Optimization in Chemical Engineering
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Lecture – 39
Introduction to Linear Programming (Contd.)

Welcome to lecture 39. This is week 8 and we are talking about Introduction to Linear Programming Problems. In our previous lectures we have introduced graphical solution of linear programming problem. Graphical solution of linear programming problem is convenient only when you have two variables. When you have more variables, the graphical solution is not a convenient method of solution. In the next week we will talk about simplex method and its various adaptations. You will see that simplest methods can be used to solve linear programming problem of any variables.

In this lecture and in the following lecture we will lay some foundations for discussions on simplex methods that we will do in the next week. So, in this lecture in particular we will talk about how to express a linear programming problem in standard form.

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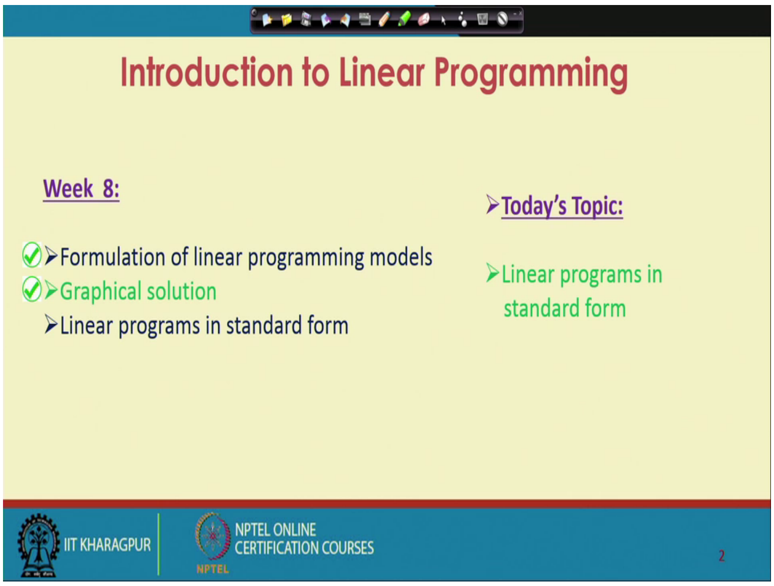

In general a linear programming problem may be either a maximization problem or a minimization problem. It may include both equalities and inequalities as constraints. It may have negative variables as well as positive variables.

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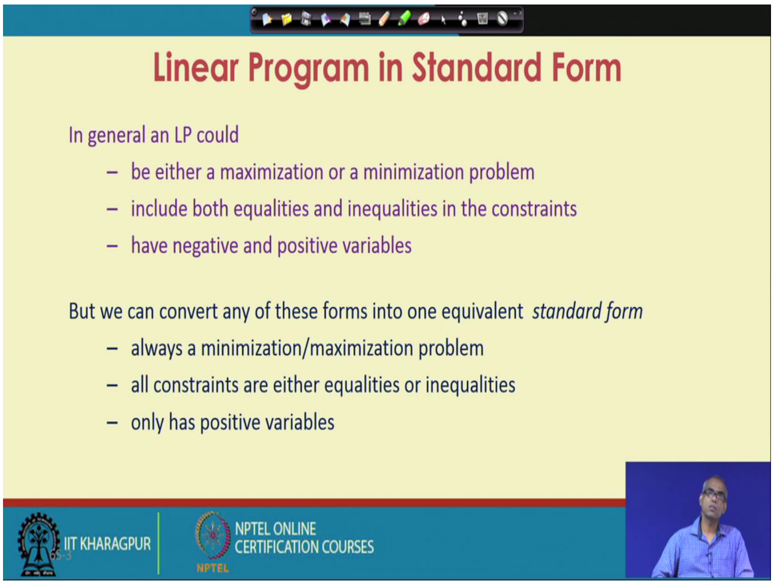

But we can convert any of these forms into one equivalent standard form where the problem is always posed either as minimization or maximization problem, where all the constraints either equalities or inequalities and all the variables are positive. So, what we are basically saying is that in general a linear programming problem may be a maximization problem or a minimization problem. It may include both equalities and inequalities as constraints. It may have some positive variables some negative variables.

But it is always possible for us to convert an LPP of any of these forms to a standard form. And by standard form I mean that the linear programming problem may be a minimization problem, all the constraints may be of are equality types, and all the variables will be positive. So, it is possible for us to convert any linear programming problem in this form where the problem may be of minimization, all the constraints will be of equality types and all the decision variables will take non-negative values. So, we will see how to do that.

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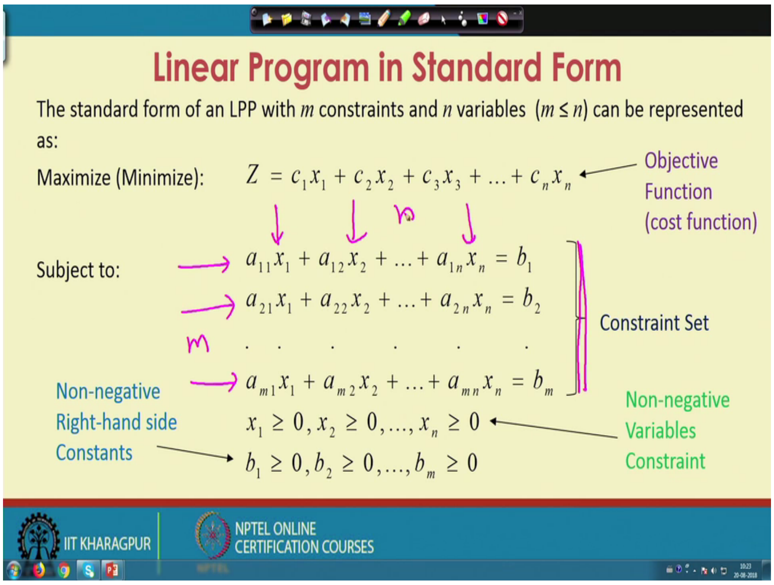

So, the standard form of an linear programming problem with m constraints and n variables, where n is greater than m can be represented as shown. Note that the objective function is a linear function of the decision variables, so either maximize or minimize often times we talk about minimization only because we know the maximization then can be obtained by just taking negative sign of the problem. So, the objective function Z equal to c 1 x 1 plus c 2 x 2 plus c 3 x 3 up to c n x n the objective function is also known as cost functions.

