Optimization in Chemical Engineering
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Lecture – 41
Linear Programming: The Simplex Method

Welcome to lecture 41. So, this is the first lecture of week 9. And in this week 9 we will talk about simplex method for solution of linear programming problems. We have talked about graphical solution method for linear programming problem and you have seen that the feasible region is the polygon. So, one way to solve a linear programming problem is to evaluate each corner of this polygon, because we know that the optimal solution will lie at one or more of these corner points.

So, this may be convenient for a very small size problem, but for a large scale problem you may have very large number of corner points and evaluating each corner point is not an easy option. So, we need more robust method for solution of linear programming problem and simplex method is one of such efficient method. So, today we will start our discussion on simplex method for solution of linear programming problems.

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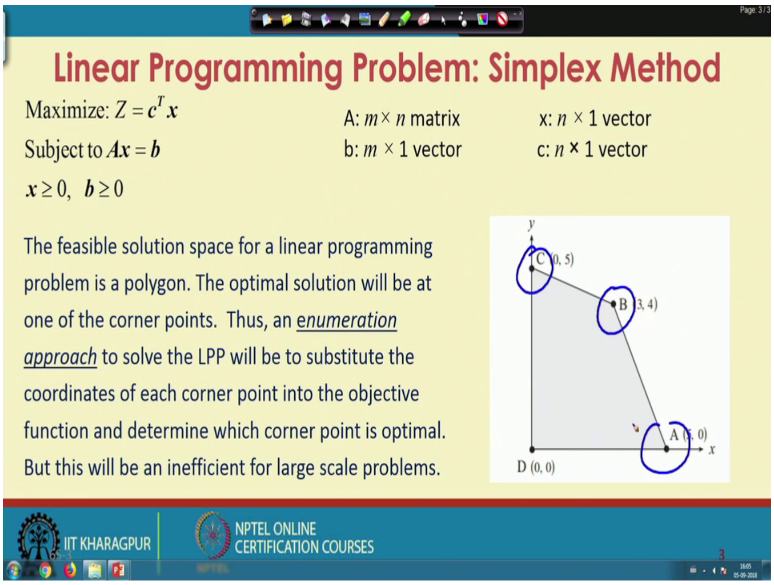

So, we have seen in case of graphical solution that, optimal solution will always be at one of the corner points. So, an enumeration approach will be to substitute the coordinates of each corner point into the objective function and determine which corner point is optimal, but this will be an inefficient approach for large scale problems.

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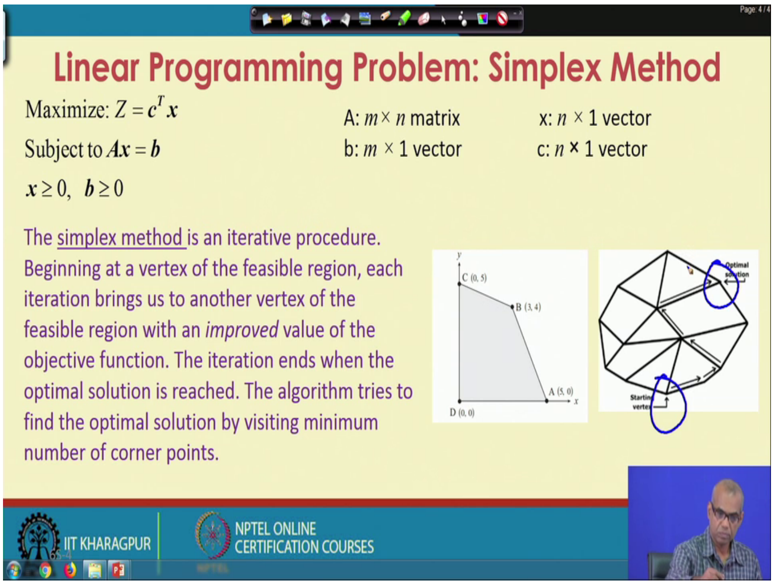

So, simplex method is an iterative procedure, it begins at a vertex of the feasible region, each iteration brings us to another vertex of the feasible region with an improved value of the objective function. The iteration ends when the optimal solution is reached; the algorithm tries to find the optimal solution by visiting minimum number of corner points. So, if you look at the figure, we start from this vertex and the way we visit each corner points is shown by the arrow mark and we reach the optimal solution by visiting a small number of corner points of the total number of corner points.

