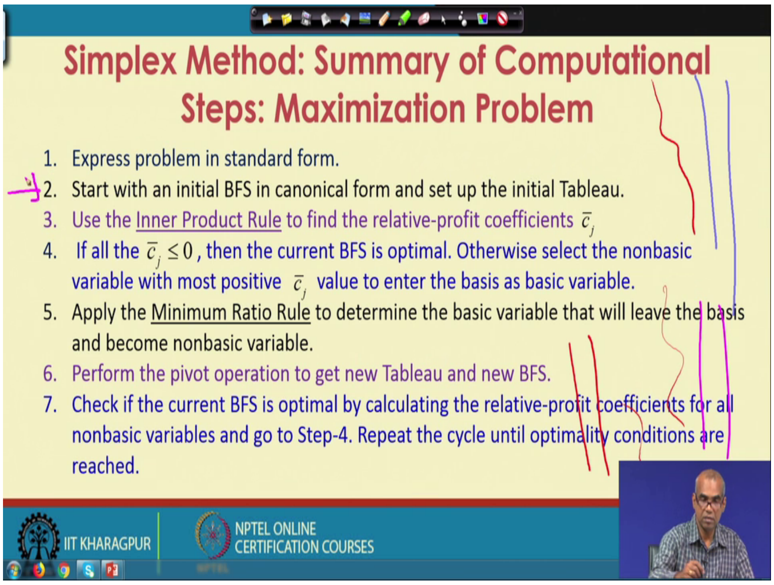

So, let us review our steps for linear programming problem solution using simplex method. First express the problem in standard form. We start with an initial basic feasible solution in canonical form and set up the initial Tableau. Use the inner product rule to find the relative-profit coefficients for all non-basic variables, for basic variables this value is 0. If all relative-profit coefficients are less or equal to 0, then the current basic feasible solution is optimal. Otherwise, select the non-basic variable with most positive relative-profit coefficient and that will enter the basis as basic variable.

Next, we apply the minimum ratio rule to determine the basic variable that we leave the basis and become non-basic variable. Now, perform the pivot operations to get new Tableau and new basic feasible solution. We check if the current basic feasible solution is optimal by calculating the relative-profit coefficients for all non-basic variables, and this cycle is repeated until optimality conditions are reached.

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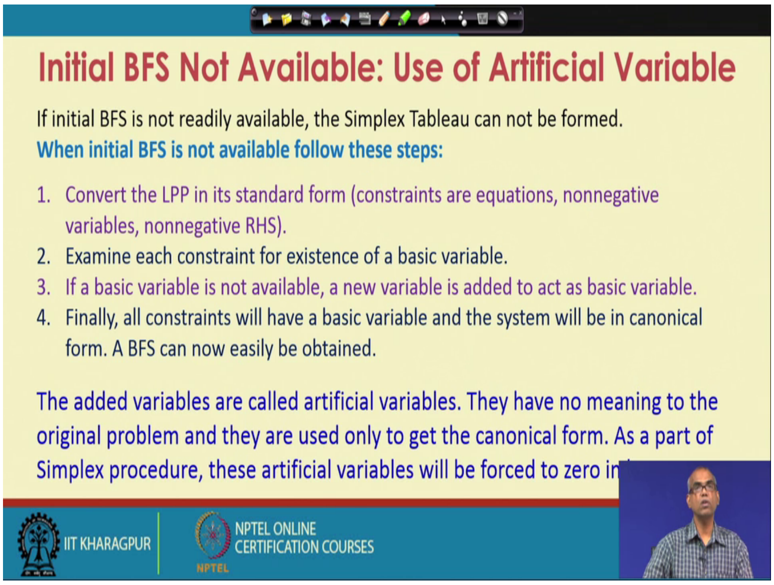

So, if you look at the second step, we say that start with an initial basic feasible solution in canonical form and set up the initial Tableau. But, if an initial basic feasible solution is not readily available, the simplex Tableau cannot be form. So, you cannot start with simplex procedures until we can find a basic feasible solution. So, when initial basic feasible solution is not available, we have to follow the following steps.

Convert the linear programming problem in its standard form that means constraints are all equations, non-negative variables, and non-negative right hand side constants. Examine each constraint for existence of a basic variable. If a basic variable is not available, a new variable is added to act as basic variable. Finally, all constraints will have a basic variable and the system will be in canonical form. A basic feasible solution can now easily be obtained.

The added variables are called artificial variables. They have no meaning to the original problem and they are used only to get the canonical form. As a part of simplex procedure, these artificial variables will be forced to zero in later steps. So, what will you do is, if initial basic feasible solution is not readily available, we will introduce artificial variables to each constant, which does not contain a basic variable. After that we will see that we can find the initial basic feasible solution to start with. And since these variables so called artificial variables have no meaning to the original problem. This should be forced to 0 in the later stages of the simplex procedure.

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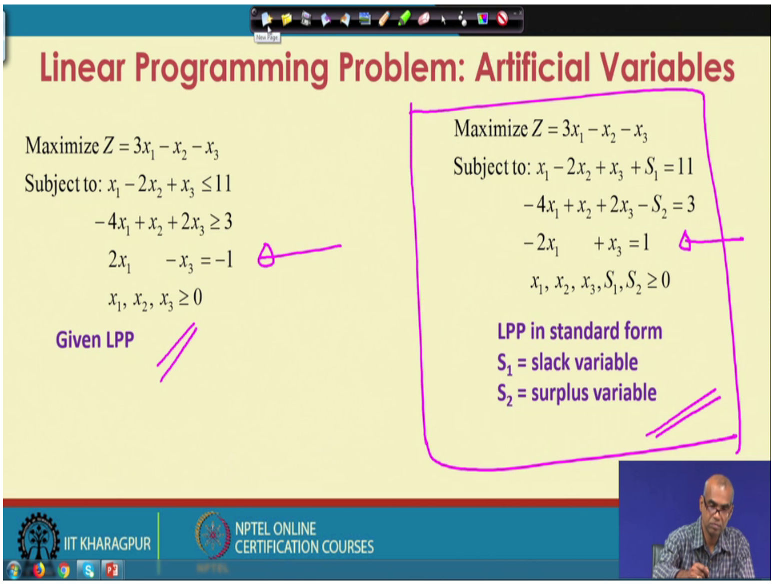

Let us consider an example. So, we are maximize an objective function subject to 3 constants. Now, note the first constant is less or equal to type, second constraint is greater or equal to type, and the third constraint is an equation equality type. We have already stated that if you have a linear programming problem with all constraints are of inequality type and less or equality type, and the right hand side constraints are positive, then we will always get an initial basic feasible solution, because the introduction of the slack variables will give you an initial basic feasible solution. In this example, shown we have a lesser equal to constant, a greater or equal to constant. And for one constraint the right hand side is negative, and that one is equality type constraint.

