Optimization in Chemical Engineering
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Lecture - 40
Introduction to Linear Programming (Contd.)

Welcome to lecture 40; this is week 8 and we have been talking about Introduction to Linear Programming problems. So, in this week we have introduced linear programming problems then, we have discussed about graphical method of solutions; which is essentially convenient only for two variable problems. In this lecture, we will talk about more on standard form of linear programming problems that we have discussed in your previous lecture. So, we will introduce certain definitions related to the standard form of linear programming problem.

And then we will discuss about solutions of set of linear equations say using Gauss Jordan elimination method that will lead to the canonical form of the systems of linear equations. These discussions, will lay the foundation for our discussion on simplex method that we will cover in the next week.

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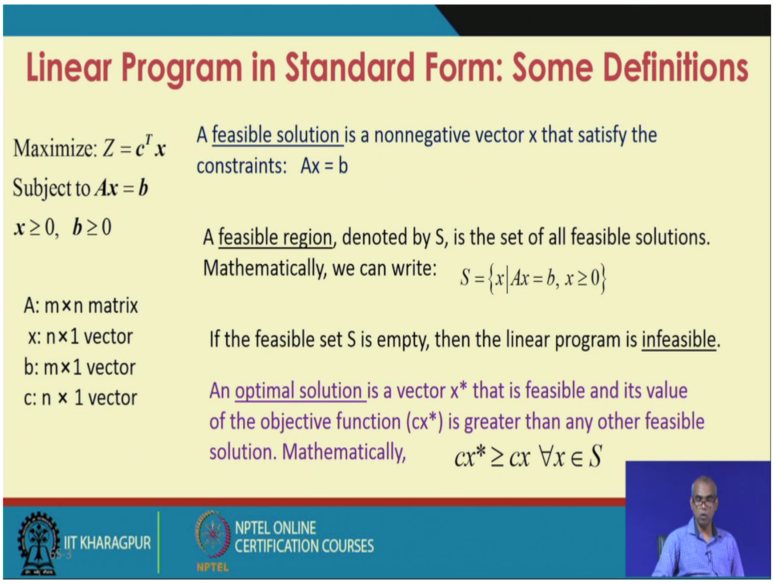

So, in matrix notation we have presented a linear programming problem in standard form as maximize or minimize Z equal to c transpose x subject to x equal to b x greater or equal to 0 b greater equal to 0 . A is m by n matrix, x is n vector, c is n vector and b is m vector.

Now we will introduce some definitions some of them you are already familiar with so, we will review some of those definitions you are already familiar with. A feasible solution is the non negative vector x that satisfies the constraints A x equal to b.

So, these notations are related to the problem formulation maximize Z equal to c transpose x subject to x equal to b; x greater or equal to 0, b greater or equal to 0 . A feasible solution is a non negative vector x that satisfy the constraints A x equal to b, A feasible region denoted by S is the set of all feasible solutions.

So, mathematically we can write S equal to x such that A x equal to b; where x is greater or equal to 0. If the feasible set S is empty, then the linear program is infeasible . An optimal solution is a vector x star that is feasible and it is value of the objective function cx star is greater than any other feasible solution. So, if x star is optimal solution; it will be feasible and the objective function value at x star that is cx star will be greater or equal to cx for all x belonging to the feasible region S .

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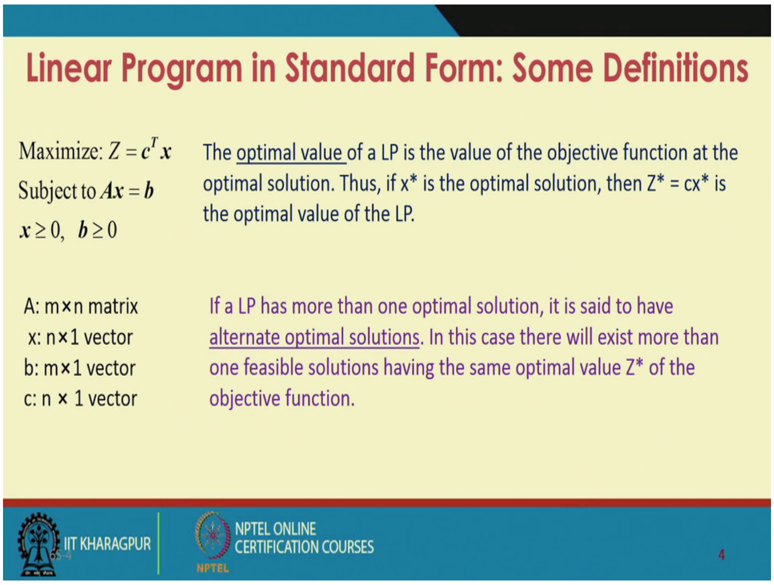

The optimal value of a linear program is the value of the objective function at the optimal solution. Thus, if x star is the optimal solution then Z star equal to cx star is the optimal value of the linear program. .