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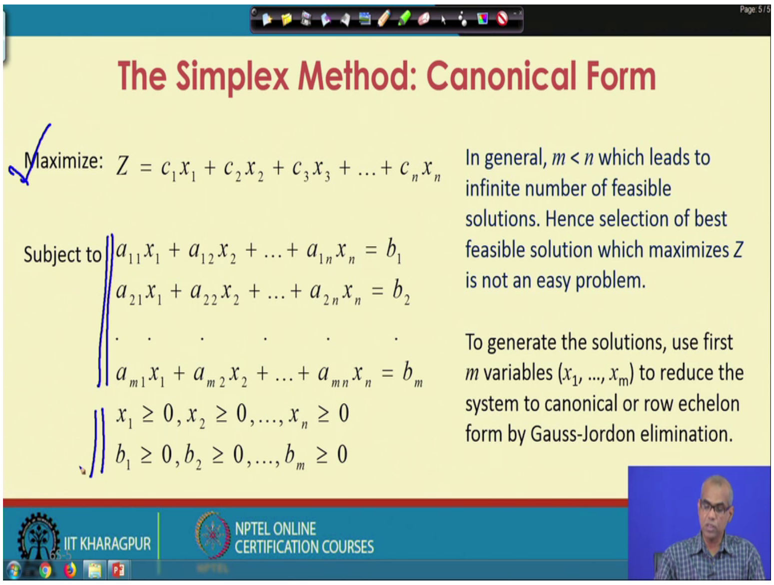

So, to start our discussion on simplex method, let us first revise what we learned in our previous lectures about Canonical form. So, what you see on the screen is, the general formulation of a linear programming problem, you have the objective function which is a linear function of n variables, you have n number of equality constants and you have non negativity constants on the decision variable and the constants on the right hand sides of the constraints.

In general you will have fewer number of constants compared to number of decision variables and this let us you in finite number of feasible solutions. Hence selection of best feasible solution which maximizes the objective function may not be an easy problem. To generate the solutions we first use m variables to reduce the system to a canonical or row echelon form by Gauss Jordan elimination.

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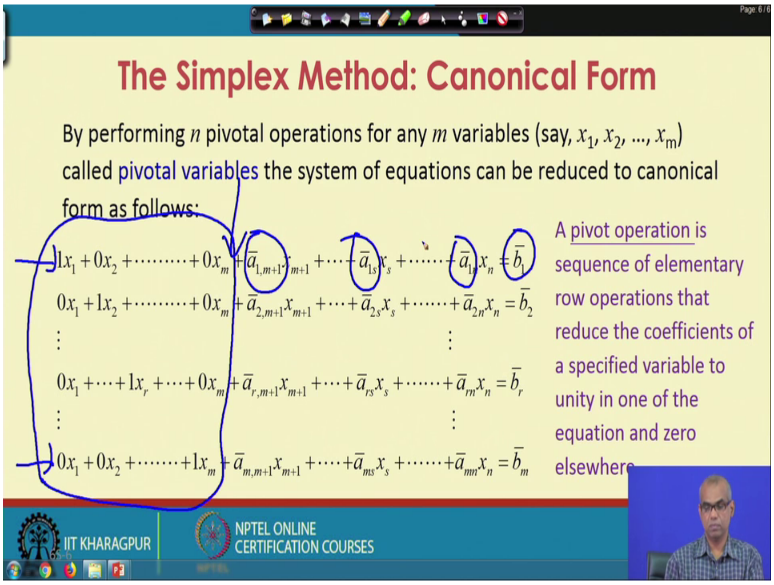

Then we can have a working solution by converting or putting the values of the remaining m plus 1 to n variables as 0. So, before that, we have to convert the set of equations into its canonical form by performing n pivotal operations. A pivot operation is sequence of elementary row operations that reduce the coefficients of a specified variable to unity in one of the equation and zero elsewhere.

So, if you look at the first equation the coefficient of x 1 is 1 and the coefficient of x 2 x 3 up to x m are all 0. So, we have considered x 1 to x m as pivotal variables and we have the remaining variables x m plus 1 to x m. So, in the canonical form you look at the x 1 to x m part of the set of linear equations, in the first equation only coefficient of x 1 is 1in the second equation only the coefficient of x 2 is 1 and in the n th equation only the coefficient of x m is 1.

So, this is known as canonical form. Please note that these coefficients as well as the right hand side coefficients are constants, all have changed from their original values due to series of elementary row operations for conversion of the system to its canonical form.

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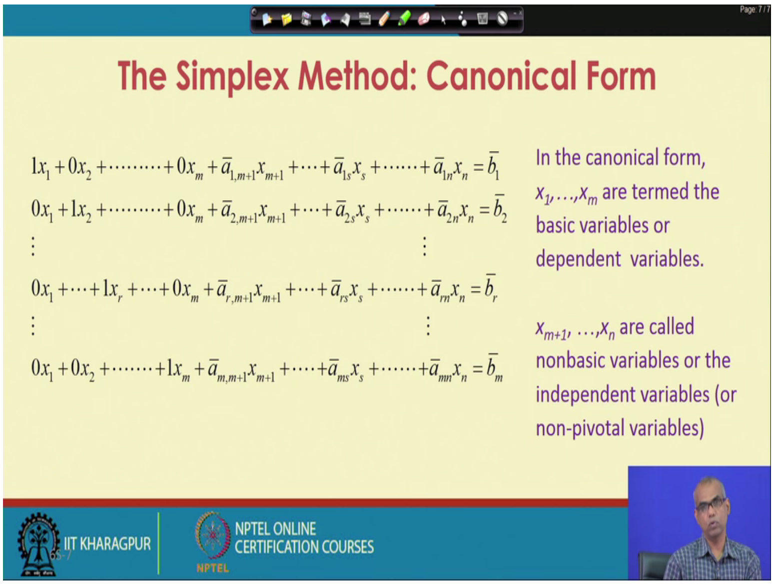

So, in the canonical form x 1 to x m are termed as a basic variables or dependent variables. An x m plus 1 to x n are called non basic variables or independent variables or non pivotal variables.

