Transport Phenomena

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Lecture 6

Example of Shell Momentum Balance (Contd.)

So we are dealing with the flow through a circular tube in which there is an imposed pressure gradient and gravity is also present. So our goal is to obtain the velocity distribution inside the pipe during the flow and to obtain an expression for the volumetric flow rate. Now this volumetric flow rate based on our understanding so far we realized that it will contain some geometric parameters. For example, what's going to be the length of the pipe? What's its radius?

There will be some operation parameters for example, what is the pressure difference thatwe have imposed from outside? That is, what is p0 minus pL? Or what is the value of p0 and pL? Itshould also contain a description of the force field body force field which is present. In this case it is gravity. Since it is vertical, it's simply going to be G. The geometric and the operational parameters namely R L p0 pL and gravity, all these would be collected with the velocity orwith the flow rate by physical parameter, a physical property.

And the physical property or more correctly the transport property in question in this case would be viscosity. So whatever expression ofvelocity or volumetric flow rate that we would get should contain all this. So we have done thisanalysis and we have come up with the governing equation. Now this governing equation has to be solved with appropriate boundary conditions. So let's start with our final form of the governing equation and see what we can do in order to obtain an expression for the velocity.

So what you have in here is this expression that we have obtained for this. If you integrate it twice what you are going to get is the expression that you are going to get out of this is, from this one.

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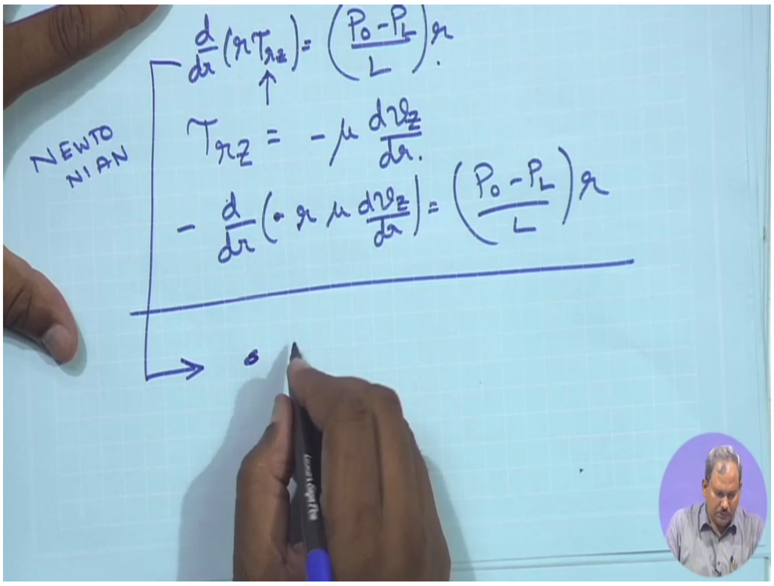

Once you integrate this, it would be p0 minus pLby 2L R square plusC1. Or in other words we can express it as taurZ is p0 minus pL by 2L times R square, R sorry R, plus C1 by R.

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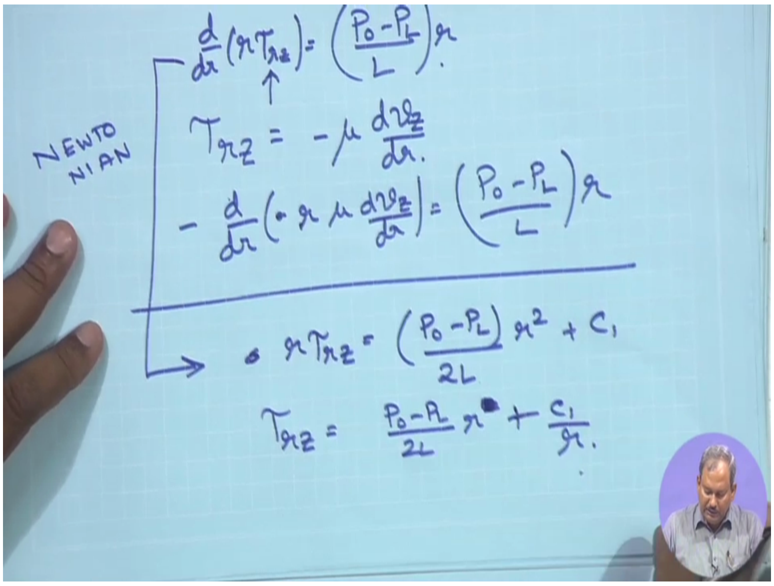

