Course on Transport Phenomena
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Module No 2
Lecture 09
Equations of Change for Isothermal Systems

We are going to start with something new this morning. So far what we have seen is that using a simplified shell momentum balance, it is possible to account for all the forces acting on the different surfaces of the shell and we can calculate, we can express the convective momentum that comes into the system, the conductive momentum, in other words, the molecular transport of momentum or the shear stress which is acting on the lateral surfaces of the control volume.

And we have also identified that the forces which can act on the control volume would be the surface forces and body forces. And in our example so far we have seen, we have used pressure force is the only surface force and gravity as the body force. If the system is in steady state, then we understand that the sum total of all these, that is rate of momentum in by 2 different mechanisms - rate of momentum out + some of all forces acting on the system must be 0 in order to maintain the steady state.

So this give us a different situation, this different situation where the smaller dimension is allowed to approach 0. In other words we have used the definition of the 1st derivative when let us say del X tends to 0. The definition of the 1st derivative would give rise to a differential equation to. So that differential equation explains or describes the motion, the laminar flow, the laminar motion of fluid layers slipping past one another, it is steady-state and there are no unbalanced forces acting on the system.

Many of our everyday examples are at steady-state and it is possible to use this shell momentum balance to obtain concrete expressions for velocity for such systems. We have used the, we have used the case of flow through a pipe, flow through a tube and we have seen how the velocity varies as the function of radius. All these cases that we have analysed are one-dimensional flow. So for example in the case of flow through a tube, on the application of pressure difference and gravity, it was a vertical tube.

The velocity was a function of radius but it was not a function of the axial distance which is Z.

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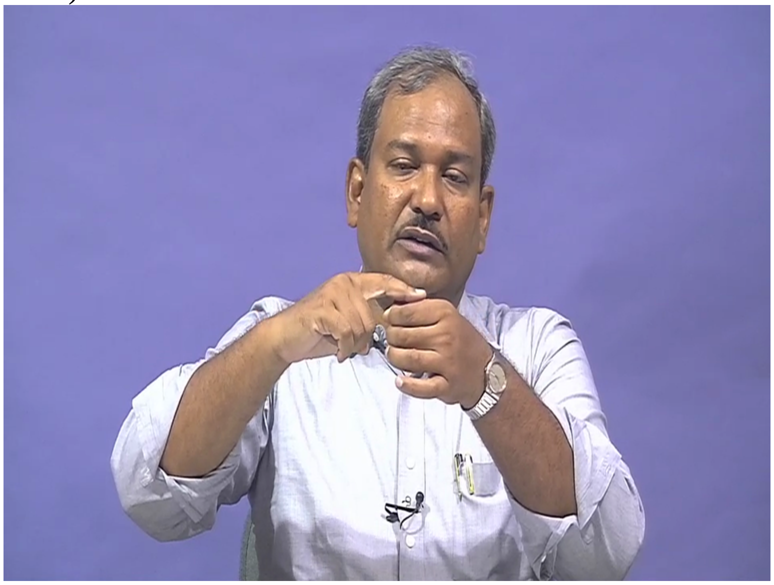

So velocity over here at a fixed R location and velocity at some other value of Z at the same R location, the these 2 velocities are same. So the velocity does not vary initially, it only varies radially. And we have also found out that the velocity distribution was parabolic in nature with the maximum at the Central line and from the expression of the velocity, this is differential approach when we get the parameter of interest, in this case velocity at every point in the flow domain and from this velocity profile we could and we could differentiate this velocity profile and if this is a Newtonian fluid, then we understand that the shear stress is simply going to be mindless mew times velocity gradient at some specific value of R, where R is the radius of the tube

So with this approach, we did find what is the wall shear stress exerted by the fluid on the walls of the tube. And the we also derived what will be the formula for average velocity. This is axial average velocity, velocity across an area which is perpendicular to the direction of motion. So it is the cross-sectional area of the tube across which we average the velocity and we obtain, we have obtained an expression for the average velocity.