Now, if you look at this canonical form, if I put the values of x m plus 1 to x n variables which are non basic or independent variables as a 0, I can readily solve the equations and the solutions will be x 1 equal to b 1 bar, x 2 equal to b 2 bar, x r equal to b r bar, x m equal to b m bar and so on and so forth.

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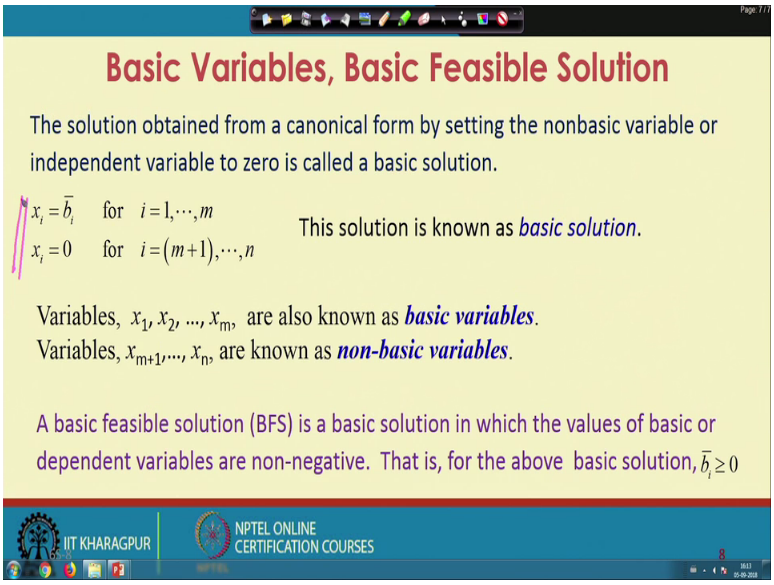

So, the solution obtained from a canonical form by setting the non basic variables or independent variables to 0 is called a basic solution.

So, the basic solution is obtained by setting the values of the non basic variables to 0. So, in general I can write x i equal to bi bar for i equal to 1 to m, which are basic variables and x i equal to 0 for i equal to m plus 1 to n which are non basic variables. So, this solution will be known as basic solution. Variables x 1 x 2 up to x m are also known as basic variables, variables x m plus 1 to x n are also known as non basic variables.

A basic feasible solution is a basic solution, in which the values of basic or dependent variables are non-negative. That means b i bar is greater or equal to 0. So, a basic feasible solution is a basic solution in which the values of the basic or dependent variables are non negative. In other words basic feasible solutions are those basic solutions for which b i bar is greater equal to 0.

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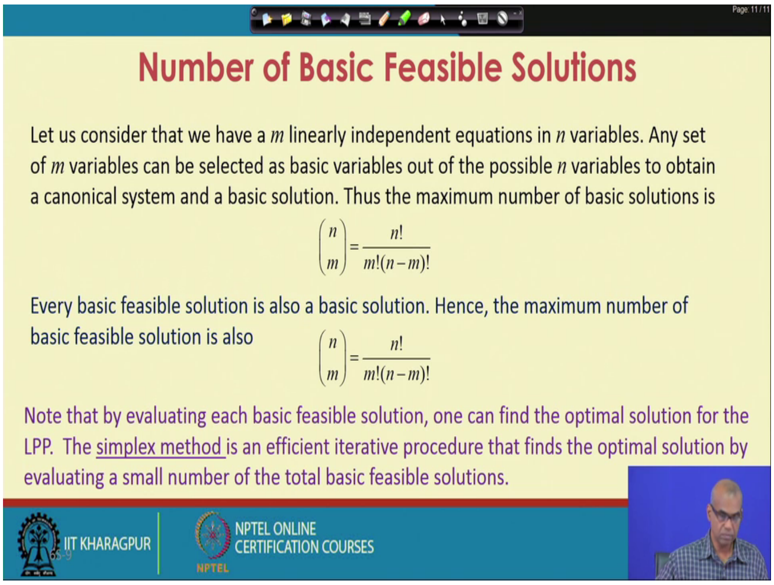

How many number of basic feasible solutions are possible? Let us consider that we have m linearly independent equations in n variables. Any set of m variables can be selected as basic variables out of the possible n variables to obtain a canonical system and a basic solution. In the previous example shown we have considered first m variables as basic variables, but if you can consider any set of m variables as basic variables out of the possible n variables. Thus the maximum number of basic solutions is n c m which is factorial n divided by factorial m into factorial n minus m.