So, c 1, c 2, c 3, c n they are all numbers associated with cost. Actually you have seen the objective functions such as Z equal to 600 x 1 plus 300 x 2, so 600 300 where the cost associated with the fertilizer problems. So, c 1 equal to 600, c 2 equal to 300 so on and so forth. So, the objective function is written as a linear function of decision variables. Objective function is also known as cost function. Then all the constraints are written as equations a 11 x 1 plus a 12 x 2 up to a 1 n x n equal to b 1.

So, you have n variables x 1, x 2 up to x n, and you have m constraints so each constraint correspond to one equation. So, there are m rows of equations and n columns of variables. So, there are m rows of equations corresponding to m constraints and there are n columns of variables corresponding to n variables. And a 11, a 12, a mn all these things are coefficients.

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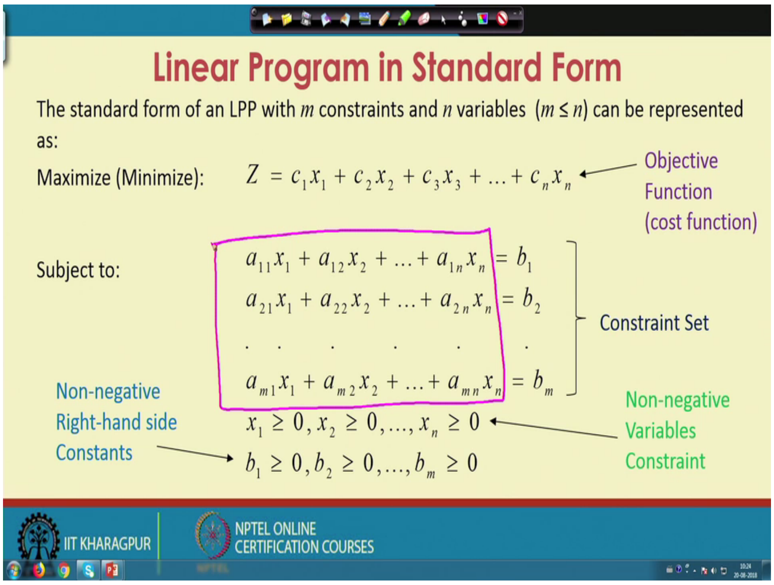

So, all these constraints which are of equally type. The left hand side of these constraints can be written as a matrix A x, where x represents the decision vector x 1, x 2 up to x n and the matrix a represents a matrix of m rows n columns and the elements are the coefficients a 11, a 12 up to a 1n, a 21, a 22, a 2n and finally, a m1, a m2, up to a mn.

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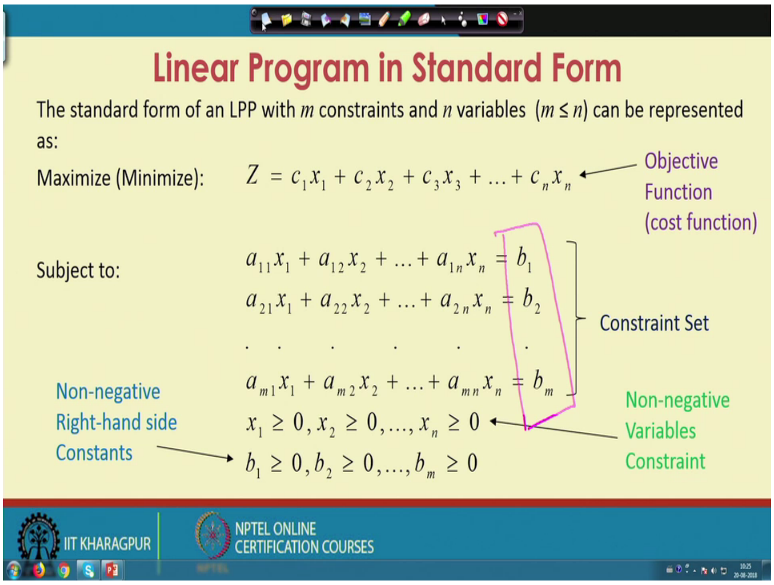

B 1, b 2 up to b m represents the right hand side part of the constraints x 1 greater or equal to 0 x 2 greater or equal to 0 up to x n greater or equal to 0 represents non-negative variable constraints.

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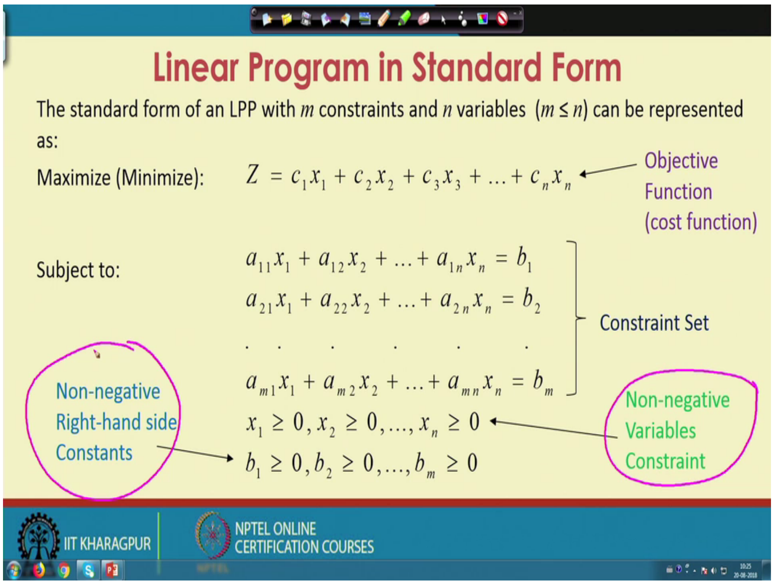

And b 1 greater or equal to 0 b 2 greater or equal to 0 up to b m greater or equal to 0 represents non-negative right hand side constraints. So, in the standard form we will express the right hand side of the constraints as non-negative.

