Optimization in Chemical Engineering
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Lecture - 37
Introduction to Linear Programming (Contd.)

Welcome to lecture 37. This is week 8, and we are talking about Linear Programming Problems. In our previous lecture we have discussed how to formulate a linear programming problem. In this lecture we will discuss graphical solution method for linear programming problem. A linear programming problem in two variables can be conveniently solved using graphical methods. So, in this lecture we will see how to solve a linear programming problem in two variables using graphical methods.

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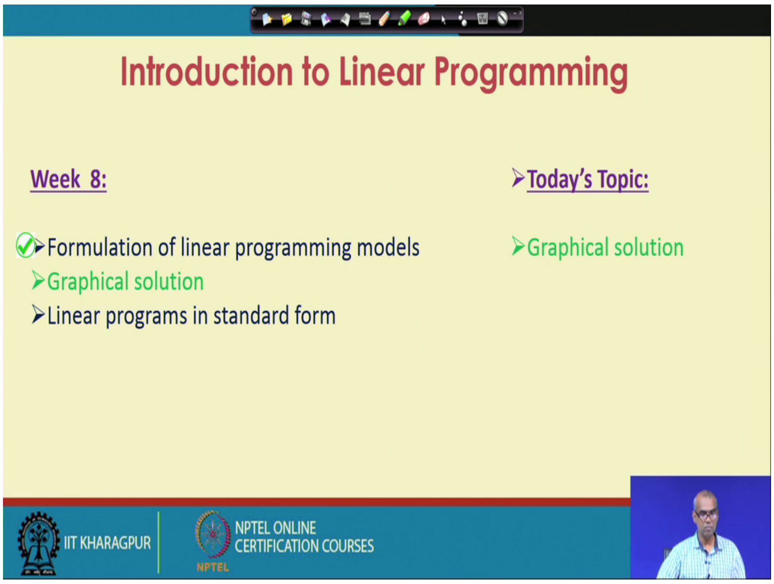

So, linear programming problems in two variables can be very conveniently solved by graphical method and following at the steps to solve a linear programming problem using graphical method.

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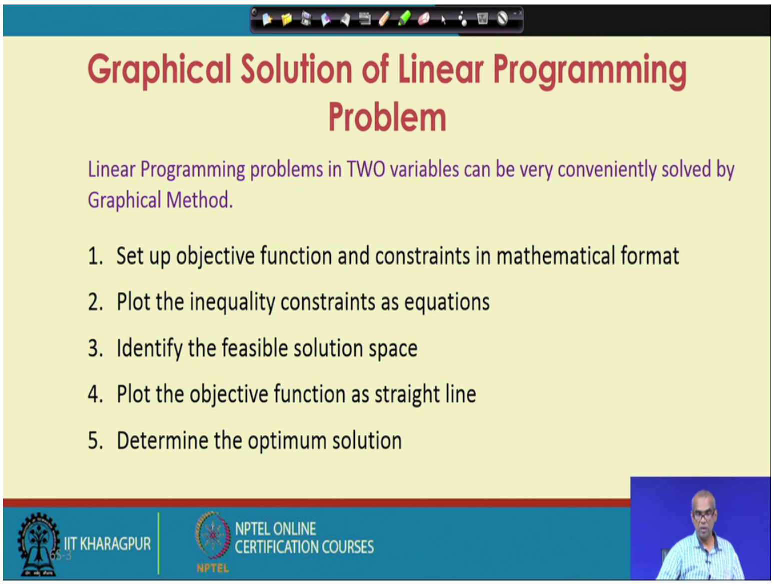

First set up objective function and constraints in mathematical format that means, formulate the linear programming problem. And we know that how to do that the objective function will be a linear equation in decision variables and all the constraints will be equality or inequality type again in decision variables.

Next we plot the inequality constraint as equations then you identify the feasible solution space which is defined by the constraints associated with the problem. Then you plot the objective function as a straight line it is a linear equation in decision variable, and then we determine the optimum solution.

So, these are the steps and you will see how this steps are followed to solve a linear programming problem in two variable using graphical method.

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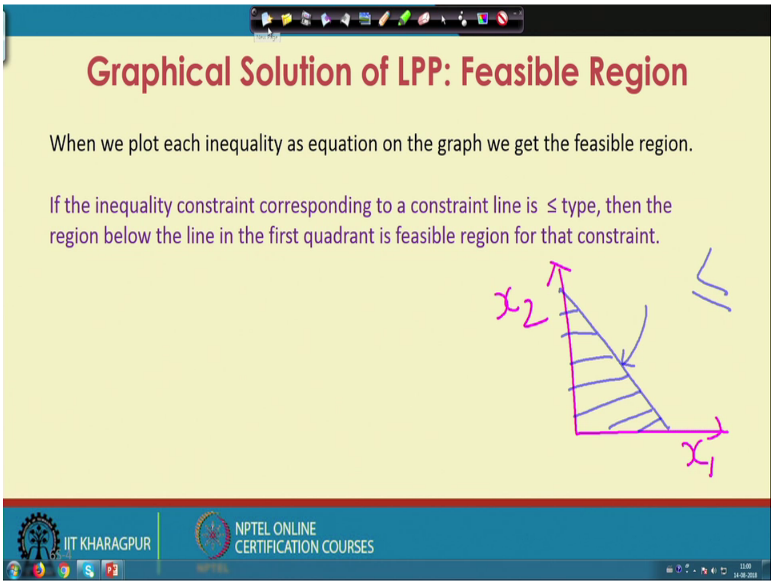

So, let us first define the feasible region. When you plot each inequality as equation on the graph we get the feasible region. If the inequality constraint corresponding to a constraint line is less or equal to type than the region below the line in the first quadrant is feasible region for that constraint.