If a linear program has more than one optimal solution it is said to have alternate optimal solutions. In this case, there will exist more than one feasible solutions having the same optimal value Z star of the objective function.

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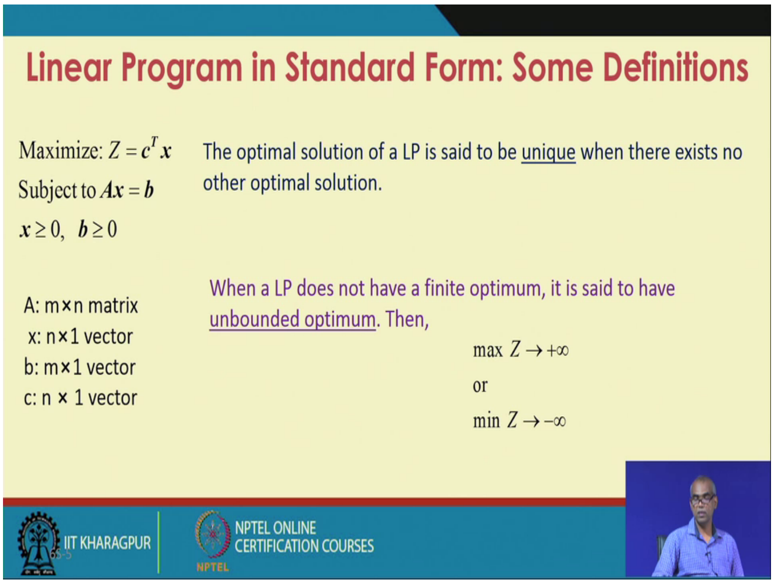

The optimal solution of a LP or Linear Program is said to be unique when there exists no other optimal solution. So, there is only one optimal solution, when a linear program does not have a finite optimum, it is said to have unbounded optimum.

Then maximization of Z will lead to plus infinity minimization of Z will lead to minus infinity. You can go on increasing or decreasing the objective function without any bound depending on whether you have solving a maximization problem or minimization problem.

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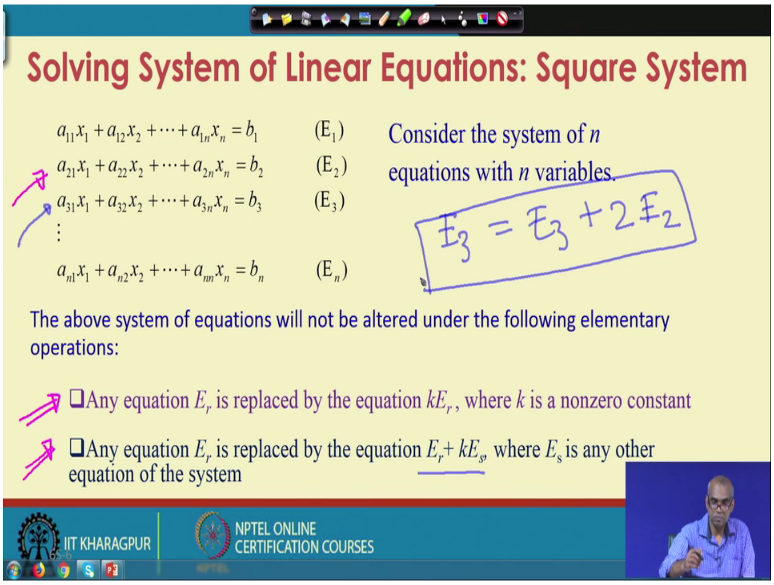

Now, let us discuss the solution of system of linear equations. First let us consider a square system, by square system; I mean I have a system of n equations with n variables. So, there are n variables and then n equations.