So, first let us introduce the slack variables, surplus variables. The first constant is an less or equal to type, so it needs a slack variable S 1. The second constraint is greater or equal to type, so it requires S 2, a surplus variable. So, after introduction of S 1 and S 2, my linear programming problem is in standard form. Note that the 3rd equation does not need any slack or surplus variable that is already an equation.

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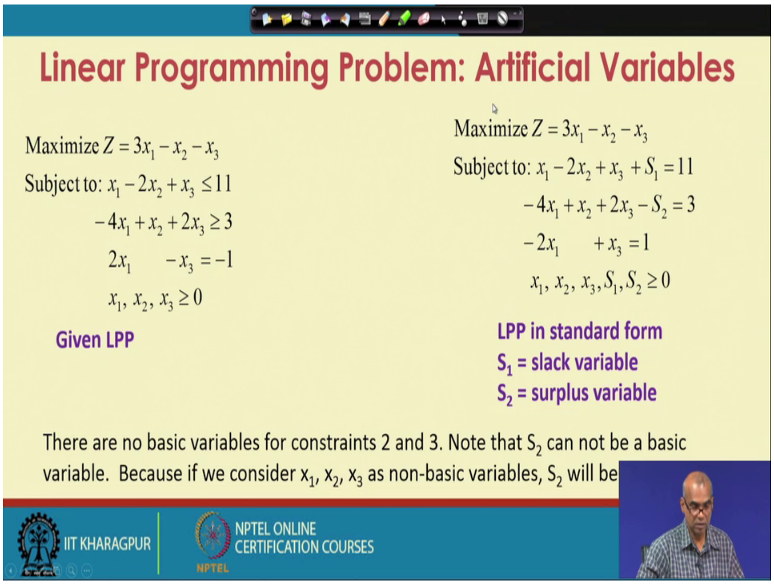

But, since the right hand side was negative, we have multiplied throughout by minus 1 to make the right hand side positive. So, this problem now is in standard form, but we do not have a basic feasible solution to start with. So, what do you do, we add artificial variables. Note that constraint 2 and 3 does not have any basic variables. In the constraint 2, we cannot consider S 2 as a basic variable. Because, if we consider x 1, x 2, x 3 as non-basic variable, minus S 2 will be equal to 3 that means S 2 will be equal to minus 3, so that will violate the constraint on non-negativity on S 2. So, you do not have a basic feasible solution. So, we have to introduce artificial variables.

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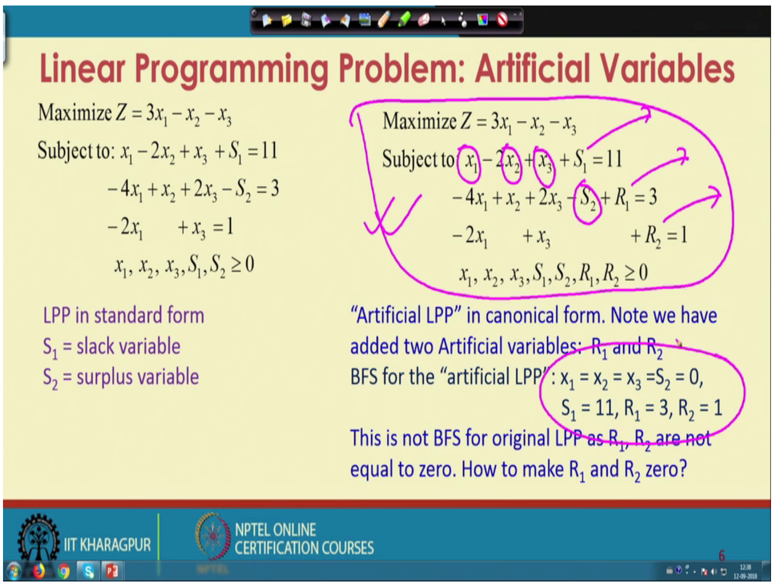

So, let us introduce two artificial variables R 1 and R 2. We introduce R 1 to constraint number 2; we introduce R 2 to constraint number 3. And now the set of constraints that we get for the given linear programming problem, basically gives us an artificial linear programming problem and this one is in canonical form. Artificial, because R 1 and R 2 has no meaning to the problem, but introduction of R 1 and R 2 helps me to find a basic feasible solution.

Now, look let us look at this artificial linear programming problem. Let us consider x 1, x 2, x 3 as non-basic variable as well as x 2. So, I consider x 1, x 2, x 3 as well as S 2 as non-basic variables, and set their values to 0. So, S 1, R 1, and R 2 are my basic variables. And the solution S 1 equal to 11, R 1 equal to 3, and R 2 equal to 1 so, we get a basic feasible solution for the artificial problem, but for the original problem R 1 and R 2 must be 0. So, this basic feasible solution is only for the artificial linear programming problem. This basic feasible solution is not a basic feasible solution for the original linear programming problem, because in the original linear programming problem R 1 and R 2 must be 0. So, how do I make R 1 and R 2, 0?