Every basic feasible solution is also a basic solution, hence the maximum number of basic feasible solution is also n c m which is factorial m divided by factorial m into factorial n minus m. Note that by evaluating each basic feasible solution one can find the optimal solution for the linear programming problem. The simplex method is an efficient iterative procedure that finds the optimal solution by evaluating a small number of total basic feasible solutions.

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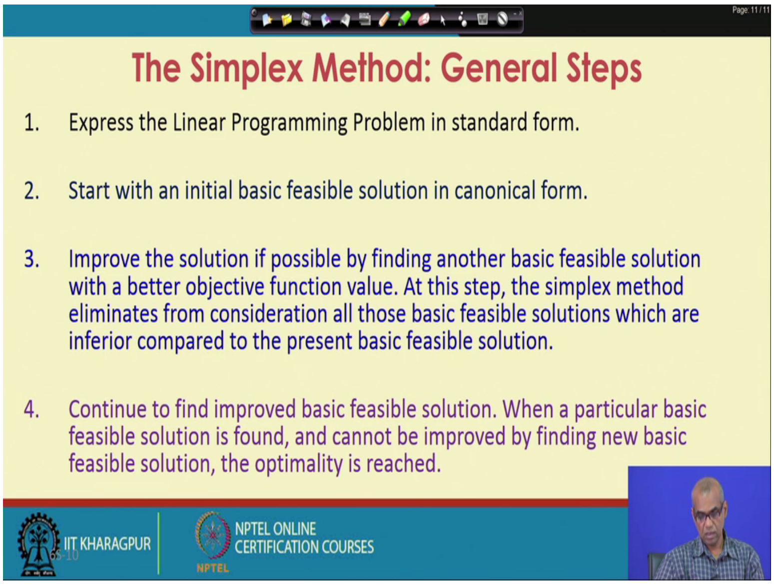

So, here are the general steps for the simplex method. First we have to express the linear programming problem in standard form. We know what is standard form, by standard form you all the constants will be equality type, decision variables and the right hand side vector will be non-negative. After expressing the linear programming problem in standard form we start with an initial basic feasible solution in canonical form. Once we have a starting basic feasible solution we would like to improve the solution by finding another basic feasible solution with a better objective function value.

So, if I am maximizing a problem, I would like to find another basic feasible solution with an objective function value, which is higher than the current basic feasible solution. At this step the simplex method eliminates from consideration all those basic feasible solutions which are inferior compared to the present basic feasible solution. We continue to find improve basic feasible solution, when a particular basic feasible solution is found and cannot be improved by finding new basic feasible solution the optimality is reached.

So, this is in general the steps of a simplex method. So, the key steps are you start with an initial basic feasible solution and then we have to find at each iteration and improved basic feasible solution then the current basic feasible solution. When we cannot do that optimality is reached and we stop.

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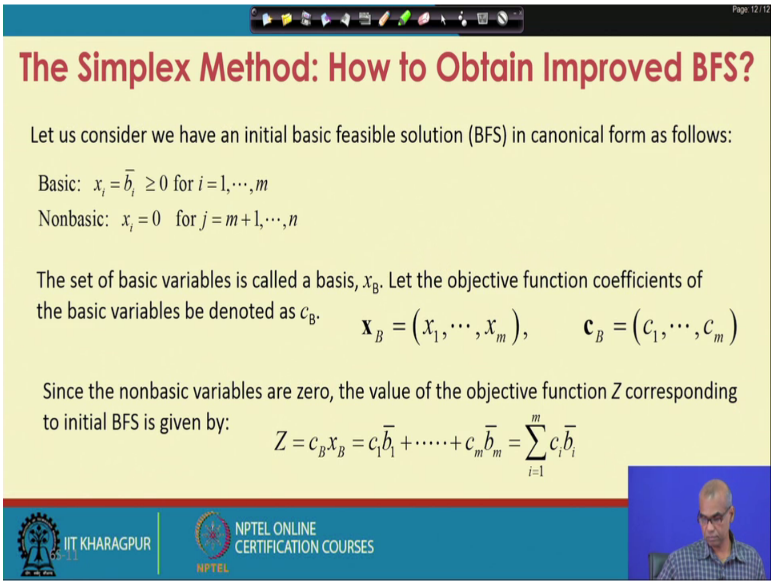

So, the question that we ask now is, how to obtain the improved basic feasible solution. So, let us consider we have an initial basic feasible solution in canonical form as shown. Basic variables are x i equal to b i bar and bi bar are all greater equal to 0 so, that their basic solutions are feasible.

So, I equal to 1 to m basic variables and non basic variables xi equal to 0 for j equal to m plus 1 to m. The set of basic variables is called a basis and we represent this basis vector by X B. Let the objective function coefficients of the basic variables be denoted as C B. So, X B is basis vector. So, its components are the m basic variables and objective function coefficients of basic variables are denoted by C B.

So, C B is the vector with components objective function coefficients for basic variable C 1 C 2 up to C m. Since the non-basic variables are 0 the value of the objective function z corresponding to the initial basic feasible solution can be obtained as C 1 b 1 bar plus C 2 b 2 bar up to Cm bm bar; that means, C B into X B vector product of C B and X B.