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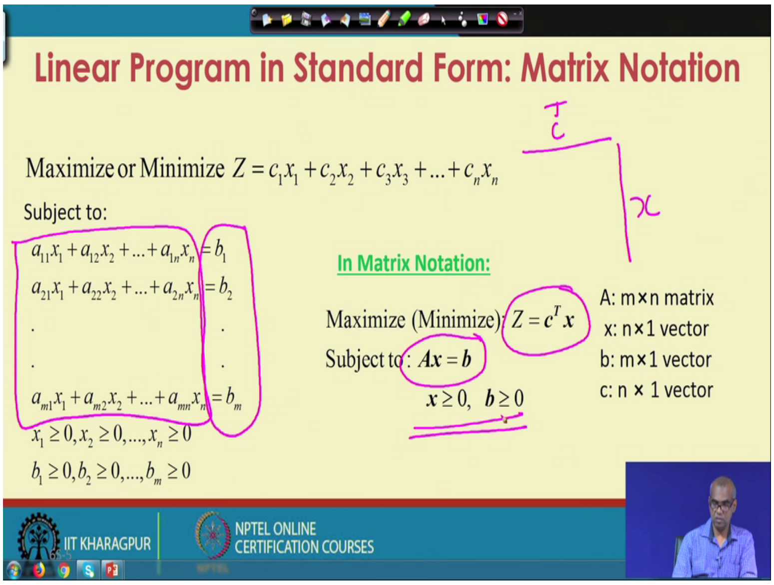

So, you can represent these equations compactly using matrix notations. Note that using matrix notation we can present the linear programming problem as maximize or minimize Z equal to c transpose x, c is a vector n cross one vector n vector and x is also an n vector.

So, Z equal to c 1 x 1 plus c 2 x 2 up to c n x n will be valid when we take the c transpose and x 1 x. So, this is the c transpose vector and this is the x vector. So, the product will be c 1 x 1 plus c 2 x 2 up to c n x n. So, this is the objective function Z equal to c transpose x subject to A x equal to b x is n vector a is m cross n matrix which comes from the left hand side part of the equality constraints, all the constraints has been written as equality constraints, and b is the right hand side part of the constraints which will be written as non-negative.

So, if any of these b is negative you have to multiply both sides by minus 1 so that the right hand side part becomes positive, and then you have these non negativity statements x greater or equal to 0, b greater or equal to 0. So, a is m by n matrix x is n vector c is n. So, n vector and b is the right hand side part of the m equality constraints. So, b is m vector.

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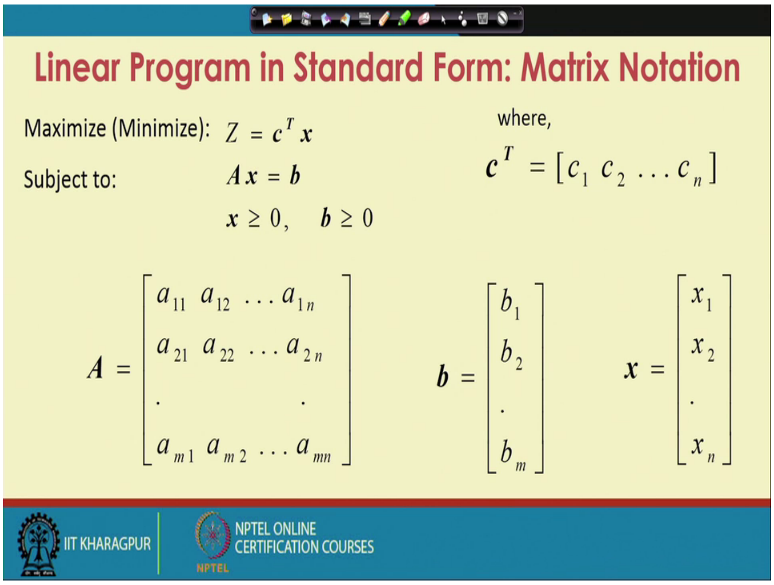

So, the matrix notation is again presented here more elaborately.

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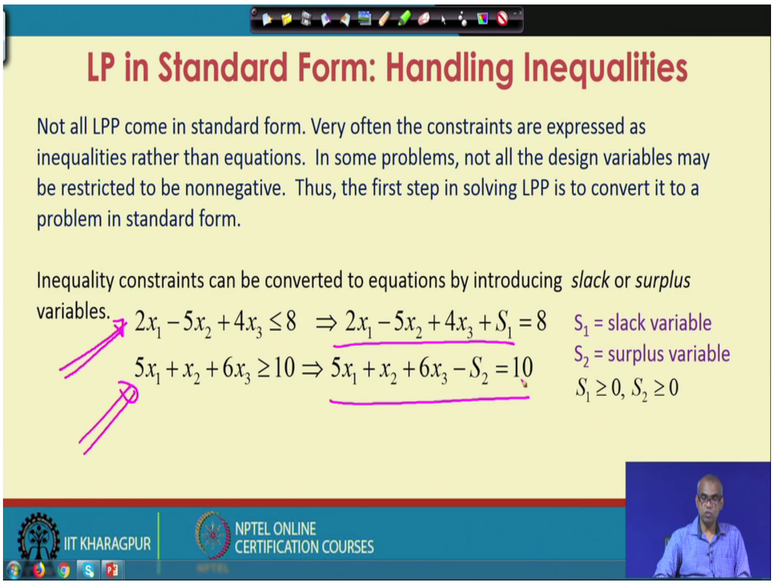

Now, all linear programming problem will not come in standard form. Very often the constraints are expressed as inequalities rather than equations in some problems not all the decision variables may be restricted to be non-negative. There are cases where some decision variables may be unrestricted in sign that means, there is no sign restriction on the values of the variable that means, that variable can take positive values as well as negative values, including 0.

So, in other words all linear programming problem will not come in standard form. So, the first step in solving a linear programming problem is to convert it to a problem in standard form. And in standard form we will express the linear programming problem let us say as minimization problem or may be maximization problem. We will have all the constraints as equations and the right hand side of all these equations will be non-negative. There will not be any variable which is unrestricted in sign because we will have non-negativity constraint on all variables.