Now we realize that this gives us an opportunity to say something about the value of the integration constant C1. We understand that tau rZ must be finite at R is equal to 0. Shear stress has to be finite at R equals to 0. And this can only happen if C1 is 0. So the fundamental condition that the shear stress cannot be indefinite at any point in the flow field will give me definite value for one all the boundary conditions.

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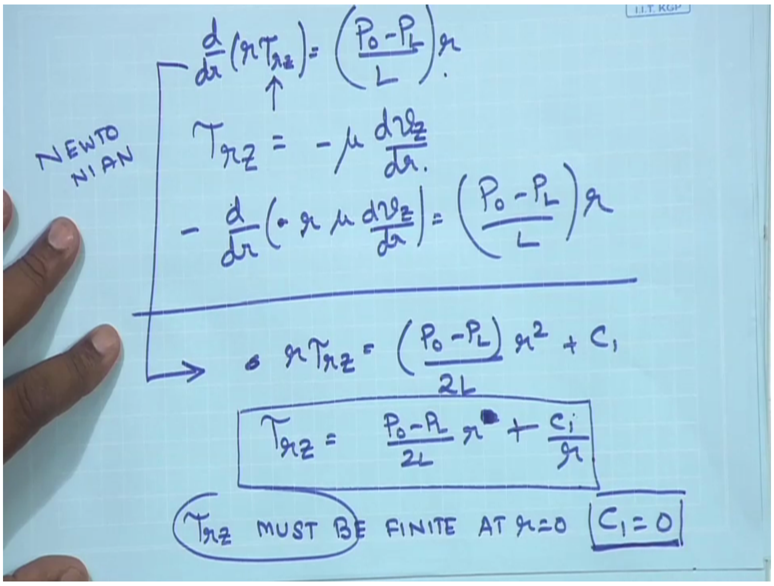

So in certain cases the physics of the problem has to be kept in mind not just blindly, no slip and no sheer at two interfaces. In some cases you can make a definitive statement about the nature of the velocity, nature of flow, nature of shear stress, which would give you an additional physical boundary condition which in this case we have used to obtain the first integration constant C1. So once you know C1 equals to 0, thentheremaining part of the equation can be integrated to obtain what is velocity distribution.

So let's see what we do that in. So your expression once you do the integration from that point onwards thereis your taurZ would simply now become p0 minus pLby twice L times R. And use Newton's law of viscosity. So your tau rZ is minus mu times D vZ dR. So when you plug this in here, what you have is then, D vZ by dR is equal to minus p0 minus pL by 2 mu L times R. And upper integration it will give you, vZ equals minus. So C2 is thesecond integration constant.

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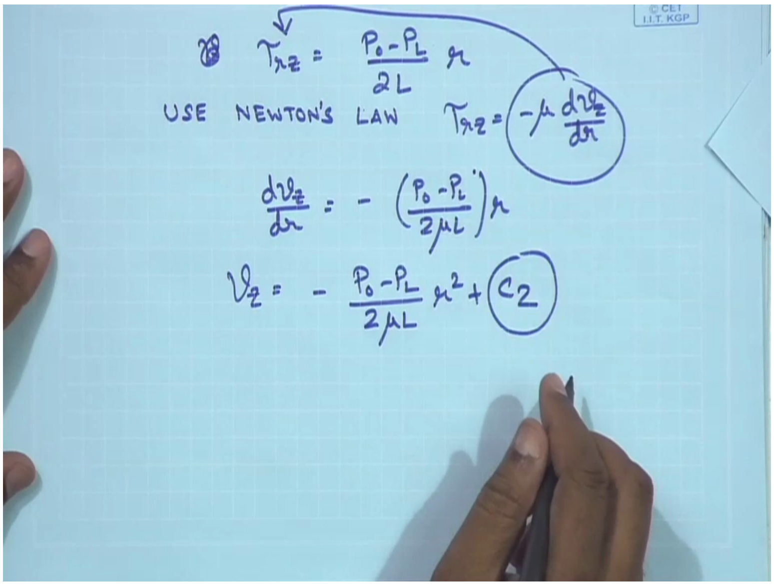

So how do you evaluate C2? You need a boundary condition and the boundary condition that is available to you is, no slip at small r equals to capital R. Because in this shell,when smaller becomes capital R then you essentially have a liquid-solid interface and at a liquid-solid interface, the relative velocity is zero. So you would have vZ is equal to 0 at small r equals capital R.