So, the system can be solved uniquely, the system can be easily solved because you have n variables and n equations. So, the equations are presented as a 1 1 x 1 plus a 1 2 x 2 up to a 1 and xn equal to b 1 and so on and so forth; a n 1, x 1 plus a n 2 x 2 plus a n n; x n equal to b n. .

It is a square system; so, there are n variables and n equations. So, you will have n rows of equations and there will be n columns of these variables. This system of equations will not be altered under a certain set of elementary operations. There exists a set of elementary operations, which when we apply on this set of linear equations; the system will not be altered ; that means, the solutions of the transformed set of equations and the solutions of the original equations will be same . So, let us discuss these 2 elementary operations.

Any equation E r is replaced by the equation k into E r, where k is a nonzero constant. So, pick up any equation; let us say I have picked up the second equation, if I replace this equation by multiplying this equation by a nonzero constant, the system of equations will not be changed.

So, this is one elementary operations that you can replace a row by a multiple of that row. Second elementary operation any equation E r is replaced by the equation E r plus k into E s where k is a nonzero constant and E s is any other equation of the system. So, this tells you that I can replace this equation number 3 or 3rd row by sum of third row and let us say a multiplication of the second row or any other row. So, the third equation or the 3rd row can be can be replaced as sum of 3rd row; plus let us say 2 into 2nd row, if I do this the transformed set of equations and the original set of equations we will have same solutions.

And that is what we mean when I say the system of equations will not be altered. So, these two elementary operations can be performed on the set of linear equations without altering the solutions. So, first one is any equation can be replaced by a multiple of that equation. .

You have to multiply that equation by any nonzero constant and can the equation can be replaced by a multiple of that equation. Second elementary operation is any equation can be replaced by a sum of that equation and a multiple of any other equation.

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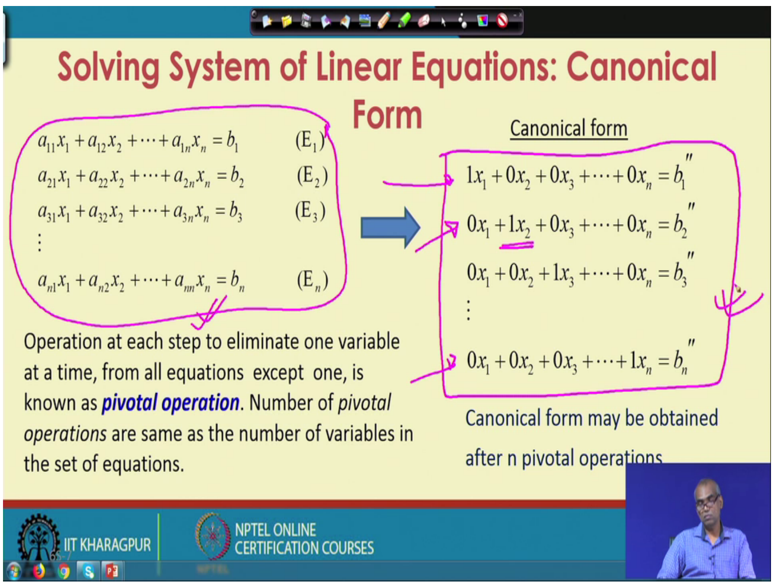

So, when I have this set of linear equations and I perform those two elementary operations; it is possible for me to express this set of equation in a form known as canonical form.

And if you closely look at in this canonical form; the first row; that means, the first equations has coefficient 1 for variable x 1 that is first column and all other coefficients are 0. Right hand side b 1 has now been depressed as b 1 double prime because those elementary row operations has been performed.

So, essentially the first equation is x 1 equal to b 1 double prime. Similarly in the canonical form the second row has been transformed to a row where only x 2 has coefficient equal to 1 all other variables has coefficient 0.

So, essentially making this equation as x 2 equal to b 2 prime, this continues and the last row that is nth row because, you now have n cross n matrix ; the nth row has all the coefficients 0 coefficients for x 1, x 2 up to xn minus 1 0 and the coefficient of xn is 1.

So, this equation is essentially xn equal to bn double prime. So, it is possible to transform the original set of linear equations to a canonical form as shown. Operation at each step to eliminate one variable at a time from all equations except one is known as pivotal operation.