So, the problem may be of minimization type all the constraints are equations non negativity constraints on all the variables as well as the right hand side vector. Inequality constraints can be converted to equations by introducing slack or surplus variables. Suppose I have a linear programming problem, where I have inequality constraints the inequality constraint may be less or equal to type or greater or equal to type. So, I have to convert this less or equal to type inequality constraint or greater or equal to type inequality constraint to equations and that I can do by introducing slack or surplus variables. For example, let us consider this inequality constraint 2 x 1 minus 5 x 2 plus 4 x 3 is less or equal to 8. So, the left hand side part is less or equal to 8. So, to make the left hand side equal to 8 I must add some non-negative value to the left hand side so that it becomes equal to 8.

So, I add S 1 as a slack variable to the left hand side. So, my constraint becomes 2 x 1 minus 5 x 2 plus 4 x 3 plus S 1 equal to 8. So, S 1 is the slack variable which when added to the left hand side part of the original inequality constraint will be converted to an equality constraint the left hand side will be equal to 8. Similarly if I have a constraint of greater or equal to type for example, 5 x 1 plus x 2 plus 6 x 3 is greater or equal to 10.

So, to make the left hand side equal to 10 I must subtract apart from the left hand side. So, I introduce a surplus variable S 2 and rewrite the constraint as 5 x 1 plus x 2 plus 6 x 3 minus S 2 equal to 10. Note that both S 1 and S 2 must be non-negative, so S 1 greater or equal to 0 S 2 greater or equal to 0. So, by introducing to non-negative slack variable or surplus variable it is possible for me to convert an inequality constraint to an equality constraint.

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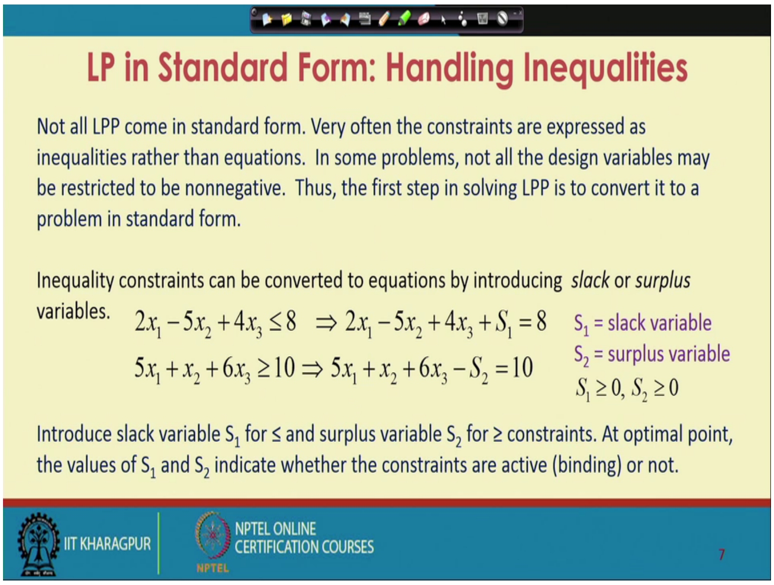

When the inequality constraint is less or equal to type I have to introduce slack variable and when the inequality constraint is of greater or equal to type I have to introduce surplus variable. Note that the slack variable is added and the surplus variable is subtracted. At optimal point the values of this slack variable and surplus variables indicate whether the constraints are active that means, binding or not. Remember that a constraint is active when it is satisfied as 0.

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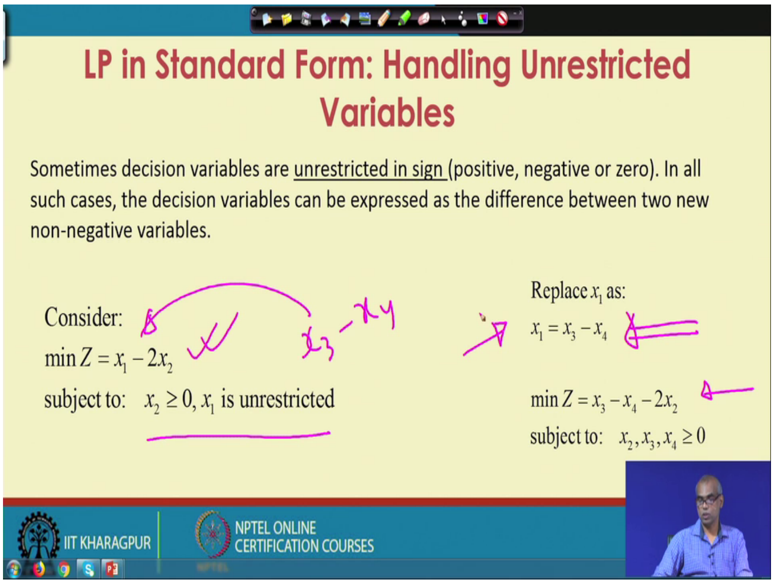

Now, how do I handle a decision variable that is unrestricted in sign? So, a decision variable which is unrestricted in sign is known as unrestricted variable. Sometimes decision variables are unrestricted in sign that means, they can take positive negative or 0 values in all such cases, the decision variables can be expressed as the difference between two new non-negative variables.

Note that any variable that is unrestricted in sign can be expressed as difference of two non-negative variables. You can choose two non-negative variables appropriately such that their difference may be equal to 0 it may be less than 0 or it may be greater than 0.

So, a variable which is unrestricted in sign that means, unrestricted variable can be expressed as a difference of two non-negative variables. So, this is how I can handle unrestricted variables. For example, consider minimize Z equal to x 1 minus 2 x 2 subject to x 2 greater or equal to 0, but x 1 is unrestricted. So, what I have to do is I first have to express x 1 as a difference of two non-negative new variables. So, I replace x 1 as x 3 minus x 4. So, I introduce two new non-negative variables x 3 and x 4 and replace x 1 as x 3 minus x 4.

So, now, my decision my objective function Z equal to x 1 minus x 2 becomes x 3 minus x 4 minus 2 x 2 and of course, the non-negativity constraints. Now, become x 2 greater or equal to 0 x 3 greater or equal to 0 x 4 greater or equal to 0. Note that the new problem formulation does not contain x 1 because x 1 has been replaced as difference of two new non-negative variables x 3 minus x 4.