This average velocity profile multiplied by the area would give us the volumetric flow rate. You know that is the known, that is the well-known Hagen-Posse equation. So there are some few other examples that we have solved in previous classes the flow along vertical, flow along an inclined plate or the flow along the outer wall of a tube when the flow comes from below comes to the top of the tube, spills over and starts to fall along the wall, along the side walls, along the outer sidewalls of the tube.

Progressively what we have seen is that the approach that we have used so far is appropriate for simple systems were the flow is one-dimensional, where the flow is laminar, where the flow is steady and where the geometry is rather straightforward but the problems soon start to creep in when these conditions are not met. So we have increasingly felt the need to express these or need to formalise our treatment of transport through any system even at unsteady state cases.

In order to do that, one must 1st start with some of 2 or 3 basic definitions of how do we define the derivatives of a system. In this class and the next class, I will talk about what is going to be the partial time derivative, the total time derivative and the substantial time derivative, these are important concepts which must be clarified before we get into the next part which is the simple continuity equation.

The derivation of the continuity equation, I will not talk about the entire derivation in this class, I will tell you the textbook where it is available. And then we will go into the equation of motion or a special form of equation of motion of a fluid which is known as the Navier stokes equation. Now once we have been Navier Stokes equations with us, then we would see that all the problems that we have handled so far can be handled quite easily by choosing the appropriate Navier Stokes equation for the flow situation.

And once that is done, we will solve a few more involved problems and then move into the next chapter.

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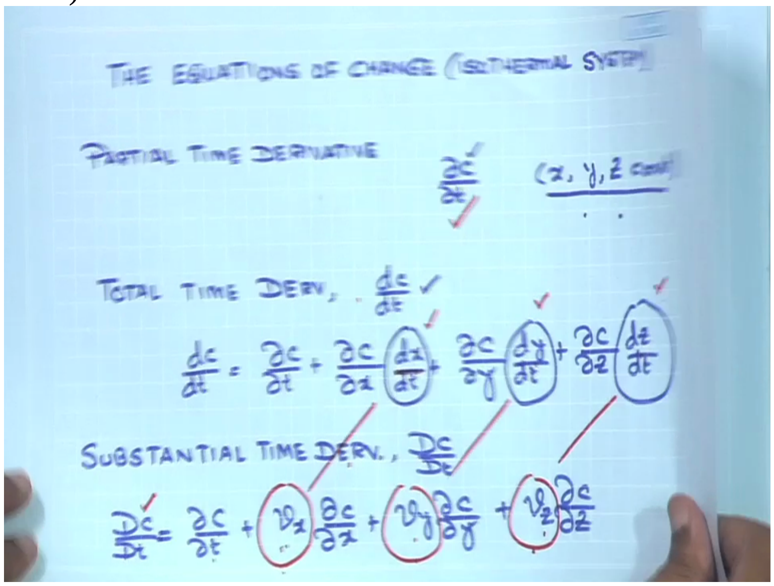

But before we before we start, the the topic of our today's descript, today's lecture is the equations of change for isothermal systems. Now when we talk about equations of change, there are certain definitions which needs to be clarified. So the concepts of different derivatives okay, you you are trying to measure something as a function of time and depending on where you are, what are you doing, the values of the quantity that you measure as a function of time can be can greatly vary.

So I will try to give you an example so that you can have a clearer understanding of these different derivatives. Let us see, you pick the busiest intersection of some place in your town where roads have come from all sides at that crossing. Now you are standing right at the centre of this intersection and you have been told that please count the number of people who are wearing a blue shirt. Okay?

So you are standing at the middle of a crossing and counting people with, who are wearing blue shirt. You are not moving, you are static at that point okay and you are measuring the number of such people as a function of time. So every 2nd, you try to see how many blue shirts you can see while you are stuck at a position. So you are at the centre of the reference frame which is static, which does not move and any quantity that you measure as a function of time is known as the partial time derivative.

Let us say this is the C denotes the number of number of persons who are wearing a blue shirt, variation of that with respect to time where X, Y, Z are constant. So your location is constant, XYZ is constant. You are right at the centre of the intersection and measuring what is the value of C. Now let us say you are, while standing there for some time, you are I mean you are definitely bound to get bored.