Note that in the canonical form each row has only one variable because, all other has been reduced to 0. In the canonical form if you pick up any row only one variable has coefficient 1; all other variables has coefficient 0. Operation at each step to eliminate one variable at a time from all equations except 1 is known as pivotal operation number of pivotal operations are same as the number of variables in the set of equations. So, the system of n equations in n variables were converted to canonical form after n pivotal operations.

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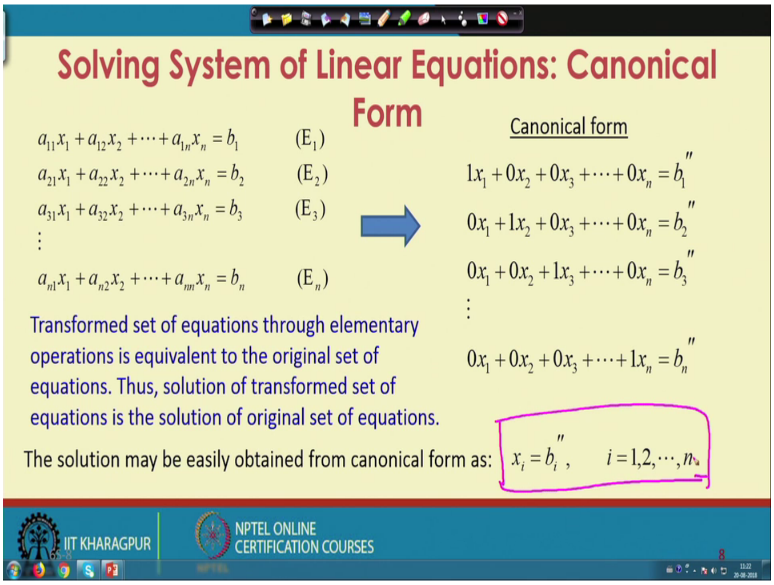

Transform set of equations through elementary operations is equivalent to the original set of equations. We discussed that these elementary operations do not alter the set of equations. In fact, they do not alter the solutions of the original equations.

Because, the transform set of equations that we obtain through elementary operations is equivalent to the original set of equations. Thus the solution of transform set of equation is the solution of the original set of equation.

Now, if you look at the canonical form the solutions are readily obtained from the 1st row; we obtain the solution as x 1 equal to b 1 double prime; from the 2nd row we obtain the solution as x 2 equal to b 2 double prime, from the 3rd row we obtain it as x 3 equal to b 3 double prime. Similarly from the nth row we obtain the solution as xn equal to bn double prime.

So, the solution can be easily obtained as xi equal to bi double prime for all values of i equal to 1 to n. So, once you can convert the solution sorry; if once you can convert the system of linear equations into its canonical form, you can obtain the solution very easily.

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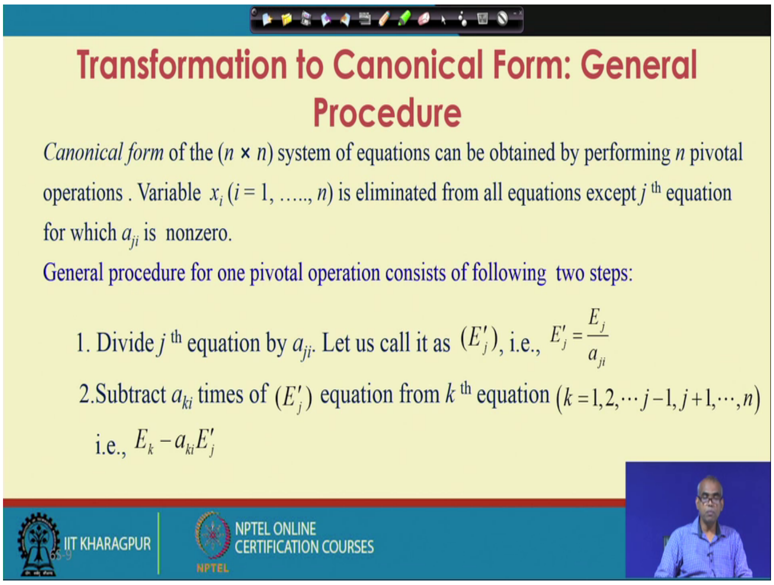

So, here is the general procedure for transformation of the set of equations to canonical form. Canonical form of the n by n system of equations can be obtained by performing n pivotal operations; variable xi, i equal to 1 to n is eliminated from all equations except jth equation for which a ji is nonzero.