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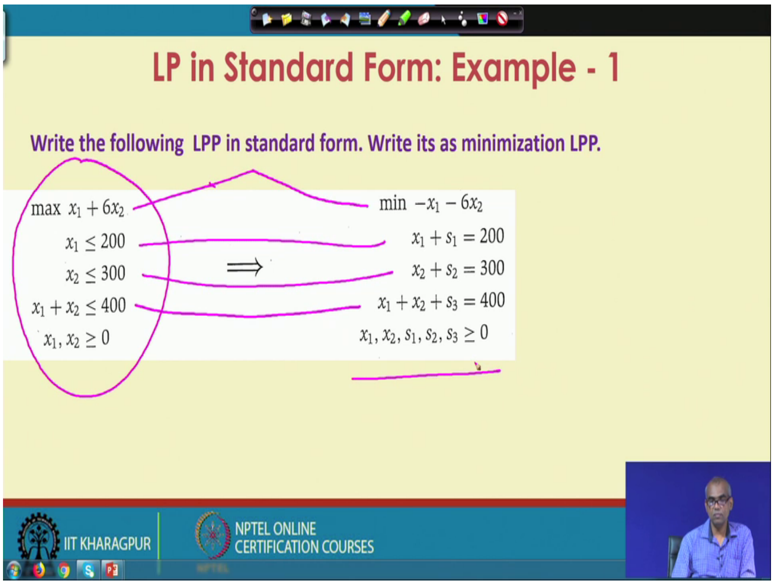

Let us now take an example of converting a given linear programming problem in standard form. Let us say the original problem is given as maximize x 1 plus 6 x 2 x 1 less or equal to 200, x 2 less or equal to 300, x 1 plus x 2 less or equal to 400 and x 1 x 2 greater or equal to 0. Let us say want to express this linear programming problem in standard form and let the standard form be minimization type.

So, maximize x 1 plus 6 x 2 becomes minimize minus x 1 minus 6 x 2 x 1 is less or equal to 200, so I have to introduce a slack variable x 1 plus S 1 equal to 200. Similarly x 2 is less or equal to 300, I introduce slack variable S 2 and write x 2 plus S 2 equal to 300; x 1 plus x 2 less or equal to 400, again I have to introduce a slack variable and rewrite the constraint as x 1 plus x 2 plus S 3 equal to 400, then I will have non-negativity constraint on x 1 and x 2 as well as S 1 S 2 and S 3.

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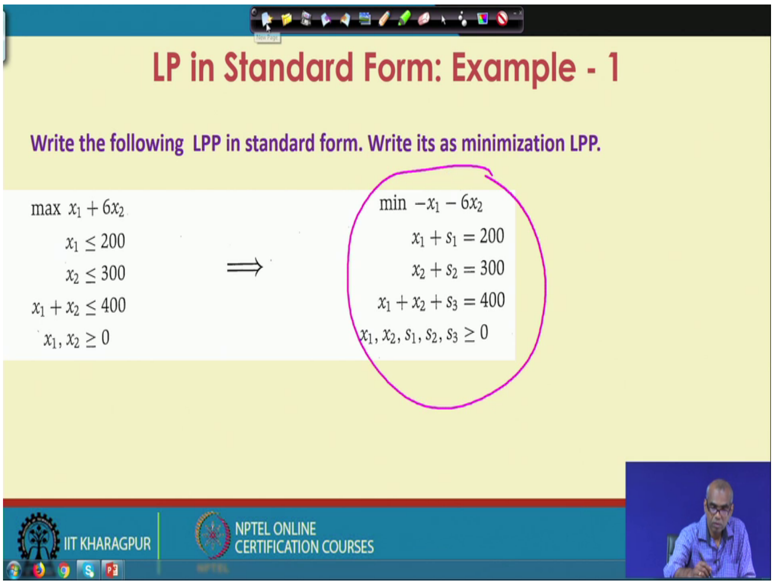

So, this is the standard form of the linear programming problem.

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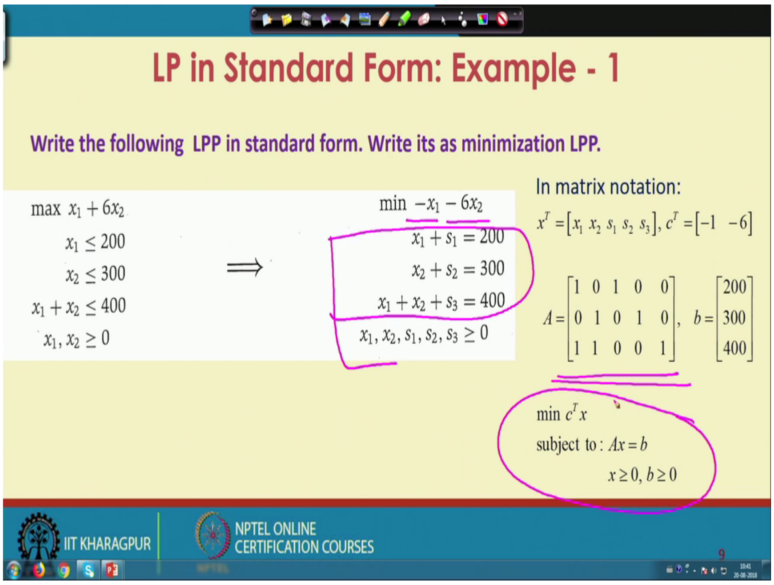

We can also express these in matrix notation. So, my decision variable vector is x 1 x 2 S 1, S 2, S 3 the cos vector is minus 1 minus 6 the A matrix can be obtained and the b matrix is 200, 300 and 400. So, the problem formulation becomes minimization c transpose x subject to A x equal to b, x greater or equal to 0, b greater or equal to 0. Let us take another example.