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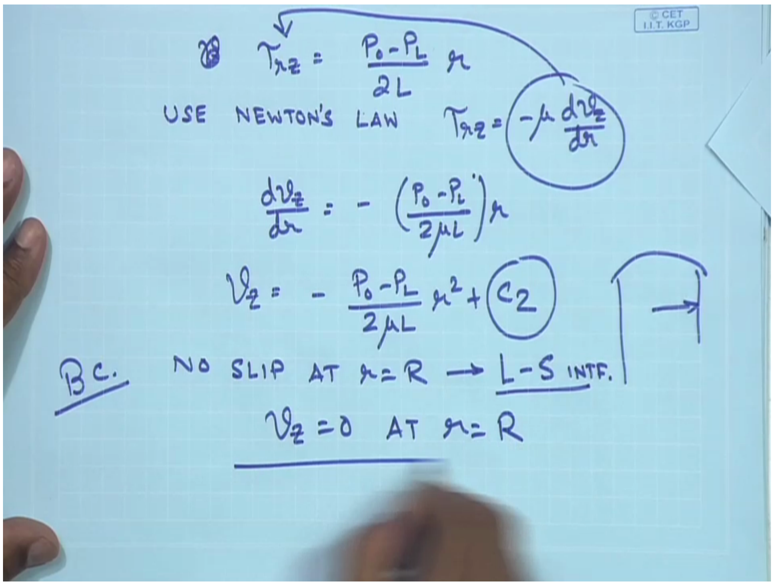

So that the second boundary condition which one can use to obtain the expression of velocity, final expression of velocity as vZ equals p0 minus pL by 4 mu L. In fact this should be for 4.4 mu L R square 1 minus r by R whole square. So this is the expression of velocity which for flow in a circular pipe, in presence of pressure gradient and gravity, which is embeddedinto it and all other parameters, the geometric parameters R and L and the transport property mu is already present in here.

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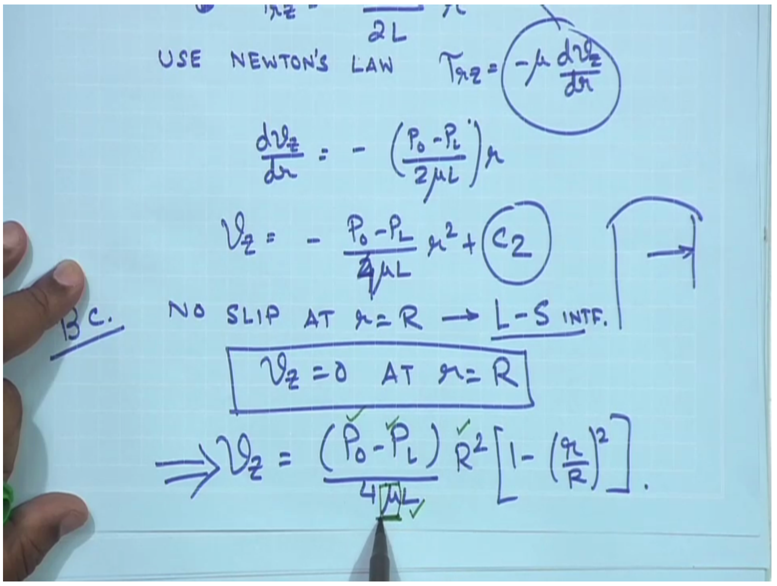

I will write it once again just to because it would help us in discussion is, vZ as p0 minus pLby 4mu L times R square times 1 minus r by capital R whole square. So operational parameters, geometric parameter and the property of the fluid.