So, what do you mean is, suppose we have a linear programming problem in two variables x 1 and x 2. Now, a constraint which is the linear type let us say I plot a constraint like this and this constraint is less or equal to type. So, the region below this line in the first quadrant is feasible region for this constraint that means, this shaded region is the feasible region for this constraint.

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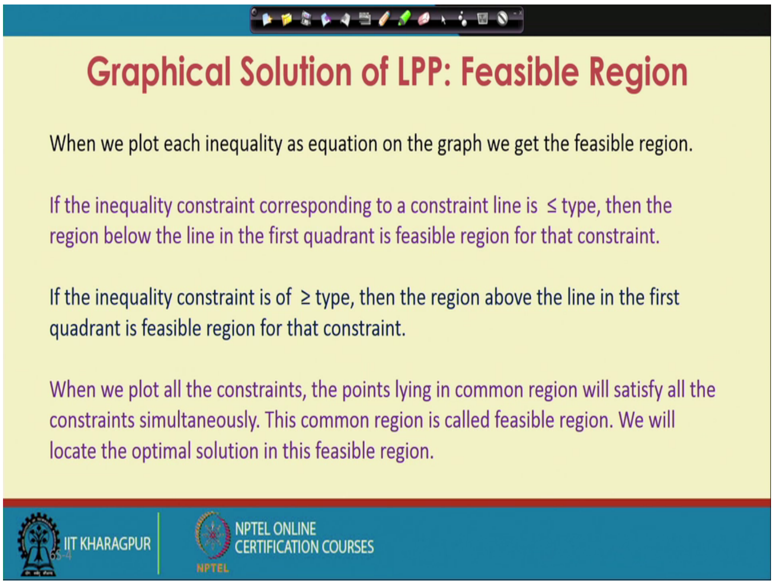

Similarly, if the inequality constraint is of greater or equal type, then the region above the line in the first quadrant is feasible region for that constraint.

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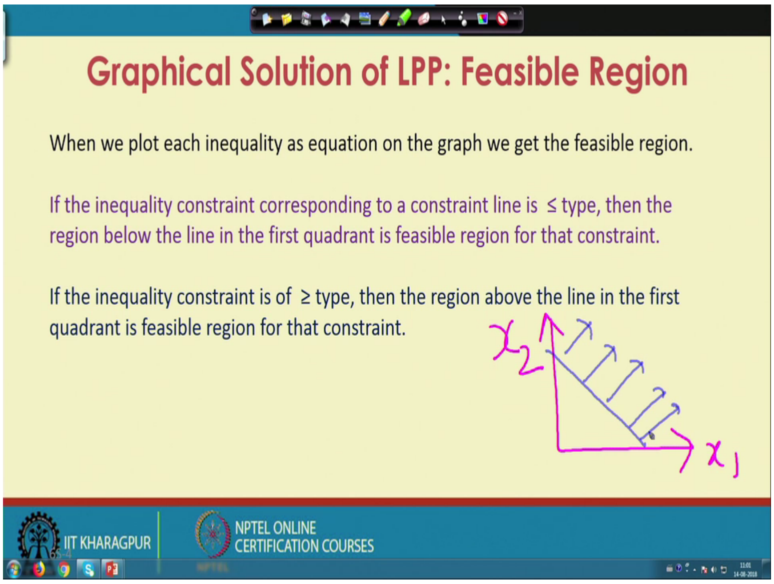

So, again what do you mean is, if you have a constraint greater or equal to type then this is the region which is feasible for this particular constraint. When you plot all the constraints the points lying in common region will satisfy all the constraints simultaneously. This common region is called feasible region. We locate the optimal solution in this feasible region.

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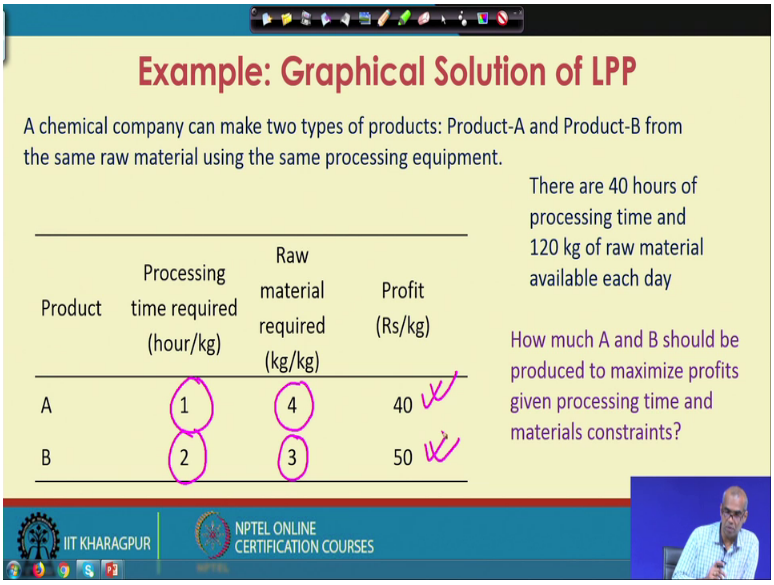

Now, let us take an example and solve the problem using graphical method. The problem is as follows a chemical company can make two types of products product A and product B from the same raw material using the same processing equipment.

Let us consider such a situation exists, that given raw material a certain raw material, the chemical company can produce either product A or product B using the same processing equipment. The processing time required to produce 1 kg of product A is 1 hour, and the processing time required to produce 1 kg of product B is 2 hours. The raw material required to produce 1 kg of product A is 4 kg, and the raw material required to produce product B is 3 kg.

The profit associated with 1 kg of product A is rupees 40 and the profit associated with product B, 1 kg of product B is rupees 50. If there are 40 hours of processing time and 120 kg of raw material available each day how much of A and how much of B should be produced to maximize the profit. And we must respect the processing time and the materials constraint that means raw constraints and processing time and the constraints and raw materials.

So, this is the problem. So, now, would like to solve it using graphical method. So, first step is to formulate the problem.