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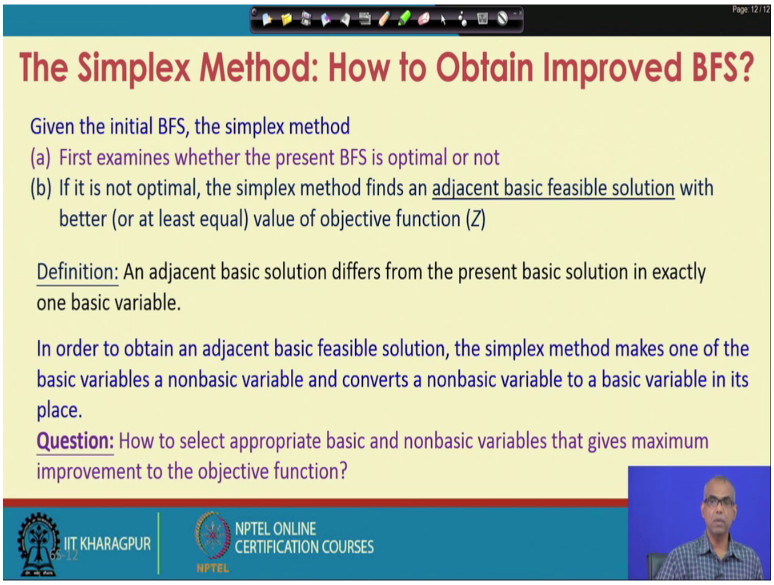

Now, given the initial basic feasible solution the simplex method first examines whether the present basic feasible solution is optimal or not. If it is not optimal the simplex method finds and adjacent basic feasible solution with better or at least equal value of the objective function.

So, when we start with an initial basic feasible solution the simplest method will first examine whether the present basic feasible solution is optimal or not. If the present basic feasible solution is optimal we stop our algorithm. If the present basic feasible solution is not optimal, we have to find an adjacent basic feasible solution with better or at least equal value of the objective function. So, what is adjacent basic feasible solution? An adjacent basic feasible solution differs from the present basic solution in exactly one basic variable.

So, basic variables will be removed from the basis and a non basic variable will enter the basis so, that the new adjacent basic solution differs from the current basic solution in exactly one basic variable. Note that you have n basic variables so, your basis contains m basic variables, the adjacent basic solution or the adjacent basic feasible solution will be another basic vector with only one component different.

So, in order to obtain an adjacent basic feasible solution, the simplex method makes one of the basic variables and non basic variable and converts a non basic variable to a basic variable in its place. So, the question we now ask is how to select appropriate basic and non basic variables that gives maximum improvement to the objective function. So, you understand that adjacent basic feasible solution differs from the present basic solution in exactly one basic variable.

So, one non basic variable will be converted to a basic variable and one basic variable will converted to a non basic variable. So, the question is how to select appropriate basic and non basic variables so that their exchange gives maximum improvement to the objective function. That means, for a maximization problem we should get the maximum increase in the value of the objective function after the exchange of basic and non basic variables.

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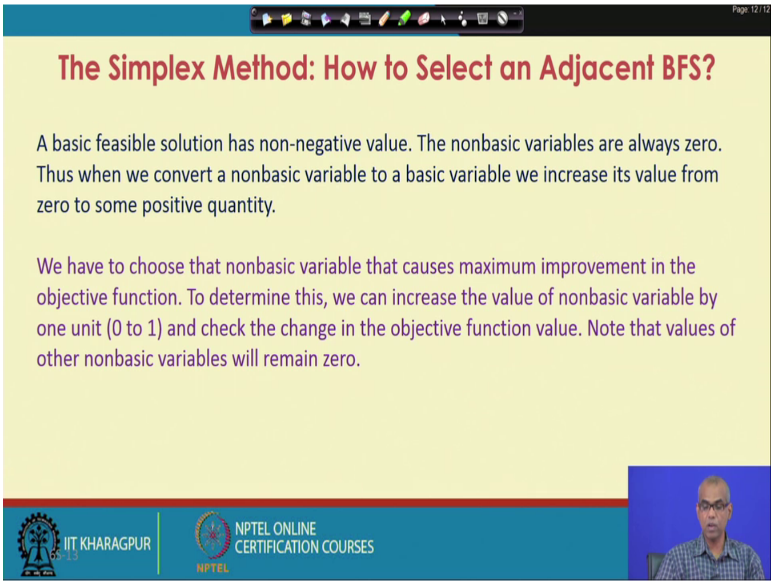

A basic feasible solution has non negative value. So, basic feasible solution is always greater or equal to 0. The non-basic variables are always 0. Thus when we convert a non basic variable to a basic variable, we increase this value from 0 to some positive quantity. Note that all decision variables are constrained to be greater or equal to 0; but a basic feasible solution can be values which are greater than 0 non basic variables are always 0.