So you get bored and you have decided that enough of it, I am not going to be at that intersection for a long team. I need to walk around, I need to walk around a bit around in in the area where I am trying to count the number of number of persons with a blue shirt. So you start to move around. You have a velocity of your own. So you can go in any direction that you want and this with some velocity.

You would still being a very conscious walker, you are still counting the number of persons with a blue shirt. But since you now you have a velocity of your own, the numbers that you are counting would definitely be different had you been struck to the place where X, Y, Z are constant. So right at the centre of the intersection. So if you measure the number of people while standing at the intersection and if you measure the number of people while you start to move around with a velocity of your own, these 2 numbers must be different.

So when you measure the variation of the number of people wearing blue shirt as a function of time while you have a velocity of your own, by definition, it is known as the total time derivative bridges denoted by DCDT and this DCDT is simply expressed in this where DXDT, DYDT and DZDT are the components of your velocity. So you being the reference frame, you start to know more around. When you start to move around, this DXDT, DYDT and DZDT are the components of your velocity and obviously the velocity of you will have an impact on the numbers that you count and that is why it is called the total time derivative and expressed as DCDT.

The next one is even just an extension of that. How long you can stand so you decided to move with a velocity. How long you can move around? At some point of time, you get tired and being a busy intersection, there is a lot of crowd which is going in all possible directions. So at some point of time, you decide that I had had enough, let us I will simply float with the crowd.

No matter which way the crowd is, the maximum number of people are moving, I will move with them with the velocity, with their average velocity and at some point I will, at each point of time I will always move with the velocity of the prevailing crowd at that point of time. But you are still counting the number of persons with the blue shirt. So now you do not have a velocity of your own.

Whatever be the local fluid velocity, that is the velocity of the reference frame and the numbers that you are counting is some sort of a derivative that DCDT, that time derivative of the number of persons is a derivative where the reference frame moves with the fluid with its average velocity. Or that way, it can also be said that it is a derivative following the motion.

So this is generally called the substantial time derivative. Innocence, it is a derivative following these following the I mean the motion of the fluid and it is expressed as del C del T + VX times del C del X VY times del C del Y and VZ times del C del Z where VX, VY, and VZ are the velocity of the fluid at that point at that instant of time. So mark the difference between these 2. Okay?

So this, these are velocity is of you that you have decided to move with this velocity whereas these velocities are that of the fluid surrounding you. So you let go yourself and float with the fluid and the numbers direct to count are now known as the substantial time derivative or the derivative following the motion. So we will, the concepts are important in our subsequent development where we find out what is partial time derivative where this stationary frame is fixed, the total time derivative where the reference frame has a velocity of its own and the substantial time derivative where the reference frame has the velocity as that of the fluid.

So with these concepts clearly understood, now we are going to, we are in a position to derive what is going to be the equation of continuity. An equation of continuity is nothing but a statement of conservation of mass. So if I define a control volume in space, fixed in space and allow fluid to come in and go out through all the possible faces, then the rate of mass of fluid coming in mind is the rate of mass of fluid that is going out must be equal to the rate of accumulation of mass inside the control volume.

So in - out must be equal to accumulation and this is nothing but the statement of conservation of mass and we are going to derive the equation of continuity for a system with a Cartesian coordinate system and the dimensions of the volume is simply going to be del X, del Y and del Z. So it is above shows dimensions are del X, del Y, del Z. It is placed in a flow and the liquid is coming in through the X face, Y face and Z face.

Liquid enters the control volume and through the face at X + del X, Y + del Y and Z + del Z, the fluid leaves the control volume. And as a result of this, there is going to be some amount of some amount of mass accumulation if possible within the system. So we are going to write the balance equation for such a system and derive the equation of continuity.