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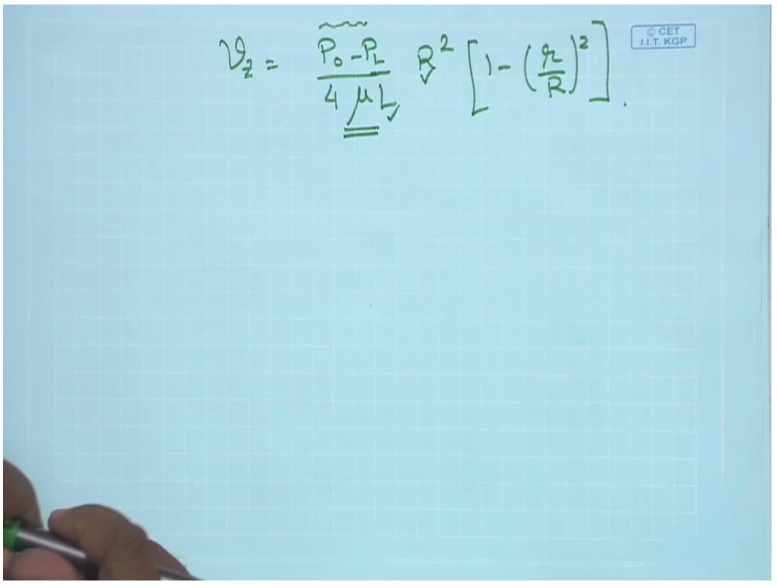

And it is obviously then the velocity distribution because of the nature, because of its R dependence,the velocity (dep) is going to be parabolic in nature. So this is your R and Z. So velocity starts at value equal to zero due to no slip at the solid wall which is a maximum and the variation is parabolic in nature.

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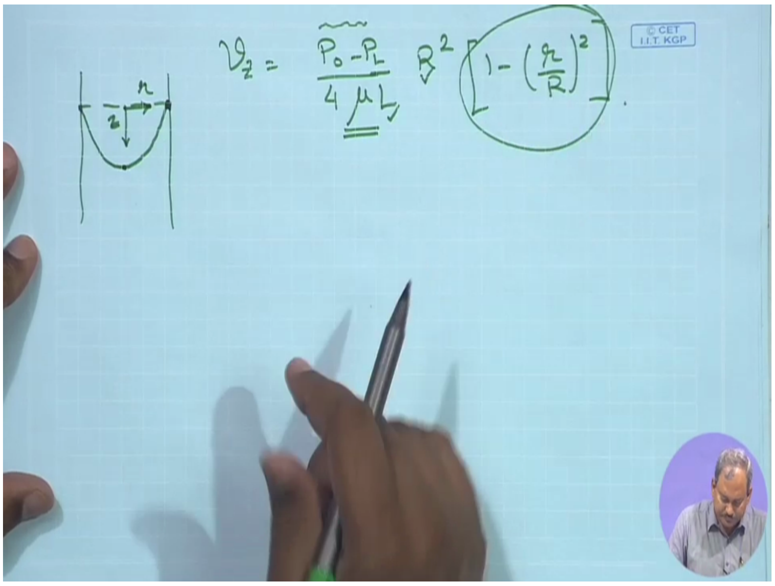

So this is the point of maximum velocity and from the expression you can simply see vZ max is essentially at R equals to 0, at this point, which would simply be equal to p0 minus pL by 4 mu L timesR square. So we can write vZ as vZ max times 1 minus r by R whole square. The expression for vZ can also be written in this way.

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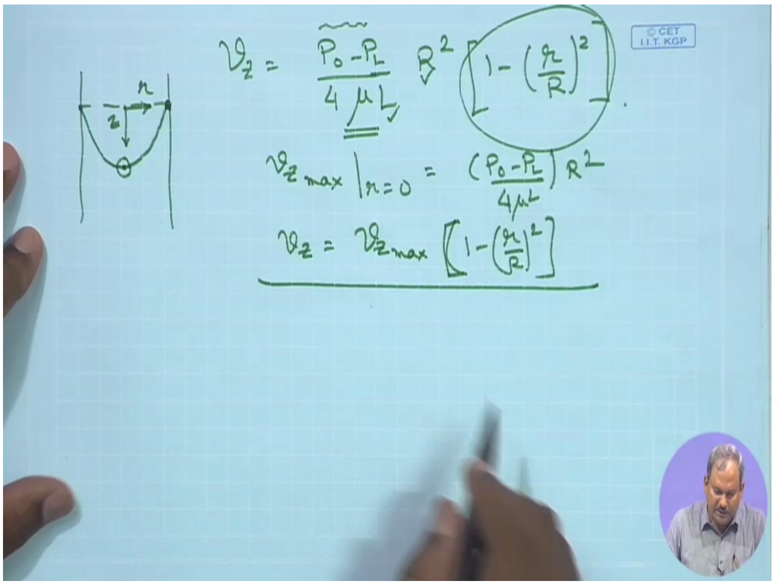