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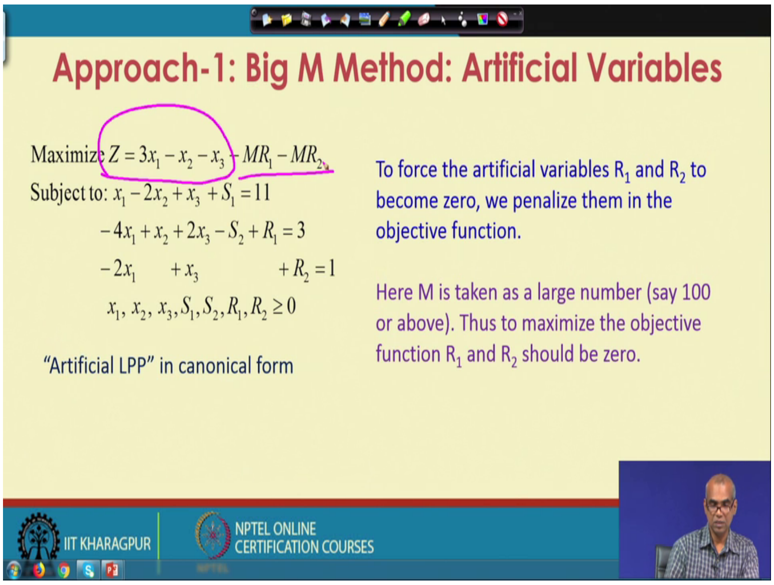

There are two approaches to handle these artificial variables R 1 and R 2. The first one, we talk about is known as Big M method. So, to force the artificial variables and R 1 and R 2 to become 0, we penalize them in the objective function. So, look at the formulation of the objective function. This was the original objective function Z equal to 3 x 1 minus x 2 minus x 3. Now, I have included minus MR 1 minus MR 2, I am solving a maximization problem. So, values of R 1 equal to 0, and R 2 equal to 0, given M a positive large number will maximize the objective function.

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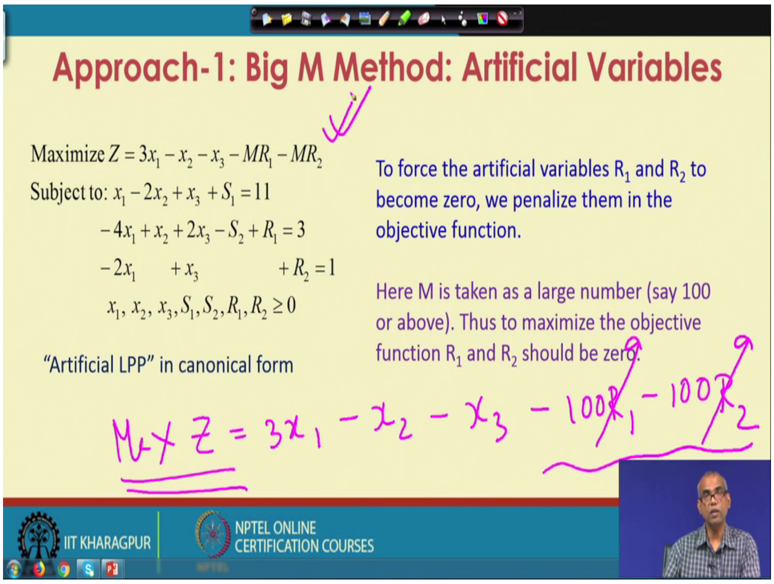

Note that the value of M is usually taken a large number say 100 or above. Then the objective function can be written as Z equal to 3 x 1 minus x 2 minus x 3 minus 100 R 1 minus 100 R 2. Let us say for example, we have taken M equal to 100. Now, if I want to maximize Z, since these artificial variables times 100 are being subtracted from Z to maximize Z, we must say R 1 and R 2 to 0. So, if I solve a linear programming problem with this objective function, I expect that at the final solution I will get R 1 and R 2 equal to 0. And then that will be a solution to the original linear programming problem.

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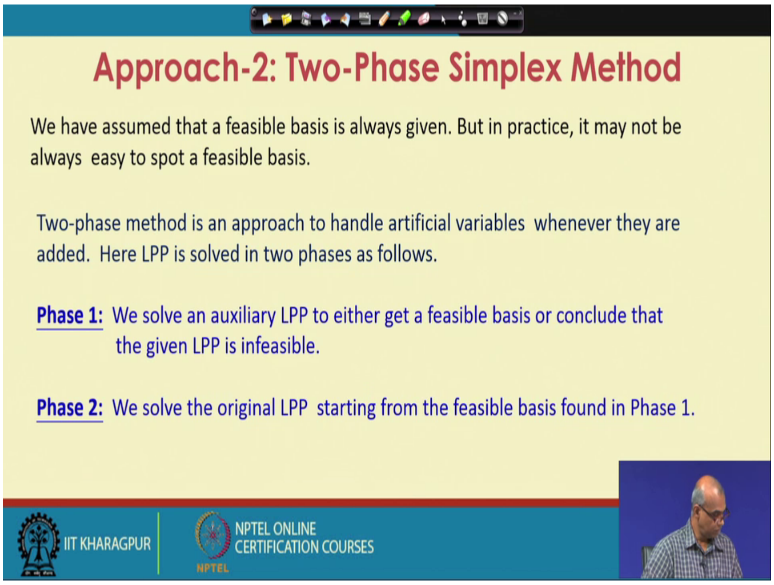

Approach-2 is known as two-phase simplest method. Two-phase simplex method is an approach to handle artificial variables, whenever they are added. In this case, the linear programming problem is solved in two phases. Note that we have assumed as of now that a feasible basis is always given. But, in practice, it may not always be easy to spot a feasible solution.

So, introduction of artificial variables will be necessary. Once we introduce the artificial variables, these variables must be force to 0 during the steps of simplex method, so that you solve the original linear programming problem, where the artificial variables has no meaning. So, you have seen the Big M method, where you penalize these artificial variables in the objective function. In the two- phase method, these artificial variables are handled in two phases.

In the first phase, we solve an auxiliary linear programming problem to either get a feasible basis or conclude that the given linear programming problem is infeasible. So, the phase one is solved either to get a feasible basis or we conclude that the given linear programming problem is infeasible. In this case, we solve an auxiliary linear programming problem. We do not solve the original linear programming problem, but we solve a linear programming problem related to the original programming problem, we call that an auxiliary linear programming problem. And the solution to this auxiliary linear programming problem will give me a feasible basis for the original linear programming problem. And if I do not get a solution for the auxiliary linear programming problem, we conclude that the original linear programming problem is infeasible.

In the phase 2, we solve the original linear programming problem starting from the feasible basis found in phase 1. So, in the phase 1, we solve an auxiliary linear programming problem related to the original linear programming problem to get a feasible basis, so that we can start phase 2. In the phase 1, if we do not get a solution to the auxiliary linear programming problem, we conclude that the given linear programming problem, original linear programming problem is infeasible. And we stop our algorithm here, phase 2 will not be necessary then. But, if we get a feasible basis in the phase 1, we start phase 2 with this feasible basis and continue with the steps that we have demonstrated so far.