So, when we convert a non basic variable to a basic variable, we increase its value from 0 to some positive quantity. We have to choose that non basic variable that causes maximum improvement in the objective function. To determine this we can increase the value of a non basic variable by one unit say 0 to 1 and check the change in the objective function value. Note that the values of other non basic variables will remain 0. So to determine, which non basic variable will be converted to a basic variable; what we can do is, we can increase the value of a non basic variable by one unit say from 0 to 1 and check the effect in the objective function value. So, for a maximization problem we would expect that this will increase the value of the objective function.

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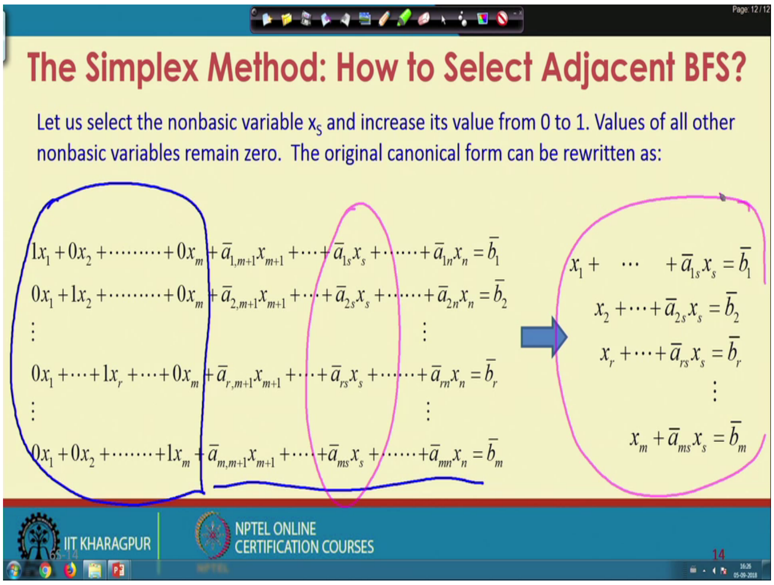

So, let us see how we can do it. Let us select the non basic variable x s and increase its value from 0 to 1. Values of all other non-basic variables remain 0. So, let us look at again the canonical form. So, first m variables are basic variables and then m plus 1 to n are non-basic variables. What we are saying now is let us increase the value of the non-basic variable x s from 0 to 1.

So, if I do that the canonical form will be rewritten as shown. It was x 1 equal to b 1 bar x 2 equal to b 2 bar etcetera now it will be x 1 plus a 1 s bar x s equal to b 1 bar x 2 plus a 2 s bar xs equal to b 2 bar and so on and so forth.

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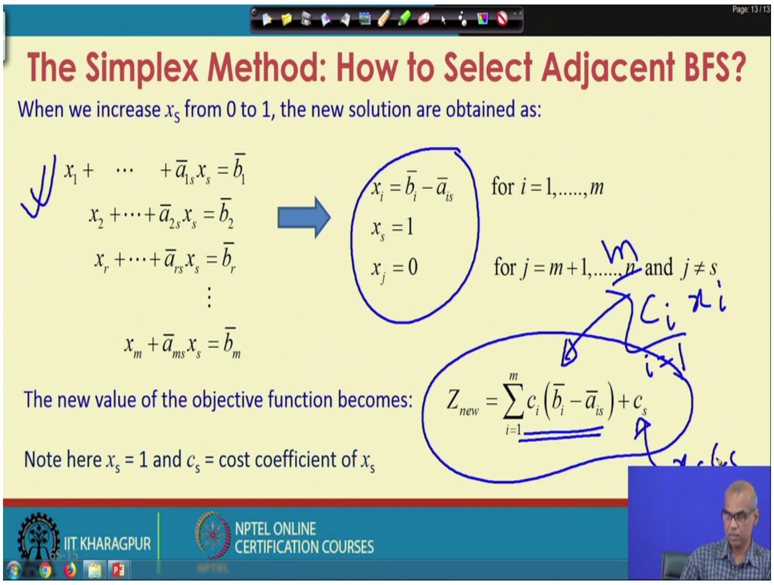

So, when we increase x s from 0 to 1, we get a new solution. Initial solution was xi equal to bi bar, but now you see the solution is x 1 equal to b 1 bar minus a 1 s bar x s x 2 equal to b 2 bar minus a 2 s bar x s etcetera.

So, the solution can be written as xi equal to bi bar minus a i's bar for i equal to 1 to n all basic variables. Xs value of x s equal to 1 and x j equal to 0 for j equal to m plus 1 to n and j is not equal to s. So, since the values of the basic variables have changed. So, the value of the objective function has also changed. So, the new value of the objective function can be obtained as shown. Note that the value of the objective functions are nothing, but the basic variable multiplied by its coefficient, objective function coefficient the cost coefficients.

So, for all i i equal to 1 to m. So, basically these are c i into xi note that this is nothing, but c i into x i for all i equal to 1 to m and then this is x s into c s where c s is the cost coefficient of x s, but x s equal to 1. So, c s into x s is same as cs. So, you now get the new value of the objective function.