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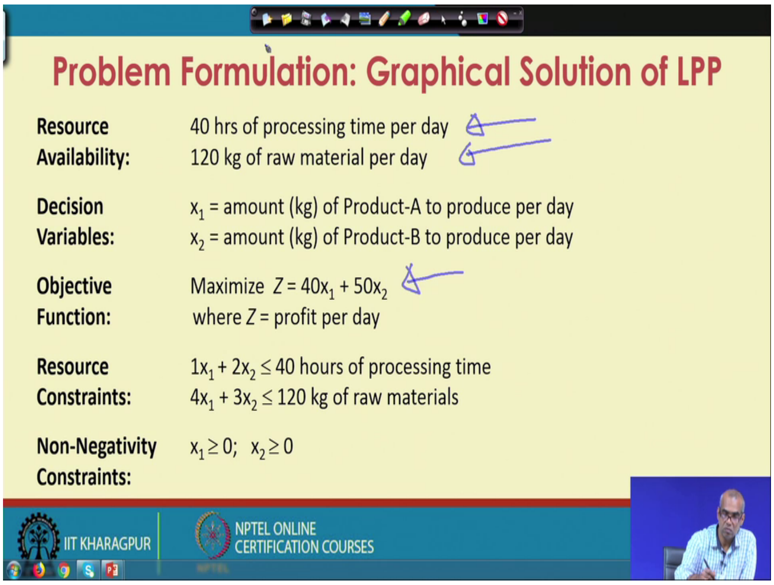

Let us look at resources available first 40 hours of processing time per day and 120 kg of raw materials per day. Let us choose the decision variables as x 1 equal to amount of product A to produce per day and amount of product B to produce per day we call it x 2. So, my production schedule that is decision variables are x 1 equal to amount of product A to produce per day and x 2 equal to amount of product B to produce per day.

1 kg of product A gives me 40 rupees of profit, so x 1 kg of product A gives me 40 x 1 amount of profit. Similarly 1 kg of product B gives a profit of rupees 50. So, x 2 kg of product B gives a profit of 50 into x 2. So, the total profit is Z equal to 40 x 1 plus 50 x 2 per day. So, this is a objective function Z equal to 40 x 1 plus 50 x 2 and we need to maximize this.

So, now look at the constraints. Constraints here two explicit constraints, one is associated with 40 hours of processing time per day other is 120 kg of raw materials per day. So, first look at 40 hours of processing time per day.

Now, look at the table one more time. 1 kg of product A requires 1 hour of processing time and 1 kg of product B requires 2 hours of processing time. So, x 1 kg of product A requires x 1 hour processing time and x 2 kg product B required 2 x 2 hours of processing time. So, the total processing time x 1 plus two x 2 must be less or equal to 40.

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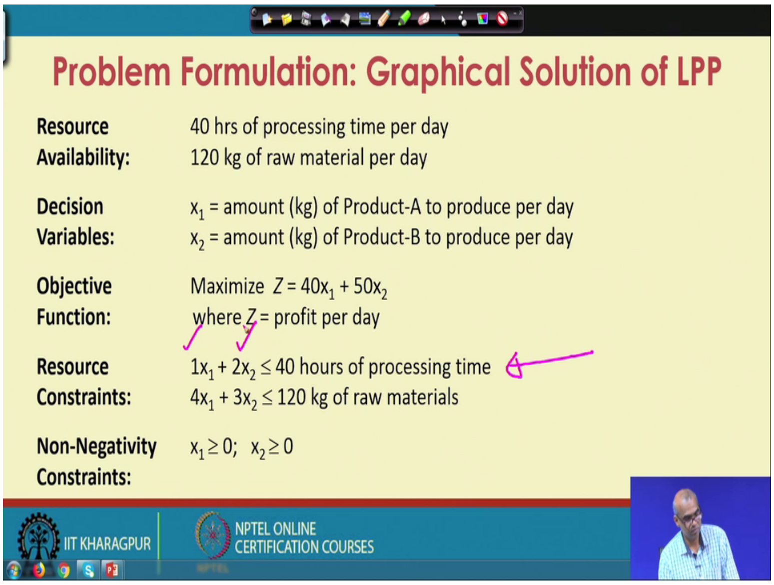

Similarly, you have constraints on raw materials 1 kg of product A requires 4 kg of raw materials. So, x 1 kg of product A requires 4 x 1 kg of raw materials and similarly x 2 kg of product B requires 3 x 2 kg raw materials. So, 4 x 1 plus 3 x 2 must be less or equal to 120 kg, which is available per day. So, these are two explicit constraints and then of course, there is non-negativity constraints because x 1 and x 2 cannot have negative values. So, x 1 is greater or equal to 0, x 2 is greater or equal to 0.

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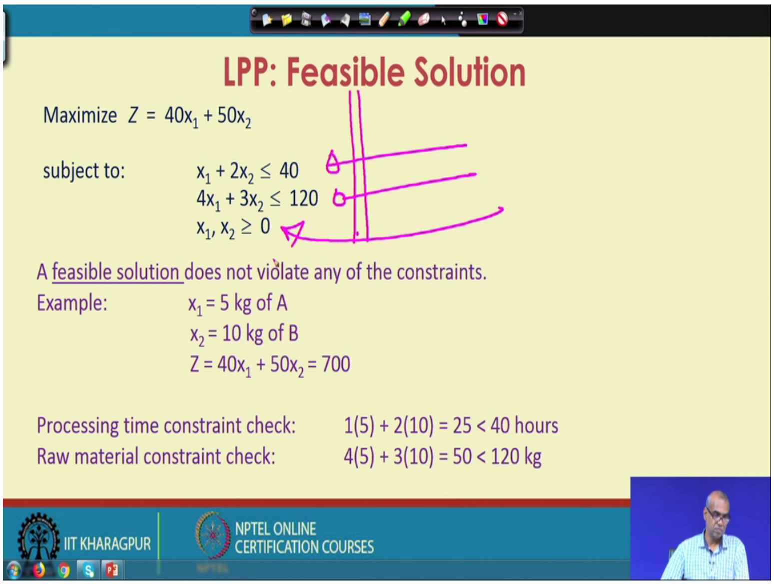

So, this part represents the complete formulation maximize Z equal to 40 x 1 plus 50 x 2 subject to x 1 plus 2 x 2 less or equal to 40, 4 x 1 plus 3 x 2 less or equal to 120 and x 1 is greater or equal to 0 x 2 is greater or equal to 0.