So, variable xi is eliminated from all equations except jth equation and for jth equation the coefficient a ji is nonzero. So, general procedure for one pivotal operation consists of the following 2 steps. First divide jth equation by a ji; let us call it as E j prime.

So, E j prime equal to E j by a ji; now subtract a k i times of E j prime equation from kth equation; that is E k minus a ki E j prime. So, by this you can convert the system of linear equations to its canonical form; that was about square system.

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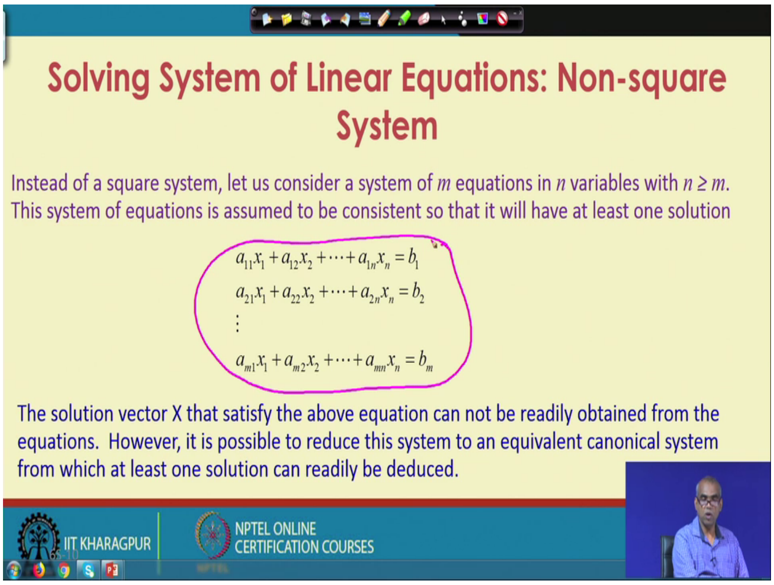

What about non square system where you have m equations in n variables and n is greater than m. So, instead of square system let us consider a system of n equations in n variables with let us say n greater than m..This system of equations is assumed to be consistent so that it will have at least one solution. So, now I have this set of linear equations note that we have now m rows, and we have now n variables.

So, if we look at the coefficient matrix, it will have m rows n columns. The solution vector x that satisfy the above equation cannot be readily obtained from the equations; however, it is possible to reduce the system to an equivalent canonical system from which at least one solution can readily be obtained. Now we are talking about a situation where you have n variables and m equations and let us consider n is greater than m. So, now, let us see how we obtain the canonical form for such systems so, that one solution can be obtained.

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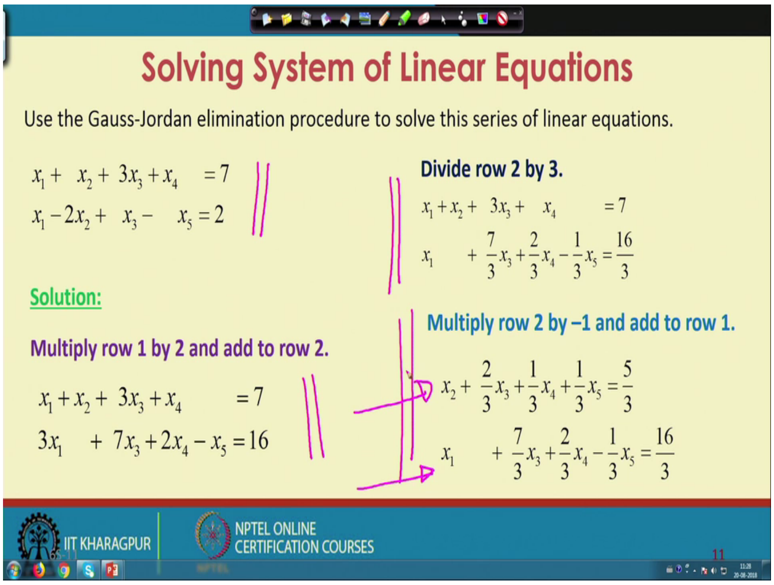

So, let us consider an example; x 1 plus x 2 plus 3 x 3 plus x 4 equal to 7 x 1 minus 2 x 2 plus x 3 minus x 5 equal to 2. So, I have 2 equations a number of variables is 5 x 1 x 2 x 3 x 4 and x 5; so, 5 variables 2 equations. So, let us convert this system of equation to its canonical form; first we multiply row 1 by 2 and add to row 2, if we do that you obtain this.