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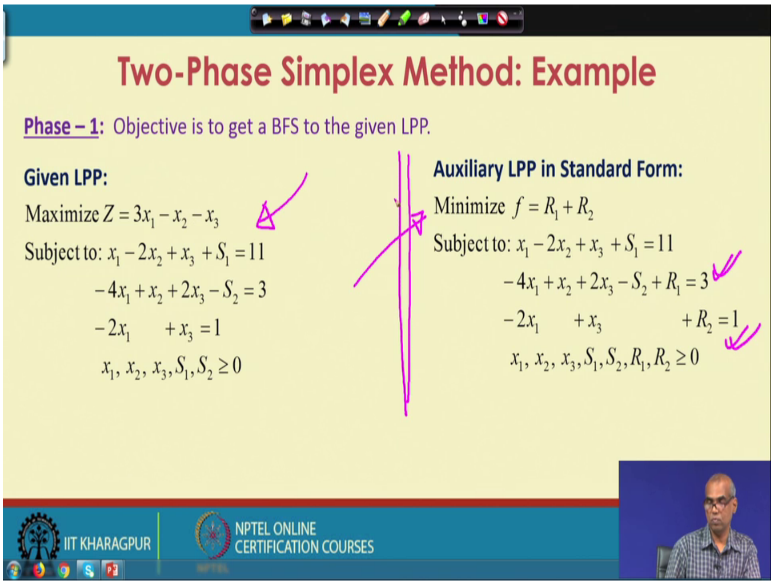

So, just to give you an example, let us consider the linear programming problem as shown. So, in the phase 1 objective is to get a basic feasible solution to the given linear programming problem. So, in this case, we first introduce two artificial variables R 1 and R 2, two constants number 2 and 3 respectively. And then solve and auxiliary linear programming problem. So, what is the difference in the auxiliary linear programming problem, the difference is in objective function.

While the given linear programming problem, the objective function is maximized Z equal to 3 x 1 minus x 2 minus x 3. In case of auxiliary linear programming problem, I consider the objective function as minimize f equal to R 1 plus R 2. So, I minimize the sum of artificial variables. Since, artificial variables are constant to be greater or equal to 0 in the auxiliary problem. You will get the minimum f when both R 1 equal to 0 and R 2 equal to 0, because R 1 is greater or equal to 0, R 2 is greater or equal to 0, both are constant to be non-negative.

So, the sum R 1 plus R 2 will be minimum, when R 1 and R 2 are individually 0. So, the solution of the auxiliary linear programming problem will give me a solution for which R 1 equal to 0 and R 2 equal to 0, so that will be a feasible basic solution for original linear programming problem. So, the solution for this will be a feasible basic feasible solution for the given linear programming problem.

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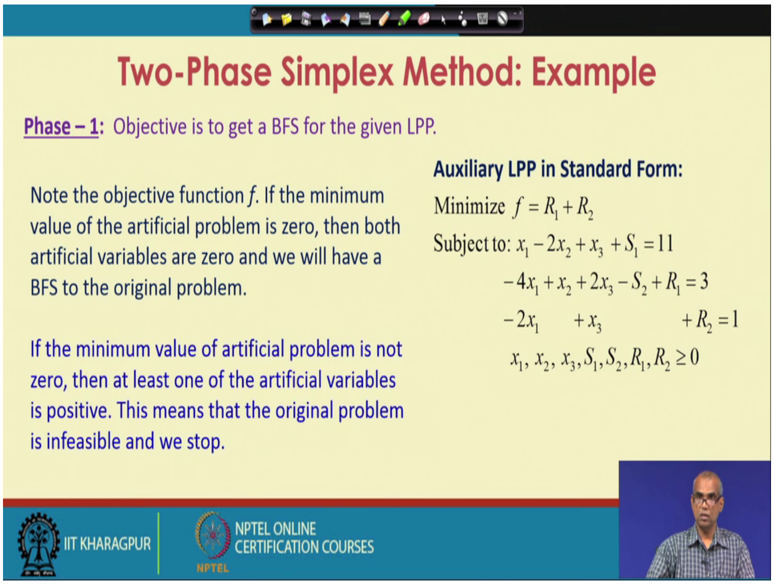

So, note the objective function f if the minimum value of the artificial problem is 0, then both artificial variables are 0. And we will have basic feasible solution to the original problem. If the minimum value of artificial problem is not 0, then at least one of the artificial variables is positive, because artificial variables can take only 0 and positive values. So, if their sum is not 0, it means at least one of the artificial variables is positive. This means that the original problem is infeasible and we will stop.

So, if the solution of the auxiliary linear programming problem is such that we have R 1 equal to 0 and R 2 equal to 0, then we have a feasible basic solution for the original linear programming problem. If the solution of the auxiliary linear programming problem is such that R 1 and R 2 are not both 0, then at least one of them is positive. And we conclude that original linear programming problem is infeasible. So, he stop our algorithm at this stage, phase 2 will not be necessary.

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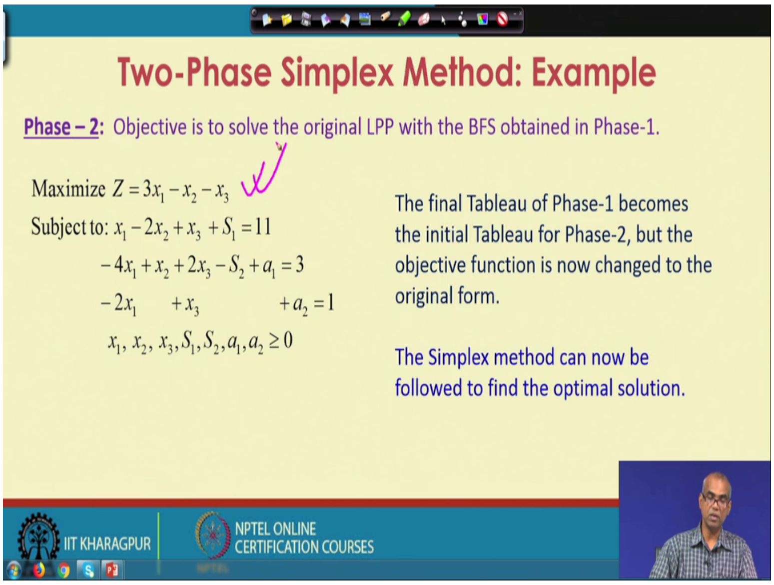

But, if the solution to the auxiliary linear programming problem is such that both R 1 equal to 0 and R 2 equal to 0, then the final Tableau of phase one becomes the initial Tableau of S 2. Because, we will start now with the basic feasible solution obtained as a solution of linear programming problem, auxiliary linear programming problem where R 1 equal to 0, R 2 equal to 0. But, we have to now change the objective function I will now consider the original objective function.