But here as engineer probably notinterested knowing, what is velocity? We are more interested to know, what is the flow rate for such a case? When I apply a pressure gradient,when I have a gravity force acting on it, in a pipe of known diameter and length, how much of fluid can I expect at the other end? Or how much of fluid I can collect which is coming out of the tube per unit time? In order to do that, the first step is to obtain an expression for the average velocity.

And to obtain the average velocity I need to integrate this velocity across some area. So what area it should be? The flow is in the Z direction and I need the average velocity and it varies with R. So the flow area which I need to incorporate in order to obtain the average velocity must be some area which is perpendicular to the flow. So if I take the cross section of the circle, then the circular area must be the flow area across which I need to integrate in order to obtain an average velocity.

All velocity is an area average velocity and all these areas are always perpendicular to the direction of flow. So it’s this area integration from zero to 2 pie, from zero totheta,then I have R dR D theta. That's the area, the cross sectional area across which the velocity need to be integrated. So that's what we are going to do next.

So average velocity which is denoted by this sign is simply equal to zero to 2 pie, zero to R, vZ R dR D theta. And by the area which is zero to 2 pie, zero to R, r dR D theta. This would be equal to p0 minus pL by 8 mu L times R square. And in, sorry this is vZ. And the vZ is essentially this one from the previous relation that you plugged it in here to obtain the expression for the average velocity.

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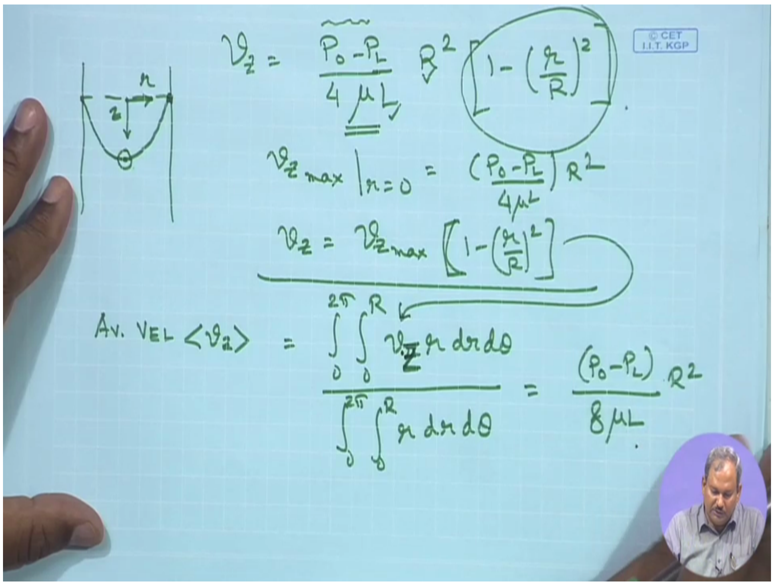

And once you have the average velocity, the flow rate Q would be equal to pie R square times the velocity. So this is meter square, this is as units of meter square, this is meter per second. So this together it becomes meter cube per second which is volumetric flow rate which would be equal to pie p0 minus pL by 8 mu L times R to the power 4. This is a famous equation which is known as the Hagen Poiseuille equation.