Next we divide row 2 by 3; you obtain this and then multiply row 2 by minus 1 and add to row 1. Now look at the system of equations that we obtained, if you look at this equation you have x 1 then 0 into x 2 no x 2. So, x 1 1 into x 1 plus 0 into x 2 and then you have x 3 x 4 x 5 and it has coefficients then that is equal to 16 by 3 .

If you look at this equation it is 0 into x 1 plus 1 into x 2 then plus x 3 x 4 x 5 you have along with its coefficients and it is a sinusoid equal to 5 by 3. So, this equation these 2 set of equations; if you look at only x 1 and x 2 they are in the canonical form.

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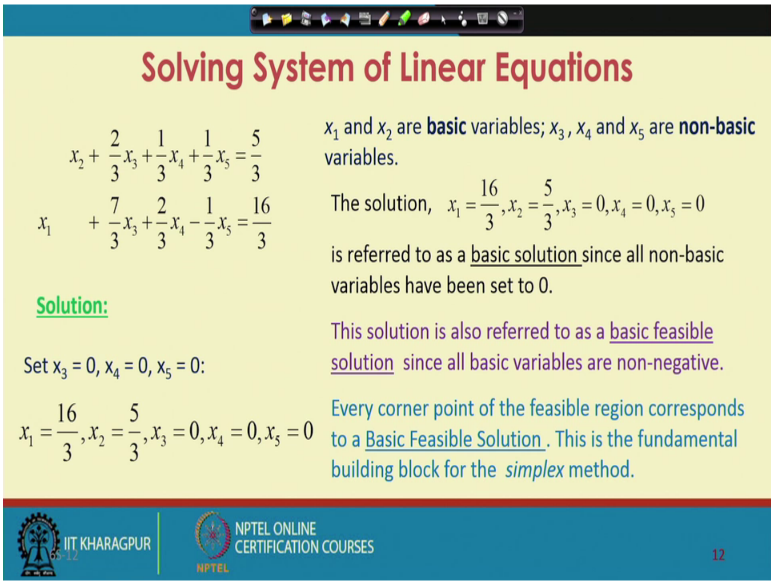

So, if I said x 3 equal to 4 x 4 equal; so, if I said x 3 equal to 0 x 4 equal to 0 and x 5 equal to 0, I can obtain the solution as x 1 equal to 16 by 3 x 2 equal to 5 by 3, note that you have 2 equations and 5 variables. So, setting arbitrarily any 3 variables to 0; I can solve for the remaining 2. So, by setting x 3 equal to 0 x 4 equal to 0 and x 5 equal to 0 ; I obtain x 1 equal to 16 by 3, x 2 equal to 5 by 3 readily .

So, this is the equivalent canonical form that we discussed for square system n by n. We have a non square systems and for this x 1 and x 2 part here, I have that canonical form. So, the solution is x 1 equal to 16 by 3 x 2 equal to 5 by 3, x 3 equal to 0 x 4 equal to 0 x 5 equal to 0 ; so, this is one solution; x 1 and x 2 are basic variables; c 3 x 4 and x 5 are non basic variables.

So, the variables whose values we arbitrarily set to 0 are non basic variables, here x 3 x 4 x 5 are non basic variables and x 1 and x 2 are basic variables. The solution that we have obtained by setting non basic variables to 0; that is x 1 equal to 16 by 3 x 2 equal to 5 by 3 x 3 equal to 0 x 4 equal to 0 x 5 equal to 0 is referred to as basic solution since, all non basic variables have been set to 0.

This solution is also referred to as the basic feasible solution since all basic variables are non negative. If you look at the basic solution the solution has all basic variables as non negative. So, this solution is also referred to as basic feasible solution.

Every corner point of the basic of the every corner point of the feasible region corresponds to a basic feasible solution; this is the fundamental building block for the simplex method. Every corner point for the feasible region corresponds to a basic feasible region and this is the fundamental building block for the simplex method.