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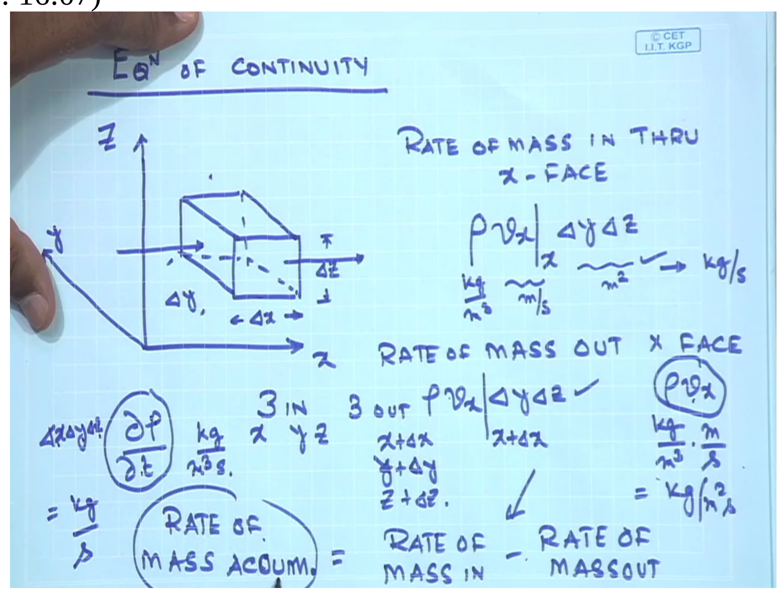

So in order to derive the equation of continuity, the way we are going to start it, we define a coordinate system X, Y, Z and we have, as I mentioned before, there is a box and this box, so this area is let us say delta Y from here to here. This is delta X and this is delta Z. So some amount of mass is coming in here through the X face in the mass that is going out. So the X face is defined as the face which has areas del Y del Z.

So this is your X face and on the other hand on this side, you have X + del X face. Similarly this face whose area is del X times del Z is known as the Y face and I have a Y + delta Y face on the other side. And this is my Z face so which has areas of del Y times del X. So we are going to write the amount of mass which comes in through all these faces. So what is the amount of load which comes so comes in through the X face?

So rate of mass in through X face can simply be obtained as VX evaluated at X multiplied by del Y del Z. So this has units of meter square, this has units of meter per second. So this becomes meter cube per second. Meter cube per second will multiplied by rho, makes it kg per second. So this is the so the unit of this, so this is kg per meter cube, essentially unit here is you are going to get kg per second.

So therefore this quantity gives you the amount of mass, the rate of mass in, time rate of mass in through the X face. And the rate of mass out through the face, again I am talking about X face only, would simply be rho VX times del Y del Z for instant of being evaluated at X, it is going to be evaluated at X + delta X which is this face. So rho VX, if you just think of rho VX, it is kg per metre cube and meter per second, so essentially this is kg per metre square per seconds.

So rho VX is nothing but the mass flux. So this mass flux, since this is kg per metre square metre square per second, so this mass flux when it is multiplied by the Ajax are responding area, would give you kg per second or the rate of mass in coming in to the control volume. As a result of so you are going to have 3 in terms and the 3 out terms. The in are at X, Y and Z face, the out are going to be at X + del X, Y + del Y and add Z + del Z faces.

As a result of this in and out, you are going to get the, there would be some amount of accumulation inside the system. So your governing equation is rate of mass accumulation must be equal to rate of mass in - rate of mass out. So if I such to dude these to in here and think of what is rate of mass accumulation, rate of mass for for in order to in order to have a mass circulation inside the system, the density of it must change.

So this is del rho directly is the change intensity of the fluid contained within the control volume. So what is del rho del T is the change in fluid density of the fluid contained within the control volume. So what is del rho del T? Is kg per metre cube per second. So this must be multiplied with del X del Y del Z to make it kg per second and therefore del X del Y del z times del rho del T would give you the time rate of mass accumulation of inside the control volume.