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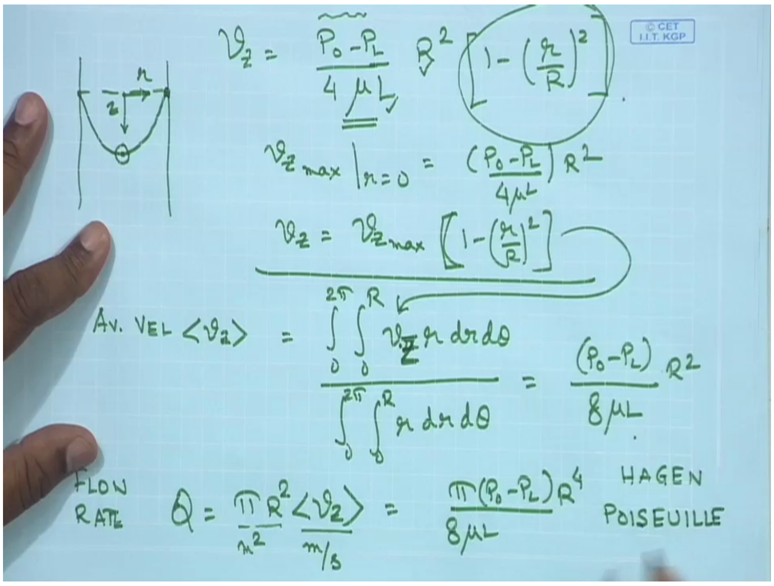

So this HagenPoiseuille equation gives you the volumetric flow rate for a fluid which is flowing through a vertical tube because of the presence of a (velo)pressure gradient and body force. So this is the imposed condition, this is a geometry, these two other geometrics and this is the thermo-physical property or transport property. So now you can clearly see how we explain the principle of capillary viscometer.

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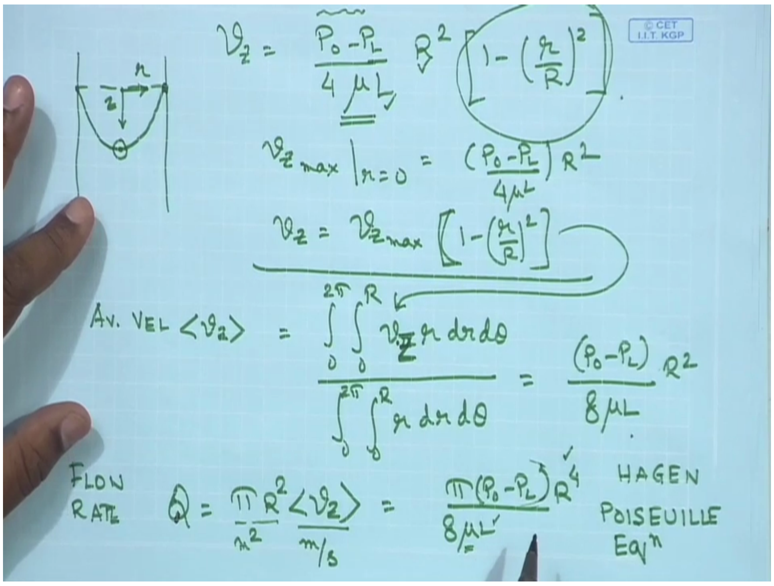

So you measured the Q, the amount of liquid that you are collecting per unit time and if it is falling freely vertically then p0 minus pL can be substituted just by the gravity force. You need to know the diameter or the radius of the capillary, the length is known, so thisyou are going to find out experimentally. All these are known to you. The only one known here is the property mu.

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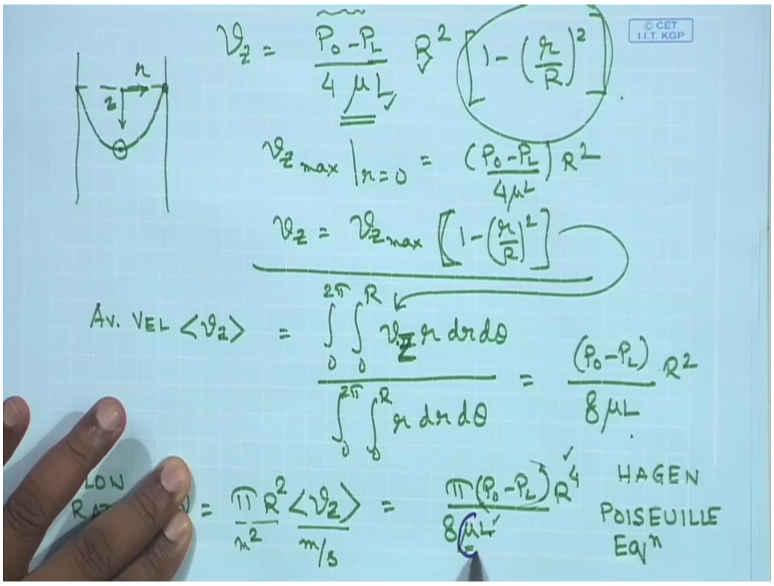