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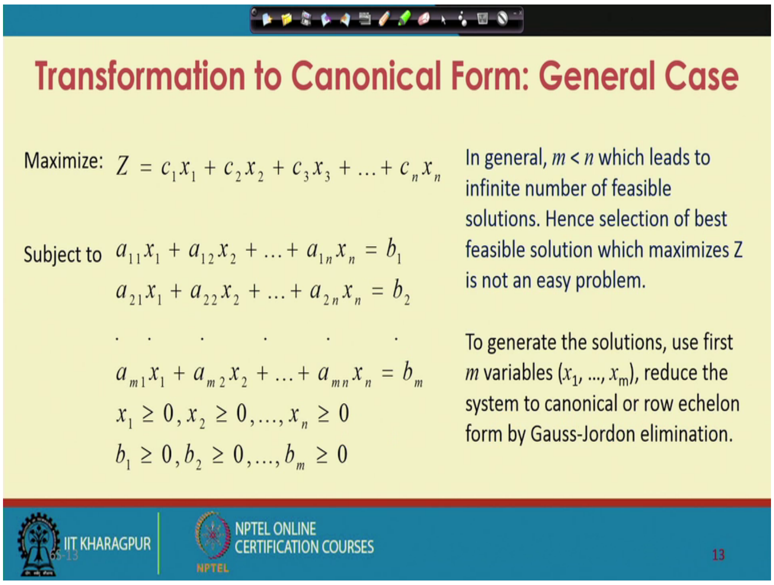

Now, let us transform a set of equations to a canonical form in a more general case. So, we have the objective function Z equal to c one x 1 plus c 2 x 2 up to cn xn. And we have m rows of constraints; that means, m equality constraints in and we have n variables and m is less than n. So, m is less than n; that means, you have number of variables more than the number of constraints.

So, these needs to infinite number of feasible solutions, so, we have a system where you have more variables fewer equations. So, this leads to infinite number of feasible solutions and therefore, the selection of best feasible solution that maximizes the objective function is not a trivial problem.

It is not an easy problem, to generate the solutions; let us first use first m variables to reduce the system to canonical or row echelon form by Gauss Jordan elimination. So, you have m constraints n variables and m is less than n. So, I can choose any m variables and then can reduce the system to a canonical form or row echelon form. So, the remaining variables I can set to 0 and can solve for m variables because, you have n variables and m equations in that case you can solve only for m variables.

So, what we do is we take the first m variables x 1 x 2 up to xn and reduce the system to canonical or row echelon form by say Gauss Jordan elimination method.

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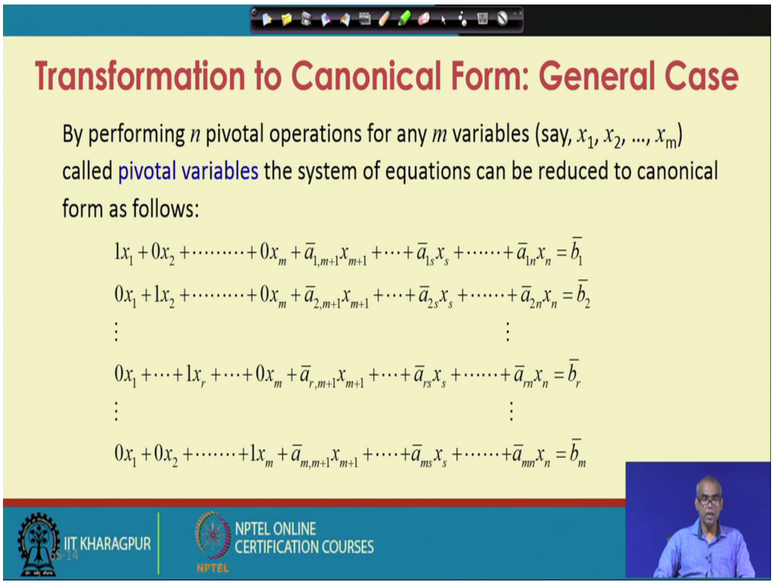

So, by performing n pivotal operations for any m variables say x 1 x 2 to xm call pivotal variables the system of equations can be reduced to canonical form as shown. So, we have n variables and m equations, so, first focus your attention on first m variables. So, this part is similar to m by n square system. So, here I will have canonical form as discussed in m by n square system and I can do this by doing the pivotal operations or doing these elementary row operations.