A feasible solution will lie within the feasible region. These two constraints and these non-negativity constraints all together will define a feasible region. Any point within this feasible region will not violate any constraint; any point outside this feasible region will violate at least one constraint. For example, if I take x 1 equal to 5 and x 2 equal to 10 that means, x 1 equal to 5 kg of A and x 2 equal to 10 kg of B. So, let us look at x 1 equal to 5 and x 2 equal to 10 whether they violate the constraints or not. For x 1 equal to 5 and x 2 equal to 10, the objective function is computed at 700.

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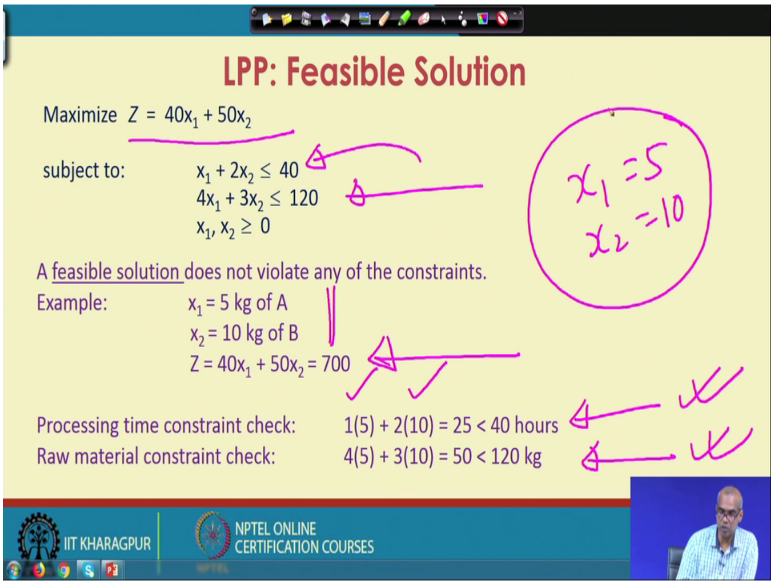

Now, x 1 equal to 5 and x 2 equal to 10 you first put in this expression. So, x 1 equal to 5 and x 2 equal to 10 it gives x 1 plus 2, x 2 as 25 which is less than 40. So, this is satisfied.

Similarly put x 1 equal to 5 and x 2 equal to 10 in the expression 4 x 1 plus 3 x 2 and you get that 50 which is less than 120 kg, so again satisfied. So, x 1 equal to 5, x 2 equal to 10 lies within the feasible region it may be a feasible solution, but it is not a optimal solution not necessarily an optimal solution, you have to find out optimal solution, we will see how to do that.

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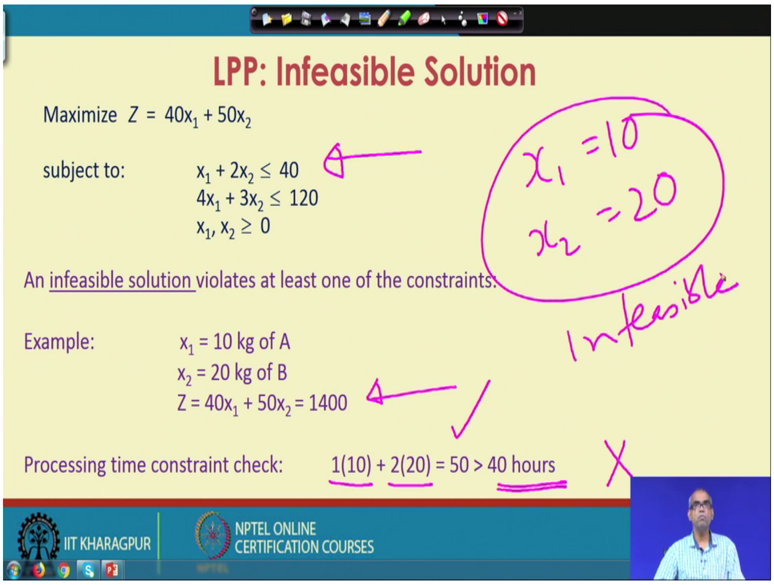

Infeasible solution will violate at least one of the constraints. For example, let us take x 2 equal to 10 and x 3 equal to 20. The objective function you can compute as 1400 which is much higher than the previous value x 1 equal to 5 and x 2 equal to 10.

Now, let us check if the constraints are satisfied. So, let us first look at processing time constraint. So, put x 1 equal to 10, x 2 equal to 20 and you get x 1 plus 2 x 2 as 50 which is greater than 40. So, constraint is violated. So, this is infeasible. Note that the objective function associated with this solution is much higher but this is infeasible solution.