So, once we have obtained the basic feasible solution for phase 1 that means, R 1 equal to 0 and R 2 equal to 0 in the final solution that solution gives me the initial basic feasible solution for phase 2. So, the final Tableau of phase one becomes the initial Tableau for phase 2, but the objective function is now changed to the original form, which is Z maximize z equal to 3 x 1 minus x 2 minus x 3. So, the simplex method can now be followed using our regular procedure to find the optimal solution.

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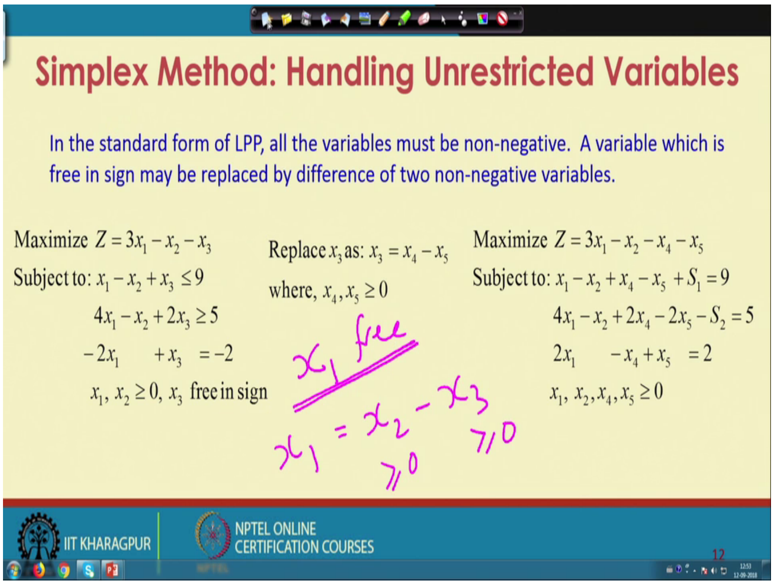

In the standard form of linear programming problem, all the variables must be non-negative. In certain situations will find that we may have variables, which are free in sign, but in the standard form the variables must be non-negative. When you formulate a linear programming problem from the statement of the problem or analysis of the physical problem, you can find that there are certain variables, which can take both positive and negative values that means, there free in sign.

But, in the standard form of the linear programming problem, all the variables must be non-negative. So, these variables, which are free in sign must be replaced by variables, which are only non-negative. We call such variables as unrestricted variables. The variables, which are free in sign are known as unrestricted variables. So, the way to handle these unrestricted variables or free variables is to express them as a difference of two non-negative variables.

Say, a variable x 1 is free in sign. Now, I can replace x 1 as difference of let us say x 2 minus x 3, where both x 2 and x 3 are greater or equal to 0. So, there will be suitable values of x 2 and x 3 such that x 1 can take both positive values and negative values. So, a variable which is free or unrestricted in sign can always be replaced by difference of two non-negative variables, so that is what we will do in the standard form of the linear programming problem for variables, which are free in sign.

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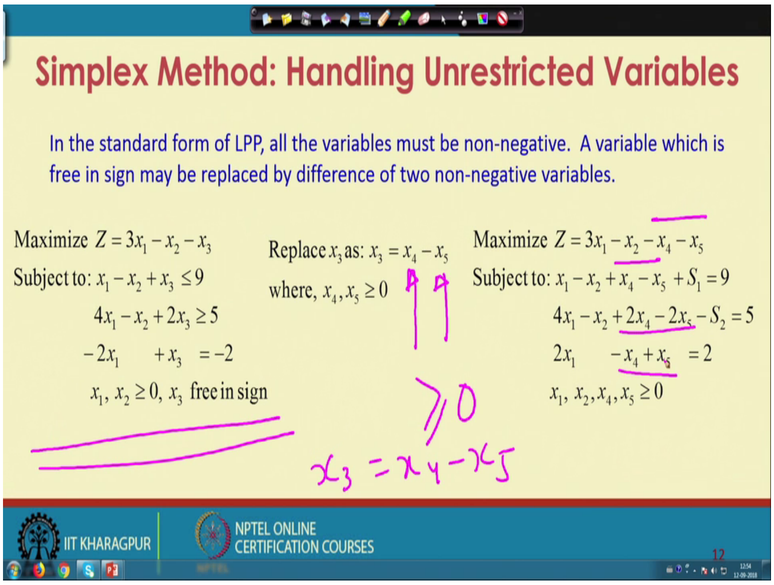

For example, let us look at the problem shown. In this case x 1 and x 2 are greater or equal to 0, but x 3 is free in sign. X 3 appears in objective function, x 3 appears in first constraint, second constraint, and third constraint, so x 3 appears everywhere. So, let us now introduce two new non-negative variables x 4 and x 5. So, both x 4 and x 5 are greater or equal to 0 type. So, then x 3 is replaced as x 4 minus x 5.

So, everywhere the objective function in the first constraint, in the second constraint, and in the third constraint, we replace x 3 as x 4 minus x 5. Also to put it in standard form, we have introduced slack variables S 1, we have introduced surplus variable S 2. We have also multiplied the third constraint throughout by minus 1, so the right hand side is positive. And then the non-negativity restrictions apply on x 1, x 2, x 4, and x 5. Note that x 3 has been replaced as x 4 minus x 5. So, this is how we can handle the unrestricted variables.