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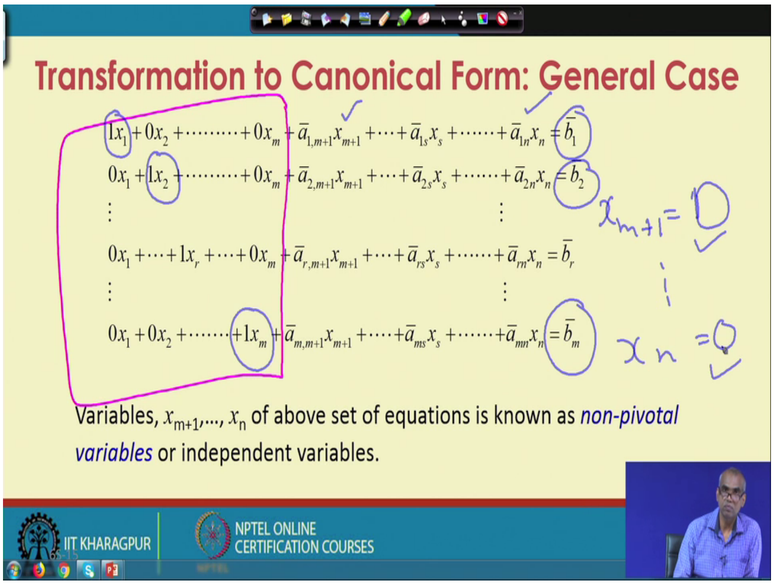

The variables xm plus 1 to xn of the above set of equations is known as non pivotal variables or independent variables; so, x 1 to xm are pivotal variables and xm plus 1 to xn are non pivotal or independent variables.

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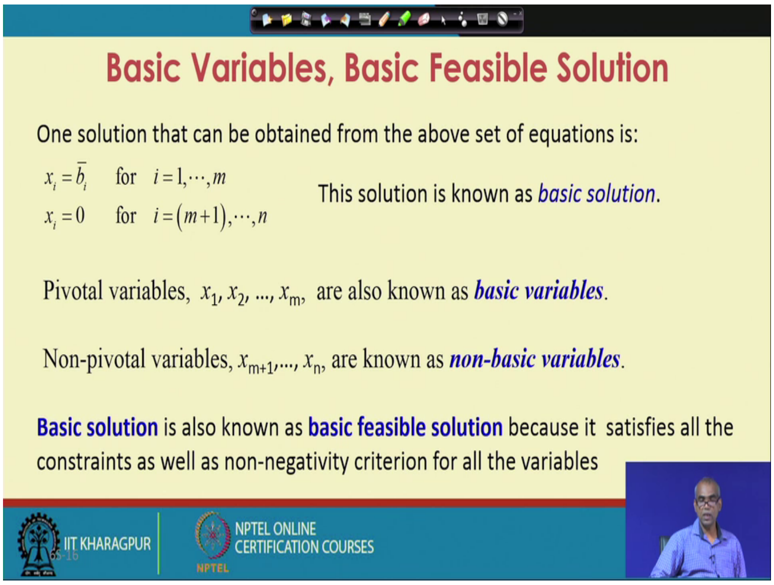

So, one solution that can be obtained from the above set of equation is ; xi equal to bi bar for i equal to 1 to m and xi equal to 0 for i equal to m plus 1 to m..

So, how all we are doing is we are setting x m plus 1 to xn as 0 . Then the solution can easily be obtained as x 1 equal to b 1 bar x 2 equal to b 2 bar xn equal to bm bar and xm plus 1 equal to 0 up to xn equal to 0.

So, one solution that can be obtained easily from the above set of equation is xi equal to bi bar for i equal to 1 to m and xi equal to 0 for I equal to m plus 1 to m this solution is known as basic solution. The pivotal variables x 1 x 2 up to xm are also known as basic variables, non pivotal variables xm plus 1 to xm are known as non basic variables. Basic solution is also known as basic feasible solution because; it satisfies all the constraints as well as non negativity criterion for all the variables

So, what we learn that we can convert a set of linear equations in n variables m constraints or n equations to an equivalent canonical form and then if I take the first m variables as basic variables; the remaining m plus 1 to n variables as non basic variables, I can set those non basic variables to 0 and can solve for the basic variables . The basic solution is also known as basic feasible solution, because it satisfies all the constraints.

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With this we stop our discussion on lecture 40 or week 8 here. In the next week, we will talk about simplex method and these concepts of basic solution basic feasible solutions we will utilize for discussion on simplex methods.