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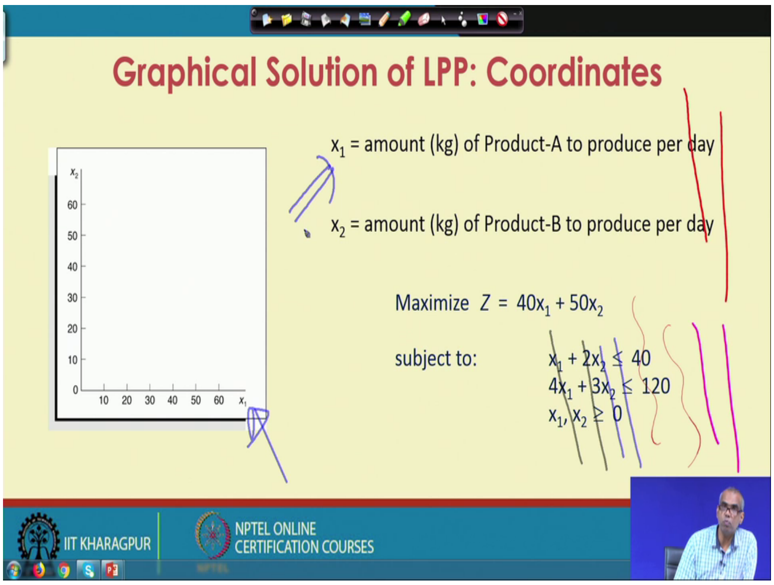

So, now let us look at how do I solve this problem using graphical method. So, let us start plotting the constraints. So, first thing is look at the coordinates. So, we have x 1 equal to the amount of product A to produce per day as decision variable and x 2 equal to amount to product B to produce per day as decision variable. So, I have two decision variables, we can very conveniently solve using graphical method. So, I have this x 1 axis and x 2 axis. So, x 1 represent amount of product A to produce per day, similarly x 2 represent amount to product B to produce per day so, on this axis I will plot the constraints.

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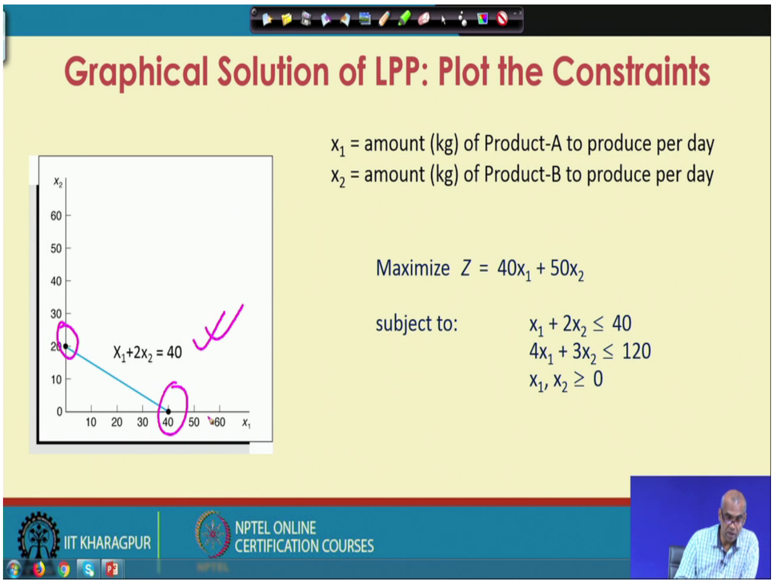

So, look at look at the first question x 1 plus 2 x 2 is less or equal to 40. So, first we will plot this constraint as equation. So, I plot x 1 plus 2 x 2 equal to 40 is very easy to solve, very easy to plot here. So, look at this point this is x 1 equal to 40 and x 2 equal to 0, so satisfied. Similarly x 1 equal to 0 and x 2 equal to 20 again satisfied. So, I have plotted x 1 plus 2 x 2 is less or equal to 40 constraint as an equation x 1 plus 2 x 2 equal to 40

Now, the region below this is the feasible region. Why? Because it is less or equal to type constraint.

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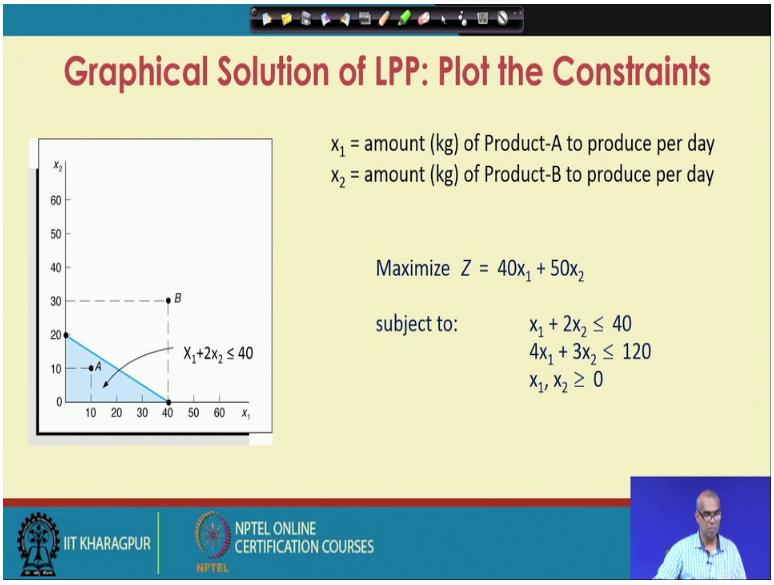

So, the region below the line representing x 1 plus 2 x 2 equal to 40 is my feasible region. If you look at the figure the point A lies within the feasible region, but point B lies outside the region, so the point B is infeasible, point A is feasible.