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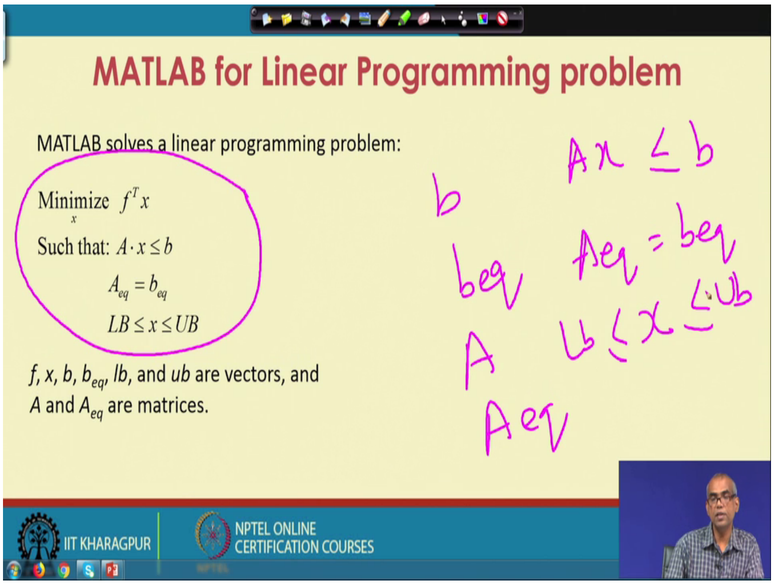

So, now we will briefly talk about how to use MATLAB function linprog for solution of linear programming problem. So, MATLAB solves a linear programming problem of the form shown. It solves a minimization problem so, if you have a maximization problem, take minus of the objective function. So, MATLAB solves minimize f transpose x such that A x less or equal to b, A equality equal to b equality, and x is bounded between lower bound and upper bound.

So, f is a vector, x is a vector, b is a vector, b equality is a vector, A is matrix, and A equalities matrix. So, A x less or equal to b is our usual inequality constraints. And if you have equality constraints in the linear programming problem that taken care of e equality equal to b equality also if I know my decision variables are bounded between lower bound and upper bound that can be specified. So, MATLAB solves a linear programming problem in this form.

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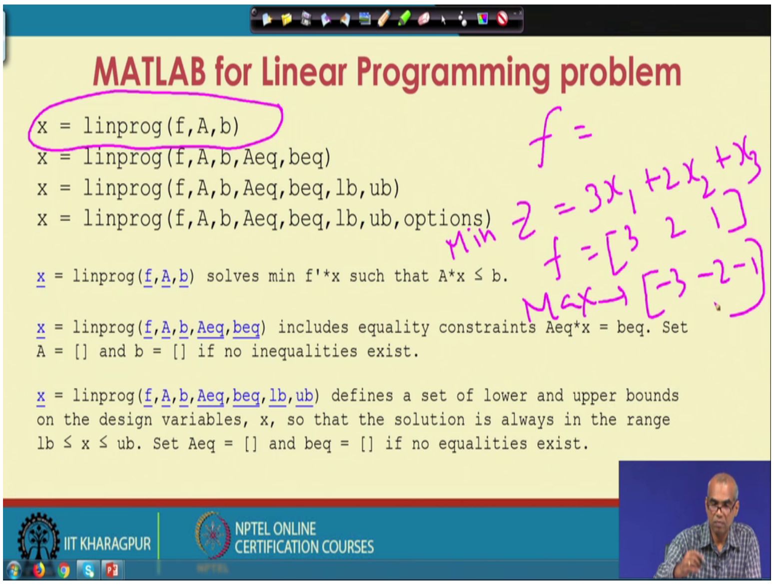

So, these are the typical syntax. So, the minimum syntax that is required is x equal to linprog f, A, b. What is f? F, if the vector for the coefficient is the objective function. So, if the objective function Z equal to a solving a minimization problem, let us say minimize Z equal to 3 x 1 plus 2 x 2 plus x 3, then f is 3, 2, 1 vector. If you want to maximize it, what I will write is for maximization, I will write as minus 3, minus 2, minus 1. A is the matrix in A x less or equal to b, and b is the constants on the right hand side. So, b is a vector, where elements at the constants for each constant that appear on the right hand side.

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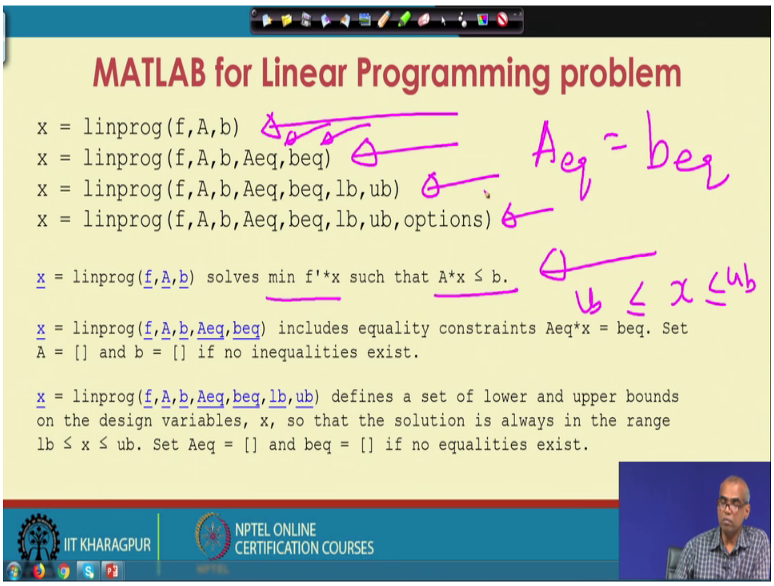

So, this is the minimum syntax. And this solves a problem f transpose minimization f transpose x such that A x less or equal to b. If you also have constants like A equal to b equality, then you also use these two as arguments to linprog. So, linprog is the function responsible for solving linear programming problem. In this calling syntax, you can mention about lower bound and upper bound for x.

So, if you know x is bounded between lower bound and upper bound, you can mention it here. Through options, you can tell linprog, what kind of algorithm it has to be it can use. So, linprog implements various algorithms such as interior point algorithm, dual simplex algorithm etcetera. So, you can specify using option structure, which algorithm you one MATLAB to use.