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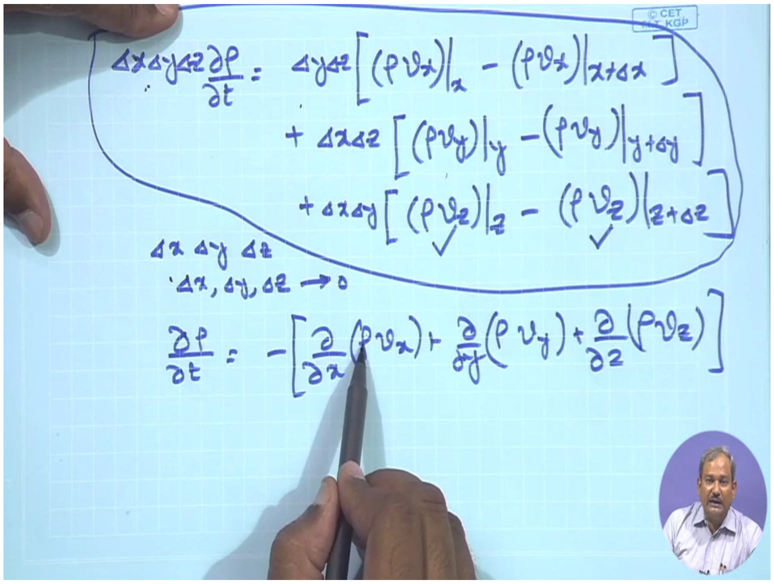

So the left-hand side is going to be equated to the right-hand side and what you get is del X del Y del Z del rho del T which is the accumulation term must be equal to del Y del Z. Then rho VX at X, this is the in term, - rho VX at X + del X, this is the out term + del X del Z rho VY which is Y, that is the in term through the face at Y - rho VY at Y + del Y, this is the out term out term through the Y face + del X del Y rho VZ at Z - rho VZ at Z + delta Z.

So this is the in term in through face through face Z, out through face Z. So in other word in difference term, this is essentially what is known as the conservation of mass. So the obvious next step would be to divide both sides by del X del Y del Z and take in the limit when all del X del Y del Z tends to 0. So dividing both sides by del X del Y del Z and and in the in this condition where del X, del Y and del Z all approach 0, you get the definition of the 1st derivative and the expression would simply be equals - del del X of rho VX + del del Y of rho VY + del del Z of rho VZ.

Please note that I did not take the rho outside because I did not yet put the put the put the special condition that the flow is incompressible. So I am allowing the flow to be both either compressible or incompressible. So that is why my rho stays inside the differentiation fine. Now this expression can be expressed in a more compact form is - del rho V where this is, rho is kg per metre cube and meter per 2nd.

So this is essentially kg per metre metre square per seconds. So this is nothing but the mass flux which is a vector. So this is the divergence of mass flux vector must be equal to the time rate of change of density inside the inside the control volume. So that is one way of expressing the equation of continuity or the conservation of mass.

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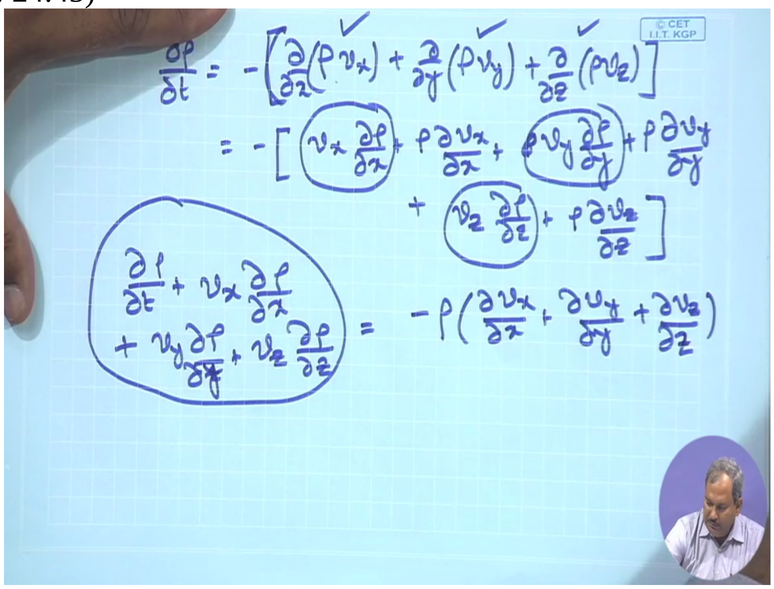