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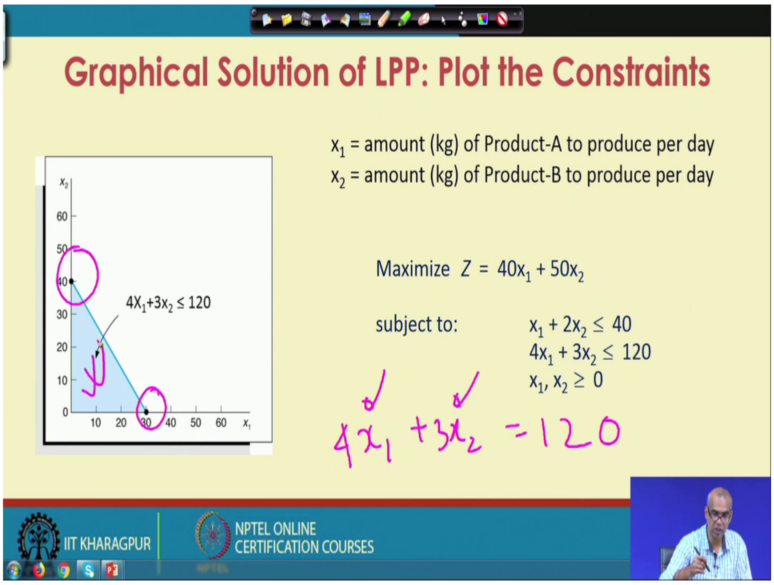

Next go to the second constraint 4 x 1 plus 3 x 2 is less or equal to 120 again I plot 4 x 1 plus 3 x 2 equal to 120. I treat the inequality as equality and plot it. So, again it is very simple to plot consider 4 x 1 plus 3 x 2 equal to 120. So, if I put x 2 equal to 0 x 1 equal to 30, similarly if I put x 1 equal to 0 x 2 equal to 40, 3 into 40 is a 120. Again the inequality is less or equal to type. So, the region below the line represented by 4 x 1 plus 3 x 2 equal to 120 is feasible region. So, this is the feasible region, the shaded region is the feasible region.

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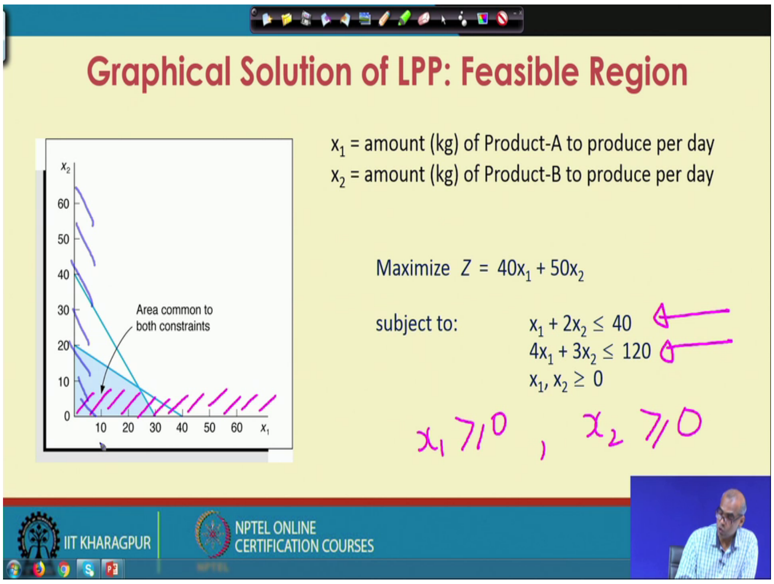

Now, let us look at the figure where we have now plotted both the constraints. So, there are two straight lines representing the two constraints x 1 plus 2 x 2 less or equal to 40 and 4 x 1 plus 3 x 2 less or equal to 120. Now, there are two more constraints x 1 greater or equal to 0 x 2 greater or equal to 0. Note that this region is x 1 greater or equal to 0 and this region is x 2 greater or equal to 0. So, the axis x 1 and x 2, the first quadrant is the feasible region for x 1 greater or equal to 0 and x 2 greater or equal to 0, these are non negativity constraints or implicit constraints.

Now, if you look at the area common to all the constraints which is shown here by this shaded region. This region is the feasible region for the problem. So, the non-negativity constraints x 1 greater or equal to 0 and x 2 greater or equal to 0 and the two constraints will together determine the feasible region as shown in the shaded region in the figure.

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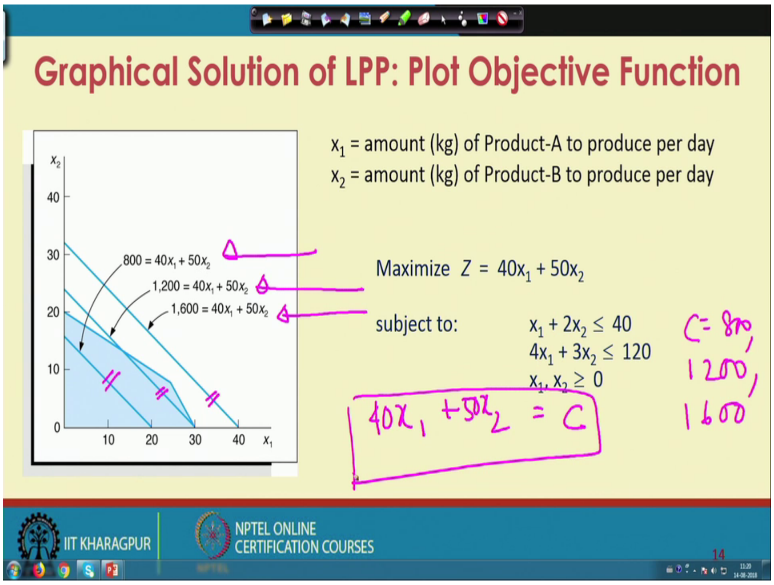

Now, let us plot the objective function objective function is 40 x 1 plus 5 x 2, Z equal to 40 x 1 plus 5 x 1 is my objective function. So, 3 different lines for objective functions are plotted here we are plotted 40 x 1 plus 50 x 2 equal to 800 as objective function. The next we have plotted 40 x 1 plus 50 x 2 equal to 1200, and then we have plotted 40 x 1 plus 50 x 2 equal to 1600. So, basically 40 x 1 plus 50 x 2 equal to C, some constraints C, here C equal to 800 in first case, then 1200 in second case, and 1600 in the third case.