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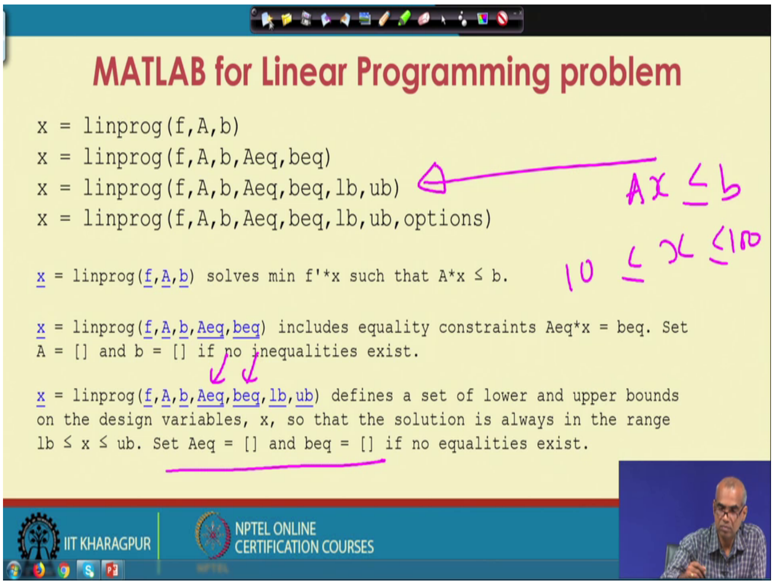

Let us look at this syntax. So, we have A x less or equal to b type, and you know x will lie between let us say 10 and 100. You do not have any constants, which are of equality type. So, what you can do is, you can put these two as null matrix.

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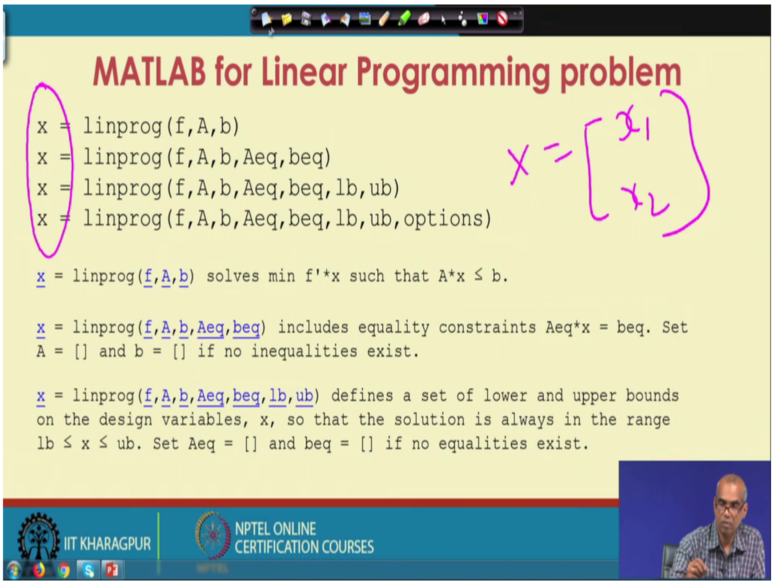

Similarly, here this x is the solution to the problem. So, the x is a vector, which contains all the decision variables as components of the vector. So, x is the solution vector. So, if you have a linear programming problem with two variables x 1 and x 2, so x will be a vector with component x 1 x 2.

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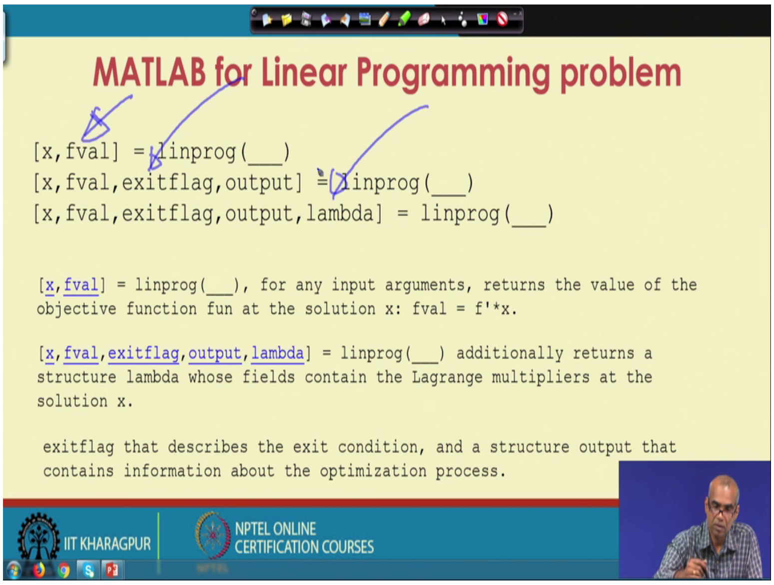

Now, along with this solution vector MATLAB may also return the value of the objective function for the optimal values of x 1 and x 2 etcetera, decision variables. If you introduce include exit flag, exitflag will describe the exit conditions, and the structure output that contains information about the optimization process.

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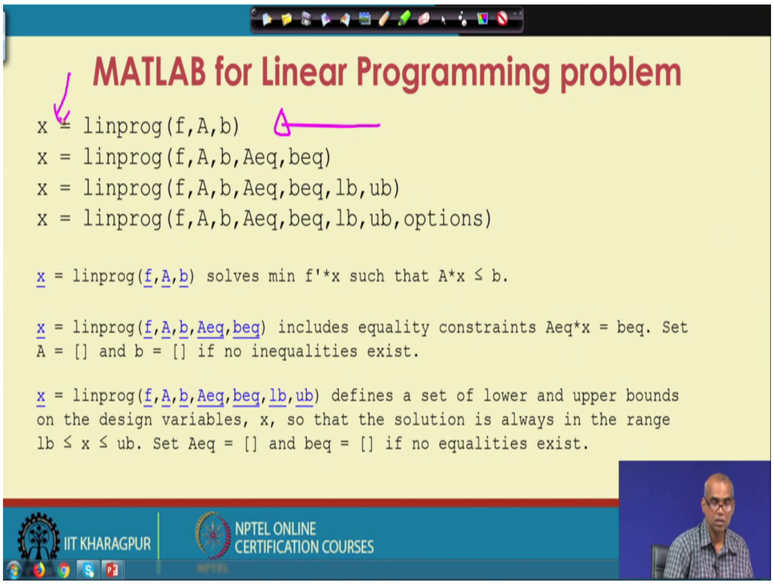

If you introduce lambda, it will contain the Lagrange multipliers at the solution x. So, all these possibilities are there. There are various ways of calling the linprog and the bare minimum is this or you can include the function value here like this. So, here you can put f, a, b, so that will be the minimum sentence required for a problem with only A x less or equal to b type constraints. Otherwise, you go on adding the arguments as per the requirement of your problem.