So using a capillary viscometer where a very thin capillary is used to obtain some flow out of the capillary, the only unknown being mu. Mu can be calculated. So that is the principle of capillary viscometer. Whenever we (ha) get such an expression, we need to be careful about what are the assumptions we have made because we need to know the assumption in order to get an idea of the applicability, the reason of applicability of in a relation or a correlation that we have developed.

The first (appro) assumption that we have made is that it is laminar flow. Straight streamline laminar flow. So you cannot have a very high pressure gradient being applied to certain length of the pipe. So that the flow inside gets disturbed and the flow becomes turbulent, does not remain lamina. If that is the case, this analysis will not be valid. This analysis also assume that it's and incompressible fluid. That is rho is constant,the density of the fluid is a constant.

So this relation is restrictedto incompressible fluid as well. So incompressible straight streamline laminar flow are some of the constraint, some of the conditions which must be met before Hagen Poiseuille equation can be used to find out how the flow rate and the conditions are related by viscosity for such a case. But this is a simple yet elegant example of the use of shell momentum balance in everyday problem.

And this Hagen Poiseuille equation has so many uses in everyday life in physics and in so many other cases. In order for the liquid column inside the pipe or tube to move with a constant velocity, we have assume that sum of all forces acting on it must be equal to zero. Otherwise if it is not the case then the column of liquid which is flowing inside the tube will either accelerate or will slow down. Now what are the forces which are acting on it? So physically we need to find out, think what are the forces which are acting on it?

The liquid column is going down because it is acted open by a difference in pressure. High pressure on the side and low pressure on the side. Which is trying to pull it in this way. There is also a gravity which is trying to make the liquid column move in the plus Z direction. So these two pressures and the gravity are acting in the same direction. So there must be an opposing force which is going to be equal to the combined effect of these two forces equal and opposite. Only then it's going to move with a constant velocity.

What is the opposing force? What is that force that opposes the motion of a liquid? There is no other way but viscosity. So viscosity is the one, the viscous force is the one which opposes any flow of the liquid imposed by some other conditions. Pressure difference or maybe gravity. So if you try to find out what is the force acting on it? Which is the opposing force that is making the liquid column move at a constant velocity?

That must be the viscous force and without solving anything heuristically we should be able to say that the opposing viscous force is simply equal to the force due to pressure and the force due to gravity which are acting on the column of the fluid. So but in many case if you like to analytically find out what is the force that is acting on it? The gravity you need to find out what's the viscous force at small r equals capital R that means at the solid liquid-interface along with the inner wall of the pipe.

So what is that (shea) force? It's taurZ, the shear stress evaluated at small r equals capital R. So tau rZ evaluated at the inner wall of the tube and this tau rZ now has to be multiplied with the appropriate area. The area on which it is acting on. So what is the area on which this wall shear is acting on? Must be equal to twice pie R times L. So twice pie R L times tau rZ evaluated at small r equals capital R, is essentially the force the viscous force that we are trying to find out.

So once you do the calculation you would see that the viscous force is going to be equal to the combined force due to the pressure and the weight of the liquid column. So this Hagen Poiseuille equation is something which we use everyday. But you can get such a neat expression starting with the simple concept of shell momentum balance. So what we are going to do now is wewill try to solve a few other problem using this simple concept of shell momentum balance.

And towards the end of these exercises we would slowly start to feel that it is getting increasingly difficult to use the shell momentum balance because we need to visualize the complex geometries and those situations in which it is not a one dimensional flow. If you have flow in two dimensions, if you have velocity in both X and Y direction, the shell momentum balance may not work or it becomes too combustion.