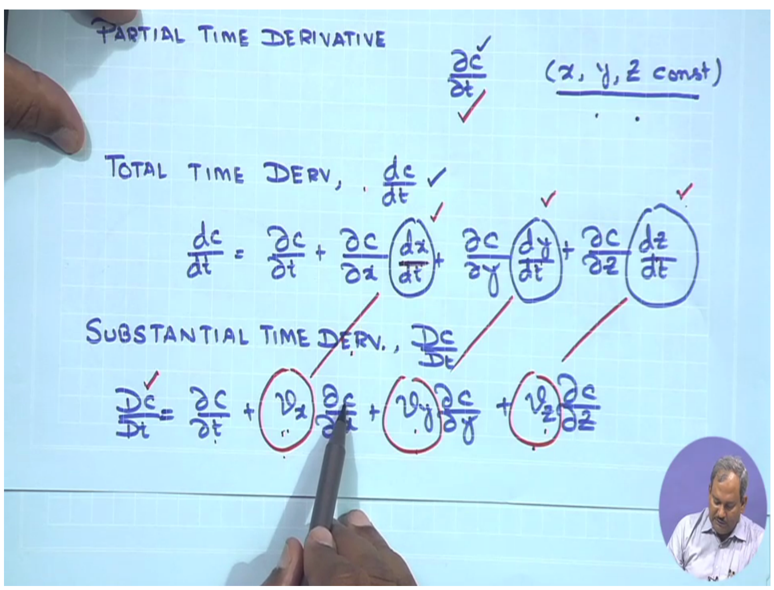

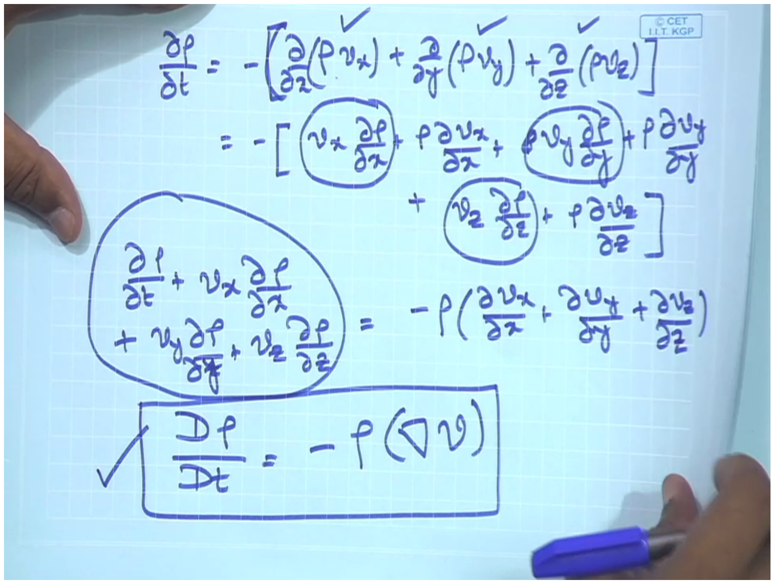

Another way if you can expand the term that is my I started with this to be equal to - del del X of rho VX + del del Y of rho VY + del del Z of rho VZ. So this was the equation of continuity which and then I can express it I I I can simply make it as VX times del rho del X + rho times del VX del X, similarly for this VY times del rho del Y + del VY by del Y + VZ Times del rho del Z + rho times del VZ del Z, simply by expanding the derivatives.

Now what I am going to do is I am going to bring this term, this term and this term on the left-hand side. So if I bring these terms to the left-hand side, it would be simply be del rho del T + VX del rho del X + VY del rho del Z + VZ is simply going to be equal to - rho del VX del X + del VY del Y + del VZ del Z. If you look at the left-hand side carefully, it is nothing but what we have written as the expression for the substantial time derivative. del C del T + VX del C del X + VY del C del Y + VZ del C del Z.

Simply replace C by rho and what you get is this expression. (())(26:48) So the entire left-hand side can be expressed as del (())(26:52) T is equal to - rho, this is another form of equation of continuity where the rho is expressed in substantial derivative form.