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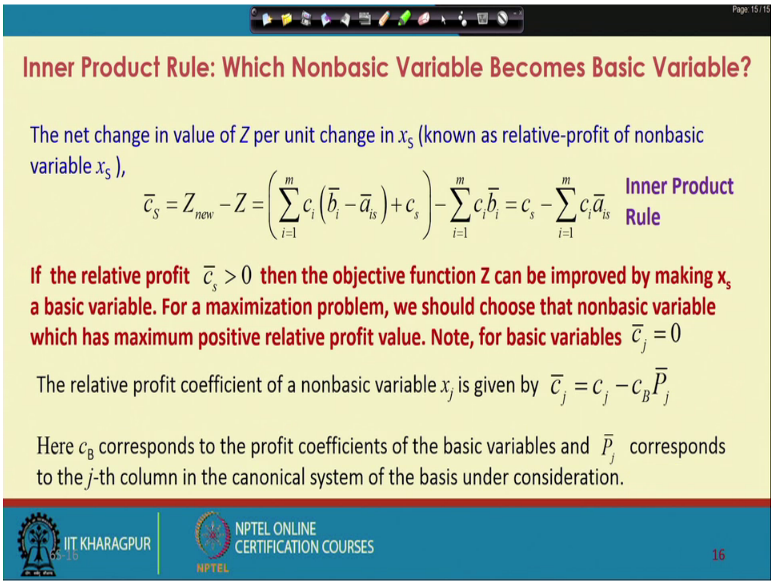

So, what is the net change in the value of the objective function? Per unit change in x s. To find this out we have to get the difference between the value of the new objective function and the value of the old objective function. So, this net change in value of objective function z per unit change in x s is known as relative profit of a non basic variable x s.

So, it is denoted as c s bar. So, c s bar is obtained as the new objective function value minus old objective function value and this is obtained as c s minus sigma i equal to 1 to m c i a is bar. This is known as inner product rule. So, if the relative profit of a non basic variables c s bar is greater than 0, then the objective function Z can be improved by making x s a basic variable. For a maximization problem we should therefore, choose that non basic variable which has a maximum positive relative profit value.

Note for basic variables this relative profit c j bar equal to 0. So, this answers the question which non basic variable will enter the basis. So, we will find out the relative profit of the non basic variables and if the relative profit is greater than 0, we understand that the objective function can be improved by making x s a basic variable. For a maximization problem we should choose that non basic variable which has a maximum positive relative profit value so, that that will cause the maximum improvement in the objective function value when we convert that particular non basic variable as basic variable. The relative profit coefficient of a non basic variable x j is given by c j bar equal to cj minus c B into P j bar here c B corresponds to the profit coefficients of the basic variables and P j bar corresponds to the j th column in the canonical system of the basis under consideration.

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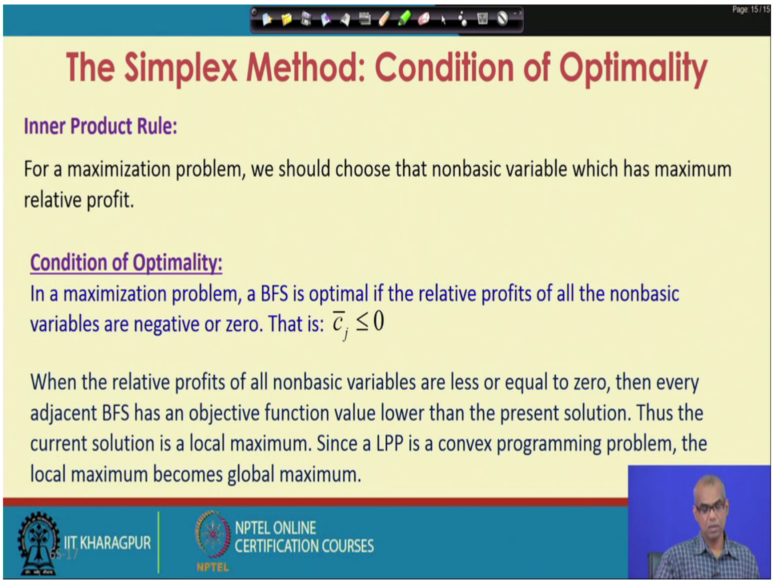

So, we understand the inner product rule which says for a maximization problem we should choose that non basic variable which has maximum relative profit.

So, now this also leads to the conditional optimality. In a maximization problem a basic feasible solution is optimal if the relative profits of all the non basic variables and negative or 0; that means, c j bar is less or equal to 0 for all non-basic variables. When the relative profits of all non basic variables are less or equal to 0, then every adjacent basic feasible solution has an objective function value lower than the present solution, thus the current solution is a local maximum.

Since a linear programming problem is a convex programming problem the local maximum becomes global maximum. So, when for a maximization problem we have the relative profits for all non basic variables are negative or 0 that means, there is no adjacent basic feasible solution, which has an objective function value which is better than the current one. So, the current solution is the local maximum and since linear programming problem is a convex programming problem, these local maximum is also global maximum.