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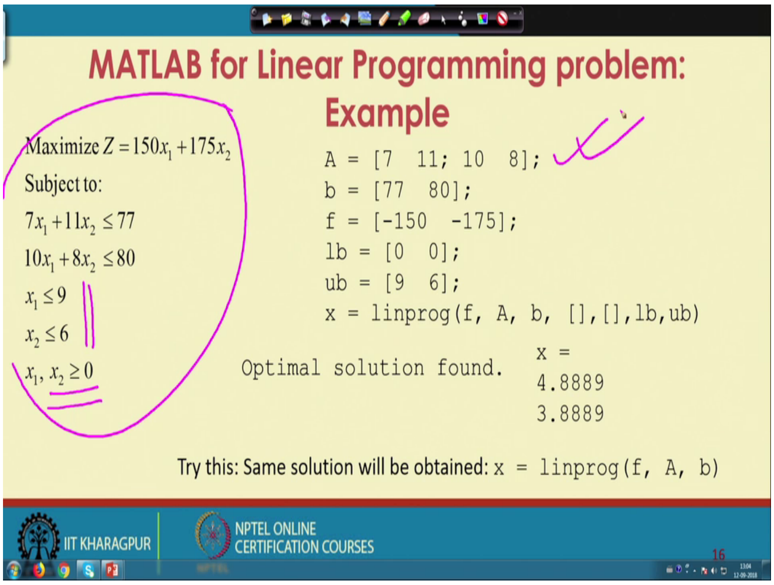

Now, let us take a very simple example and try to explain this. Consider this linear programming problem. You want to maximize 150 want to maximize an objective function objective function is Z equal to 150 x 1 plus 175 x 2, constants as 7 x 1 plus 11 x 2 less or equal to 77, 10 x 1 plus 8 x 2 less or equal to 8. So, look at these two, it is also said as x 1 less or equal to 9, x 2 less or equal to 6, but you know x 1 and x 2 are also greater or equal to 0. So, we know that x 1 will lie between 0 and 9, x 2 will also lie between 0 and 6. So, we can we can include this information in the calling syntax.

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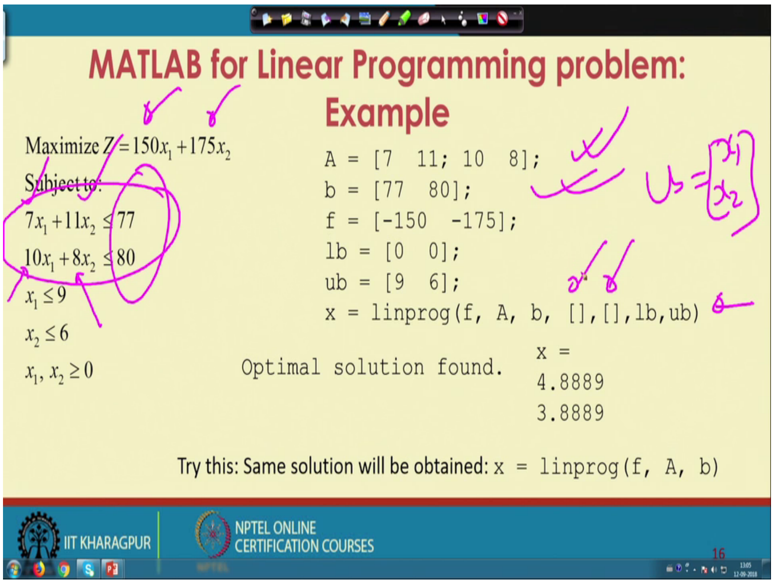

So, A is defined as 7, 11 and 10, 8 look at here. So, 7 11 and 10 8, so that is how the matrix A is defined. The b vector is the right hand side vector 77 and 80. Objective function f objective function coefficients 150 and 175, but since I want to maximize, I will take minus f. So, f equal to minus 150 minus 175. Lower bound is a vector for lower bound for x 1 lower bound for x 2, so 0 0. And upper bound is x 1 upper bound x 2 upper bound, which is 9 and 6. So, now I use this to call linprog. So, note that I do not have any equality constants so, these two null matrixes are kept there.

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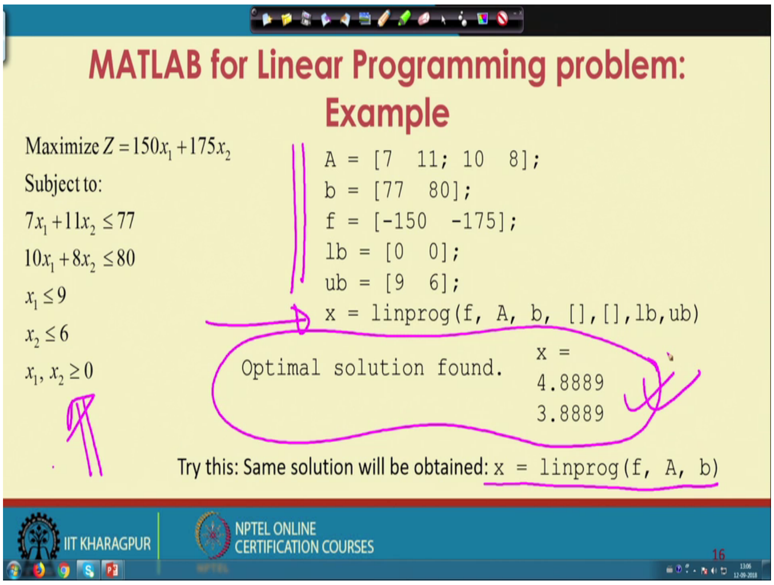

So, now if you right this, these statements on MATLAB command window or you create M file and run, it we will get optimal solution. So, you will get a command that optimal solution is found and x equal to 4.8889, 3.8889. So, this is the optimal solution to this problem. You do not have to supply the lower bound and upper bound. So, you can also try x equal to linprog f, A, b. And if you do that, you will get the same solution.

So, this will conclude our discussion on linear programming problem here. There are several other interesting topics related to linear programming problems, but because of time restrictions we will not be able to discuss everything. So, we have chosen the graphical method of solution, and the basic simplex methods that can be used to solve a linear programming problem. We have also seen briefly how MATLAB can used to solve linear programming problem. You can now takes exercises and can also use the

MATLAB solver linprog to solve linear programming problems. With this would like to conclude lecture 45 and week 9 here.