Note that with different values of C this equation 40 x 1 plus 50 x 2 equal to C we need two parallel lines. So, we get the objective functions as parallel lines. And we are trying to maximize the objective function. So, of course, we will choose that straight line among all the parallel lines, which gives me highest value for the objective function.

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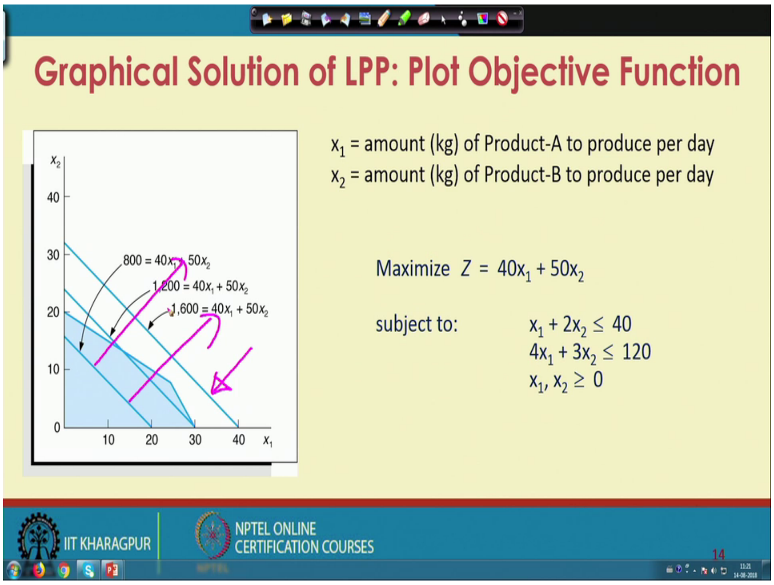

Now, in the figure shown this is the objective function line which has maximum value for the objective function 1600, but if you look at closely this line is outside the feasible region. So, this cannot be considered but what we see that if I go in this direction my objective function value is increasing objective function value increases from 800 to 1200, then 1600 it is a different story that in we need comes to 1600 we are outside the feasible region.

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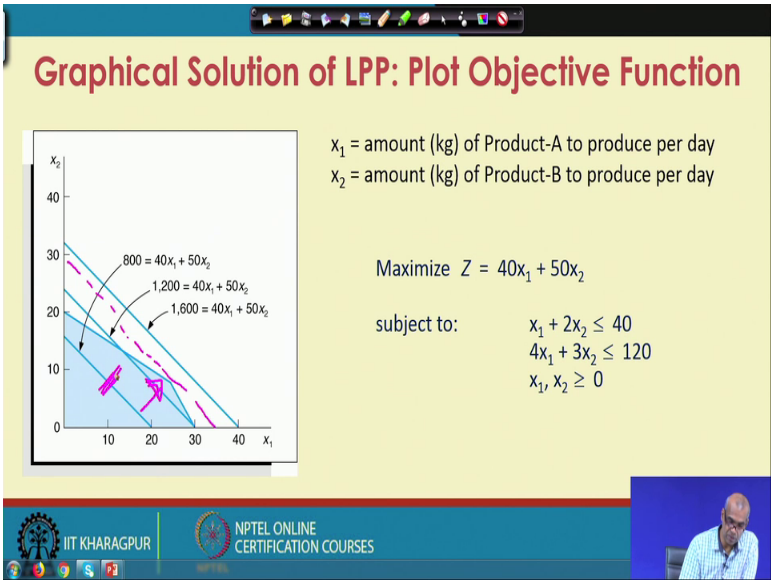

So, what we do is we start with, let us say this objective function line and slowly move in this direction, because you know if I moving in this direction my function value objective function value is increasing and then we look at that straight line which is parallel to this line and just touches a point on the feasible region because as the figure shows, as we move in the direction, in which objective function value is increasing I am slowly going out of the feasible region.

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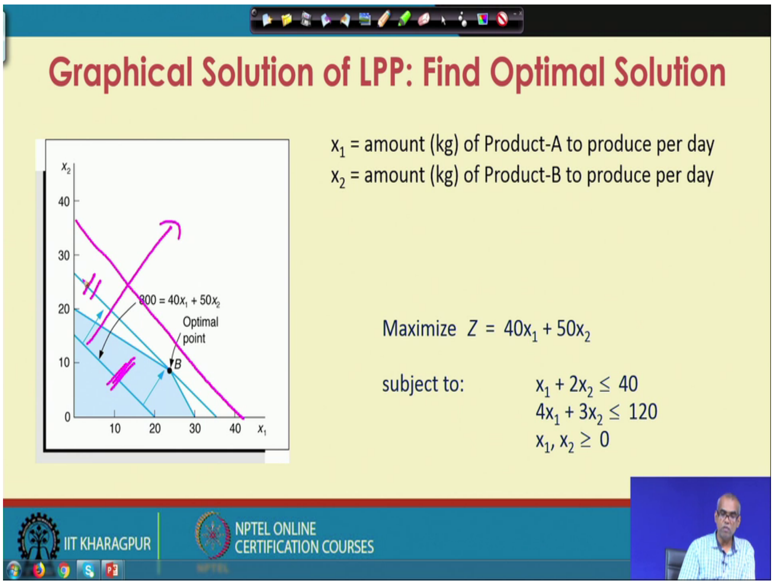

So, the situation that we will get is as follows. Now, look at the figure I am moving this objective function line in this direction. So, of the objective function value is increasing, I am always keeping the objective function parallel to this starting line. When I come to a point B I get the maximum value of the objective function which exists within the feasible region. There will be even higher value of the objective function. If I consider, another parallel line, line parallel to this but that is outside the feasible region. So, that will need to infeasible solution. But if you look at point B, the point B lies within the feasible region and gives maximum value of the objective function. So, point B is optimal.