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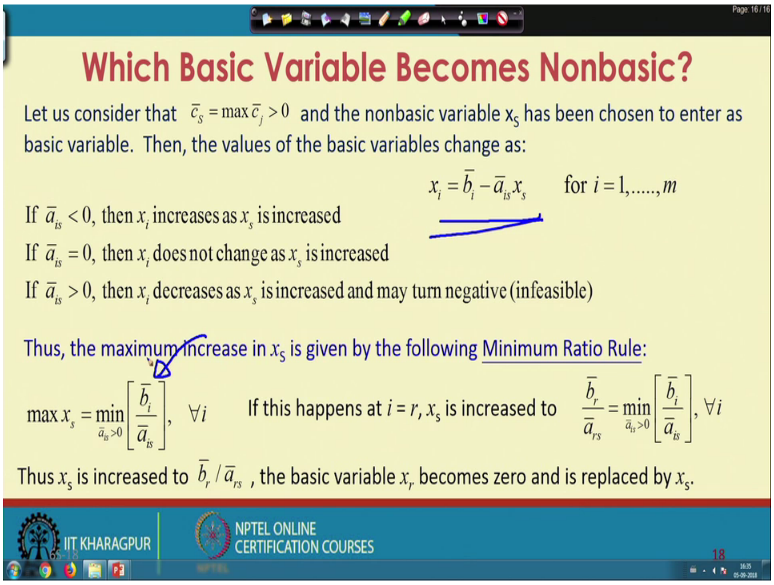

So, we understand now which non basic variable enters basis. But then which basic variable leaves basis; that means, which basic variable becomes non basic. Let us consider that c s equal to max c j bar which is greater than 0 and the non-basic variable x s has been chosen to enter as basic variable.

Then the value of the basic variables change as xi equal to bi minus a is x s. So, look at this expression now, x i equal to bi bar minus a is bar x s. So, this equation tells us that if a is bar is greater is less than 0 then xi increases as x s is increased if a is bar equal to 0 then xi does not change as x s is increased if a is bar is greater than 0, then x i decreases as x s is increased and may turn negative; that means, it may become infeasible.

So, how much increment should I make to x s? So, thus the maximum increase in x s is given by a rule known as a minimum ratio rule. So, maximum x s is minimum of all ratios which our b i bar a is bar. Note that x i equal to b i bar minus a is bar x s. So, if I put x s equal to b i bar divided by as bar, xi will be equal to 0. Beyond that if I increase it will become invisible. So, we have to find the ratio of b i bar divided by a is bar and then minimum of that will give me the maximum increment in x s if this happens at i equal to r, x s is increased to b r bar by a rs bar.

Here r means the r th row. So, thus x s is increased to b r bar divided by a rs bar and the basic variable x r becomes 0 and is replaced by x s. So, to find which basic variable will leave basis you have to find out bi bar divided by a is bar and then you take the minimum of this. So, look at the row at which this minimum happens, and that will tell us which basic variable will basis this will be more clear when you solve a problem.

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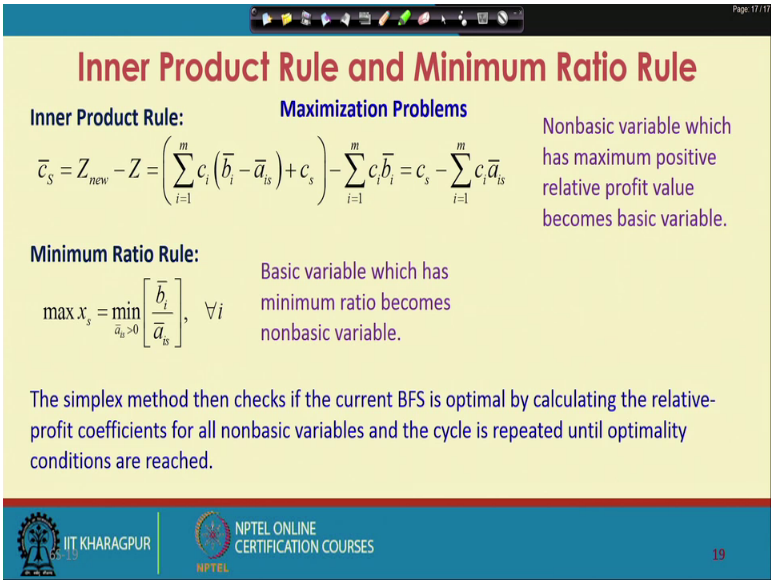

So, there are two main rules one was inner product rule and another is minimum ratio rule. Inner product rule says that non basic variable which has a maximum positive relative profit value becomes basic variable and minimum ratio rule says the basic variable, which has minimum ratio becomes non basic variable.

So, after this exchange of basic and non basic variables, the simplex method checks again if the current basic feasible solution is optimal by calculating the relative profit coefficients for all non basic variables and the cycle is repeated until optimality conditions are reached. With this we stop here and in the next lecture we will take a problem and solve using their programming problem to understand the ideas better.