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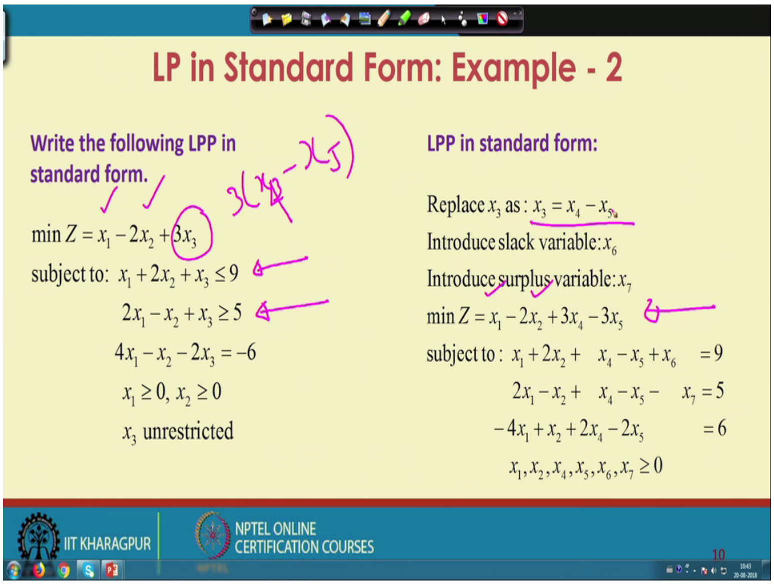

So, you want to write the linear programming problem in standard form the problem is minimize Z equal to x 1 minus 2 x 2 plus 3 x 3 subject to x 1 plus 2 x 2 plus x 3 less or equal to 9. So, it requires a slack variable. 2 x 1 plus 2 x 1 minus x 2 plus x 3 is greater or equal to 5 it requires a plus variable; 4 x 1 minus x 2 minus 2 x 3 equal to minus 6, so this requires multiplication, both sides by minus 1 so that the right hand side becomes positive. X 1 is greater or equal to 0 x 2 is greater or equal to 0, but x 3 is unrestricted, so x 3 has to be replaced as a difference of two non-negative new variables.

So, here is the solution we first replace x 3 as x 4 minus x 5 introduce two new non-negative variables x 4 and x 5. Introduce slack variables x 6 introduce surplus variable x 7. So, the objective function becomes x 1 minus 2 x 2, but 3 x 3 becomes 3 into x 3 x 4 minus x 5. So, 3 x 3 becomes 3 into x 4 minus x 5 because x 3 is replace as x 4 minus x 5. So, the objective function becomes Z equal to x 1 minus 2 x 2 plus 3 x 4 minus 3 x 5.

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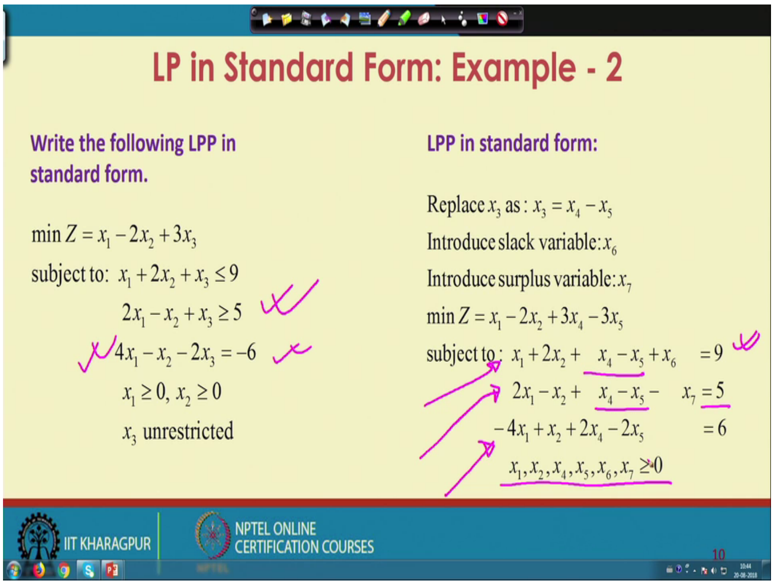

Similarly, we have introduce the slack variable x 6 as well as x 3 has been replaced by x 4 minus x 5; similarly both the equations. In the first equation we have introduce x is as surplus in slack variables and in the second equation we have introduce x 7 as surplus variable finally, the last constraint we have multiply both sides by minus 1 and then x 1, x 2, x 4, x 5, x 6, x 7 are all greater or equal to 0 because of non-negativity constraints. So, this is how you can convert any linear programming problem which is given in non-standard form to standard form.

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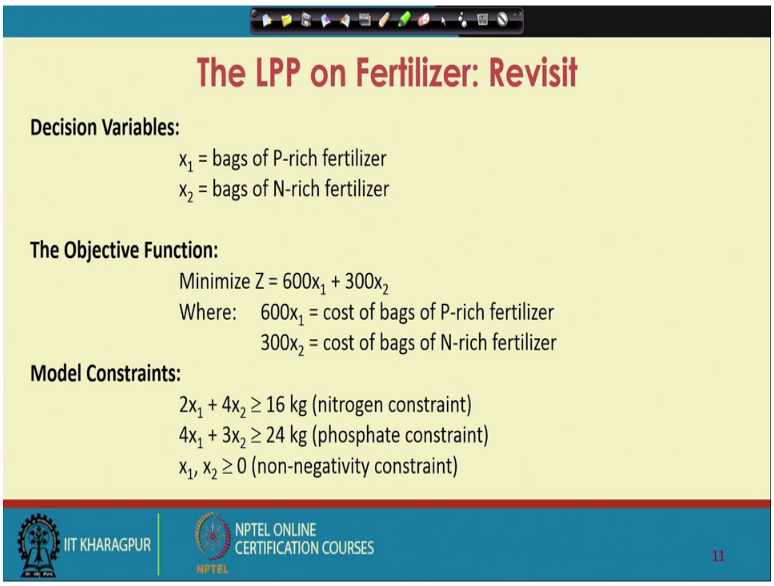

So, let us revisit the fertilizer problem. So, let us go to the problem as formulated.