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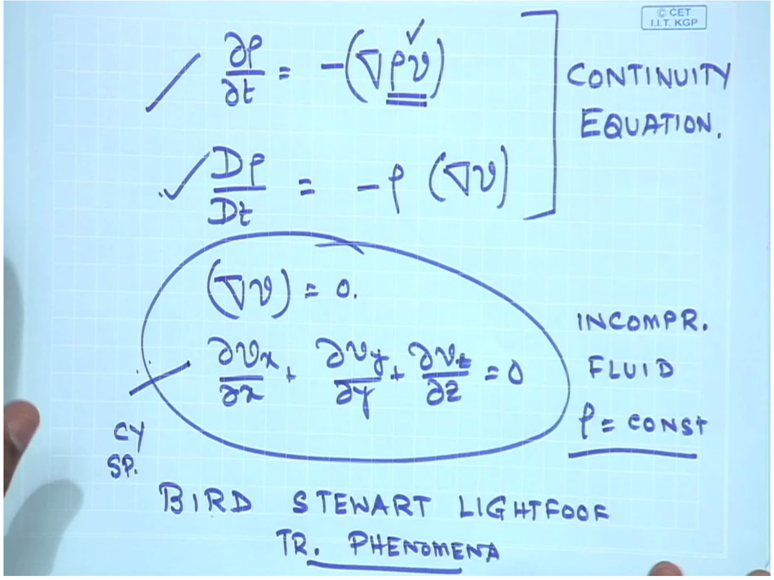

So the 2 common forms that you would get for the case of equation of continuity, one is del rho del T which is partial time derivative and which is equal to the divergence of the mass flux vector and the other one is del rho del T or the substantial derivative of density is equal to - rho times del V. So these 2 together are known as the continuity equation.

So this continuity equation plays a very important role in in fluid mechanics, in transport phenomena because it tells you about it it is nothing but the statement of conservation of mass. So rate of mass coming in - rate of mass going out must be equal to the rate of mass being accumulated inside the control volume. And so I have shown you two different forms two different forms of the equation of continuity and the 2 different forms can be expressed either in partial derivative form or in substantial derivative form.

But there is a special form of equation of continuity with which we are mostly concerned with which which which are more common, they are known as the equation of continuity further and incompressible fluid. And incompressible fluid is the one in which the rho, the density remains constant. It is not a function of X, it is not a function of X, Y or Z.

So if the density is constant, if it is an incompressible fluid, then if you look at this expression, it would it would surely give you the X the del V is equal to 0 or in other words, del VX del X +

del VY del Y + del VZ del Z to be equal to 0. So this is the form of incompressive the continuity equation for incompressible fluid which is rho to be a constant.

So these 2 equations are equivalent, this equation is a special form of equation of continuity is when there is when the when you do not have any variation of rho with with any of the any of the independent variables present in the system. So we have done the equation of continuity. In the coming class we are going to do the equation of motion and once we have the equation of continuity and equation of motion clear clear in your mind, then you would be able to solve almost any problem of momentum transfer, at least you will be able to formulate the problem.

Whether or not an analytic solution is possible, that we have to see on a case-by-case basis. For each cases, for each case, it may not be possible to obtain an analytic solution and we may have to resort to other techniques including numerical techniques but this would give you the tool at least to get the the governing equation correct. And the expression for continuity equation which I have shown you is basically for the is basically for the Cartesian coordinate systems.

You have seen similar such relations, similar such relations of equation of continuity for cylindrical systems and for spherical system. So if you look at any textbook, you would see the expression for equation of continuity in all 3 possible coordinate systems, okay? And this part what I am I am teaching you right now, is clearly mentioned in Bird Stewart and Lightfoot.

So the book that I followed for this part is from Bird Stewart Lightfoot’s book, Lightfoot’s book on transport phenomena. So the equation of continuity in in Cartesian coordinate as well as in cylindrical coordinate and in spherical coordinate, they all are provided in this this text and whenever whichever we the your system, whatever be your system, depending on that, we choose the appropriate expression for the equation of continuity. So next class, we will get into the equation of motion.