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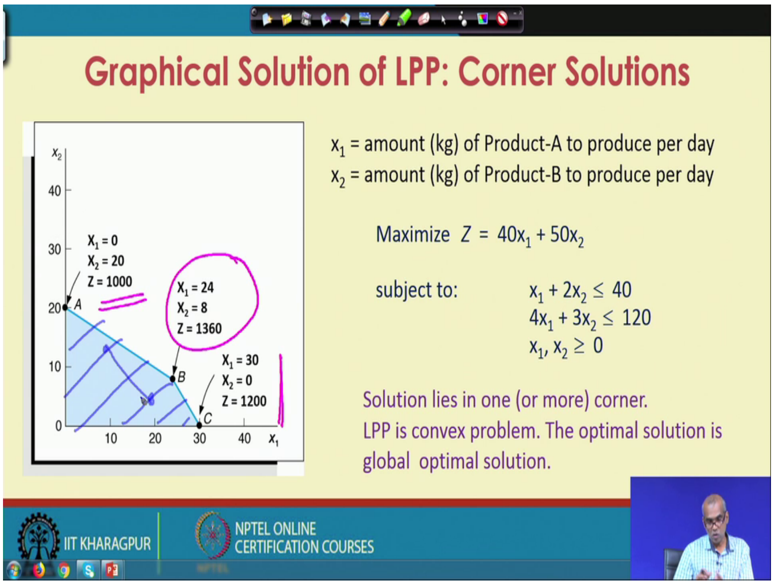

Now, there are 3 corner points A, B and C in the feasible region apart from the point 0 0. So, apart from the point 0 0 origin we have 3 corner points A, B and C, in the feasible region point A, point B and point C. We have noted that the optimal solution is located at point B. Point A corresponds to x 1 equal to 0 and x 2 equal to 20, objective function value associated with these is 1000. Point C is associated with x 1 equal to 30 x 2 equal to 0 and the objective function value is 1200. Point B is associated with x 1 equal to 24 x 2 equal to 8 and Z equal to 1360. This is the maximum objective function value and this is the optimal solution.

One interesting thing to note is that in case of linear programming problem the solution will always lie in a corner point. In fact, the solution will always lie in one or more corner point when there is optimal solution more than one optimal solutions multiple optimal solutions. We will talk about that in the next class, but right now let us try to understand that the optimal solution will always lie in a corner point.

So, if you visit the corner points you can also find out the optimal solution. When we will talk about the simplex method, you will see that the simplest method is a smart way of visiting these corner points such that optimal solutions can be found out quickly. See one can always visit all the corner points, calculate the objective function values and then pick up which one is maximum or which one is minimum; where I am maximizing a problem here. So, I can visit all the corner points and choose the corner point corresponding to maximum objective function value. It is if you have very few corner points, but if you have say 1000s of corner points then such visiting each and every corner point and finding all the objective function value may be very cumbersome.

Simplex method that we will talk about in the next class, in the next week are very appropriate problems are very appropriate solution techniques for such problem. There, we will see that we do not have to visit each and every corner point, but we just have to visit the minimum required number of the corner points so that you can get the optimal solution quickly. So, one thing we understand here is that the solution lies in one or more corner point.

Linear programming problem is a convex problem we have discussed that the objective function is linear all the constraints are linear. So, the problem is the convex optimization problem. All the constraints are linear, so you can note that the feasible region is also a convex region. Take any two points within this region; join them every point on this line will lie within this feasible region.

So, the constraint is a feasible region the objective function is convex, the feasible region is convex, objective function is convex, it is a convex programming problem. So, the optimal solution is actually a global optimal solution. So, the optimal solution for the linear programming problem is a global optimal solution because linear programming problem is a convex optimization problem.

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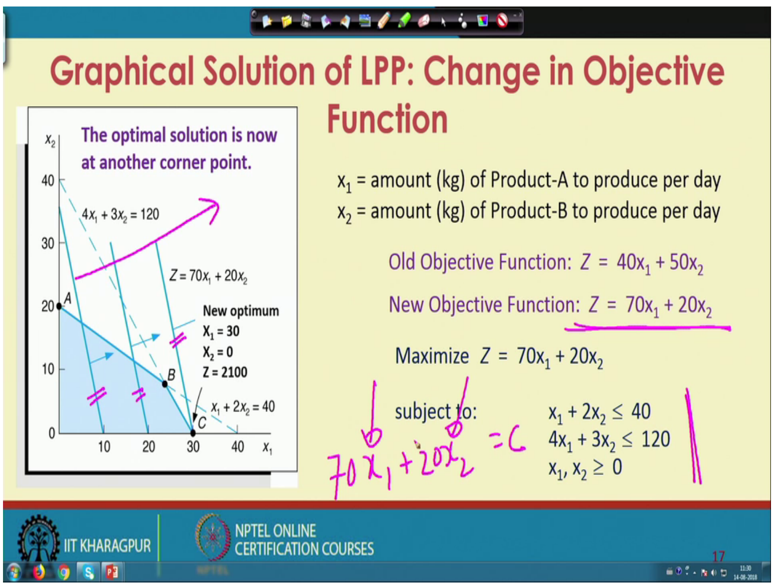

What happens if I change the objective function? In case of previous problem we have the objective function Z equal to 40 x 1 plus 50 x 2. Let us now change the objective function from 40 x 1 plus 50 x 2 to Z equal to 70 x 1 plus 20 x 2. We have kept the constraints unchanged the feasible region remains unchanged.

So, now I have to plot 70 x 1 plus 20 x 2 equal to some constraint C, as objective function lines. So, this is the objective function lines, these are the objective function lines and you see again the objective function values increasing in this direction.

Now, if you look at this figure you see point C is the new optimum point C corresponds to solution x 1 equal to 30 x 2 equal to 0 and Z equal to 2100, put x 1 equal to 30 here and x 2 equal to 0 here you get Z equal to 2100. So, when you change the objective function without changing the constraints, I get the new optimum at another corner point. So, optimal solution is always lies within the corner point. So, with this we stop our lecture 37 here.