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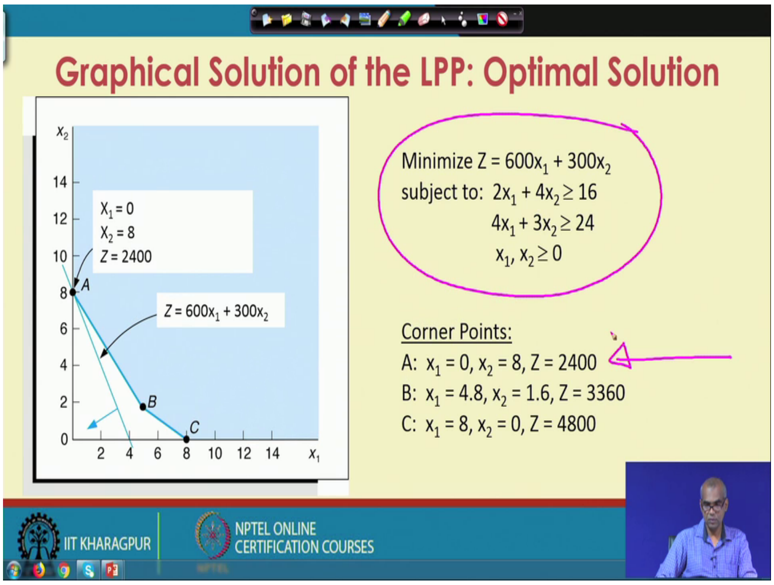

Minimize Z equal to 600 x 1 plus 300 x 2, 2 x 1 plus 4 x 2 is greater or equal to 16, 4 x 1 plus 3 x 2 is greater or equal to 24, and x 1 greater or equal to 0, x 2 greater or equal to 0. We have solved this problem and we have seen that corner point A is the optimal solution; the objective function value is minimum at point A.

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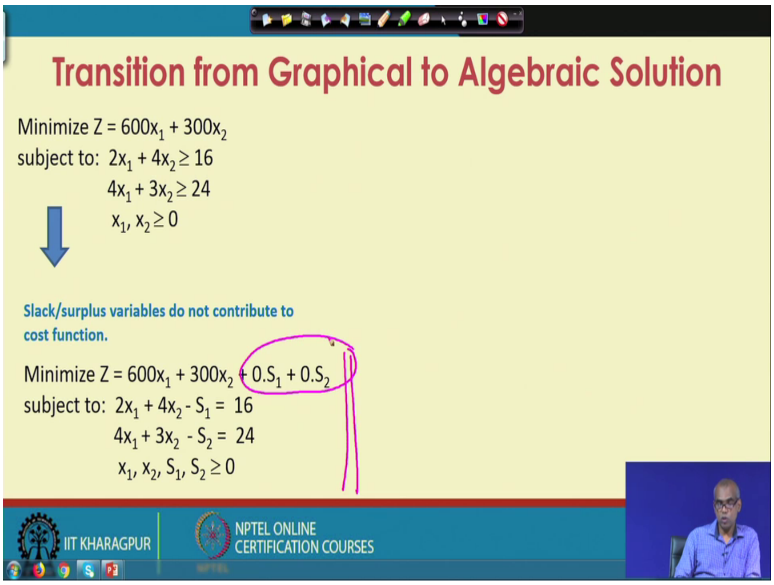

Now, let us introduce slack variables or surplus variables whichever is required to convert these inequality constraints to equality constraints. By doing that we will now see how do we transit from graphical solution to algebraic solution. This fertilizer problem we have solved in our previous lecture using graphical method. It is a problem with two variables and can be very conveniently solved by graphical method. When you have many variables the graphical method is not a convenient way of solving linear programming problem, and the simplex method which is basically an algebraic method must be used.

So, let us take this simple example and let us see how this transition is possible from graphical solution to algebraic solution. So, this fertilizer problem has to greater or equal to type constraints. So, I have to introduce surplus variables S 1 and S 2. And after introducing these two surplus variables my constraints are 2 x 1 plus 4 x 2 minus S 1 equal to 16 and 4 x 1 plus 3 x 2 minus S 2 equal to 24. Note that x 1, x 2, S 1, S 2 are all non-negative. Also note that when you introduce slack variables or surplus variables they do not contribute to cost function. So, the cost in the cost function the slack variables or surplus variables do not appear, that is why I have multiplied S 1 and S 2 by 0 there.

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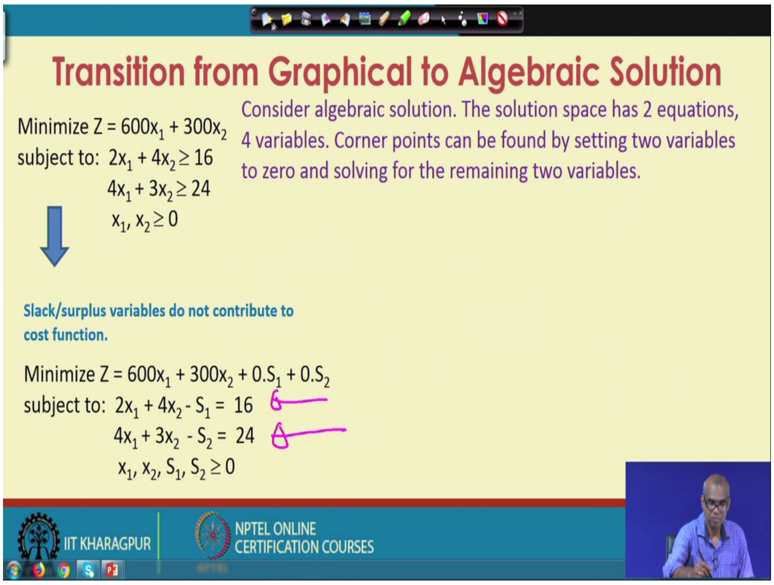

Now, we have this 4 variables now, x 1, x 2, S 1 and S 2 and I want to consider algebraic solution. The solution space has 2 equations and 4 variables, then how do I find the corner points. The corner points can be found by setting two variables to zero and solving for the remaining two variables, you have two variables, you have 4 variables and two equations we have 4 variables and 2 equations. So, you can set two variables to zero and can solve for the remaining two variables and by doing this we can find all the corner points.

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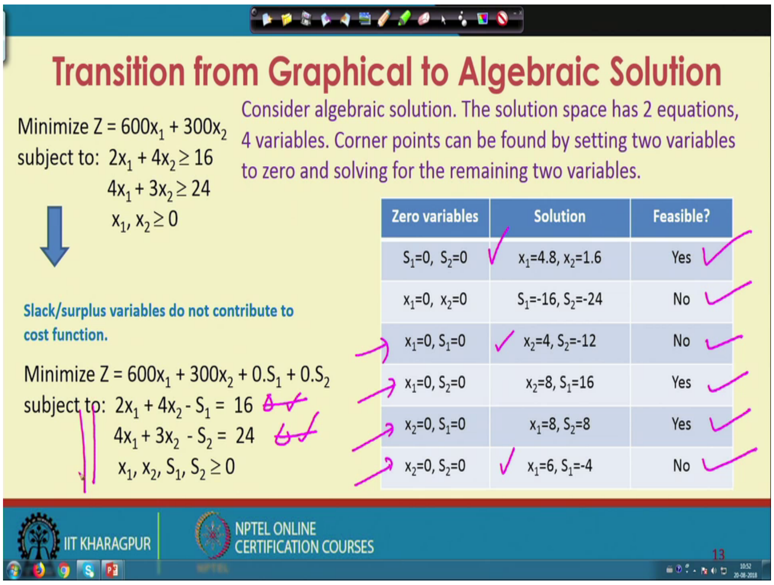

So, there are 6 combinations possible you can set both S 1 and S 2 equal to 0, you can set both x 1 and x 2 equal to 0; you can set x 1 equal to 0, S 1 equal to 0, x 1 equal to 0, x 2 equal to 0 and x 2 equal to 0, S 1 equal to 0, x 2 equal to 0, S 2 equal to 0. So, all this 6 case is possible.

And when you do this you can easily find out the optimal solution algebraically because in that case we will have two variables, two equations. So, you can completely solve. For example, if I set S 1 equal to 0 and S 2 equal to 0 in these two equations I have 2 x 1 plus 4 x 2 equal to 16 and 4 x 1 plus 3 x 2 equal to 24, two equations, two variables I can easily solve to get x 1 equal to 4.8, x 2 equal to 1.6.

The question I ask is now is this solution feasible that means, does this solution satisfy all the constraints. Yes it does, because x 1 x 2 are all greater or equal greater than 0 similarly S 1, S 2 both are 0. So, they satisfy S 1 is greater or equal to 0, S 2 greater or equal to 0. So, solution is feasible. In fact, this is the optimal solution as you have seen in previous lecture.

Similarly, if I set x 1 equal to 0 and x 2 equal to 0 the solution is straightforward S 1 equal to minus 16 and S 2 equal to minus 24 the solution is not feasible now, because both S 1 and S 2 must be greater or equal to 0. Similarly you can set x 1 equal to 0 S 1 equal to 0 and can obtain x 2 equal to 4 S 2 equal to minus 12, again it is not feasible because S 2 is negative.

You can set x 1 equal to 0 and S 2 equal to 0 and we will obtain x 2 equal to 8, S 1 equal to 16 it is a feasible solution. You can set x 2 equal to 0 and S 1 equal to 0 you get x 1 equal to 8, S 2 equal to 8, once again it is a feasible solution because all the constraints are satisfied; x 2 equal to 0 S 2 equal to 0 will give you solution as x 1 equal to 6, x 1 equal to minus 4. Now, this is not feasible because S 1 cannot have negative value.

So, it is possible for me to solve algebraically these set of equations where I have two equations 4 variables by setting two variables to 0, and solving the remaining. And I have a 6 scenarios, and I find that 3 solutions are feasible, 3 solutions are not.

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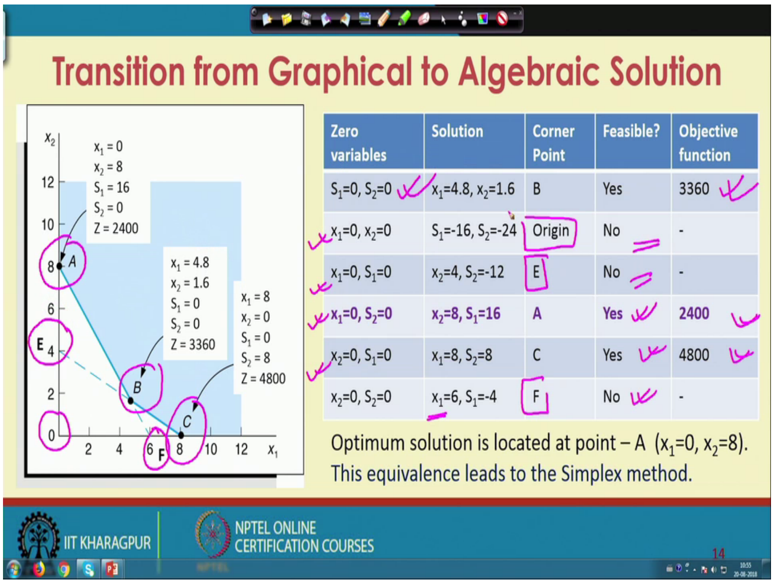

So, now, look at that graphical solution that we have seen in our previous lecture let us look at all these 6 points. S 1 equal to 0, S 2 equal to 0, this is point B which is a feasible solution with the objective function value 3360 x 1 equal to 0, x 2 equal to 0 this is the origin and it is not feasible; x 1 equal to 0, S 1 equal to 0, the solution is x 2 equal to 4 and S 2 equal to minus 12 this is point E, this is not feasible. Also look at graphically that this is outside the feasible region.

Similarly, x 1 equal to 0, S 2 equal to 0 the solution is x 2 equal to 8, S 1 equal to 16 and this is the corner point A. It is feasible and has 2400 as objective function value. In fact, this is the optimal solution. Then x 2 equal to 0, S 1 equal to 0 the solution is x 1 equal to 8, S 2 equal to 8 and this is point C which lies within the feasible region it is a feasible solution with objective function value 4800.

Finally, S 2 equal to 0, S 2 equal to 0 x 1 equal to 6 and S 1 equal to minus 4 as the solution. This corresponds to point F this is not feasible because it is lying outside the feasible region. Note that point E, point F and the origin are outside the shaded region, shaded region represents feasible region here. We get all these informations from the algebraic solution as well.

Optimum solution is located at point a as usual, and whatever conclusion we have drawn using graphical method of solution we draw the same conclusion using algebraic solution. The algebraic solution is obtained for 4 variable two equation system by taking two variables equal to 0, and remaining for the other two variables these, equivalence between this graphical solution and algebraic solution leads to the simplex method that we will discuss in our previous ways lecture. With this we stop our discussion on lecture 39 here.