So a generalized treatment, the need for a generalized treatment will become apparent as we start solvingmore and more difficult problems. But right now I willshow you one or two more examples of the shell momentum balance and then do a conceptual problem which is also very interesting.

So let's now start with another problem which is an industrial problemin which case upward moving belt. So let's say that this is a belt which rises. So this is a liquidfilm, this is liquid. Initially the belt is stationery. But let’s say at some point the belt starts to move upwards with velocity U0.

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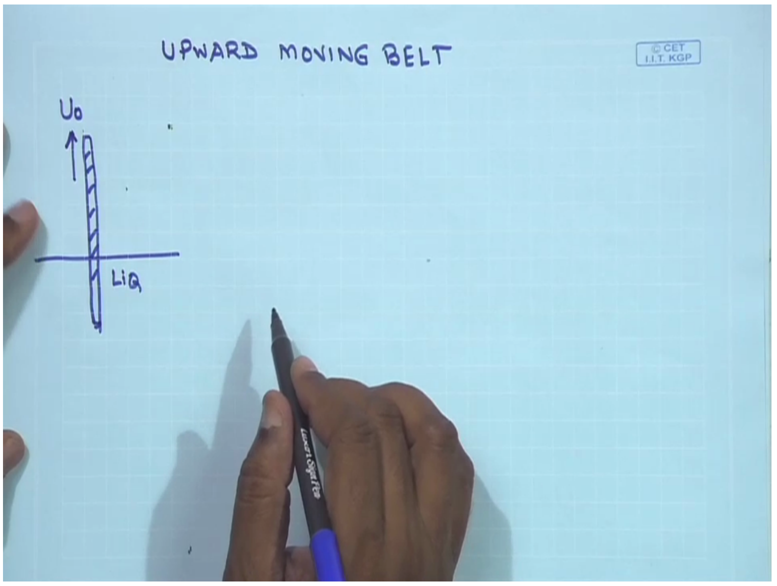

After you provide sufficient time, when it reaches steady state,the belt is going to have a thin film of liquid stuck to it. So this is the liquid. This is your belt which moves with a velocity U0 and you have given sufficient time such that this is the steady state which is reached. And let's say this is my X direction, this is the Y direction and at some point when it reaches steady state, the thickness of the film is constant and is equal to H.

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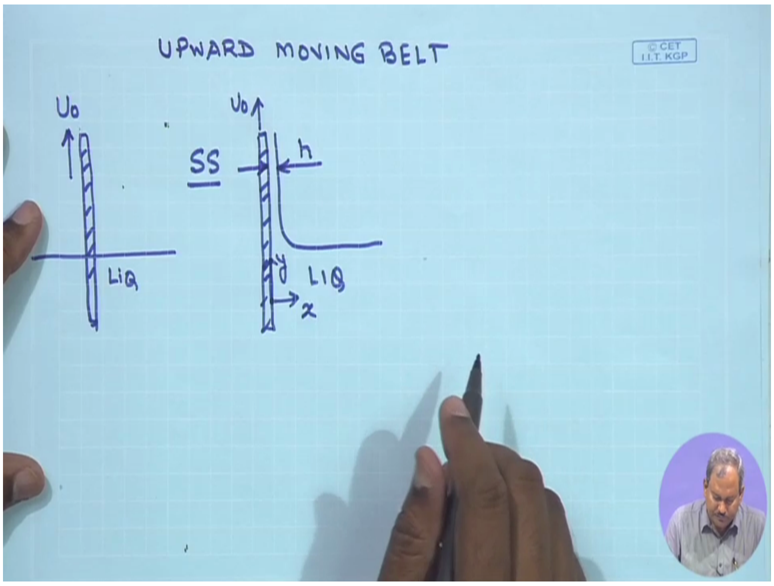

What we need to find out is, and you can see that you have vY in here. The vY is not equal to zero. VX is however equal to zero. There is no pressure gradient which is acting in this case and only a force which is acting is gravity. So the belt starts to move upward and it will carry a thin film of liquid along with it. But the gravity would like to drain the liquid in the reverse direction. So viscous forces pulls the liquid up, gravitational forces tend to drain the liquid and when the steady state is reached, let us assume that the thickness of the film is given by H.

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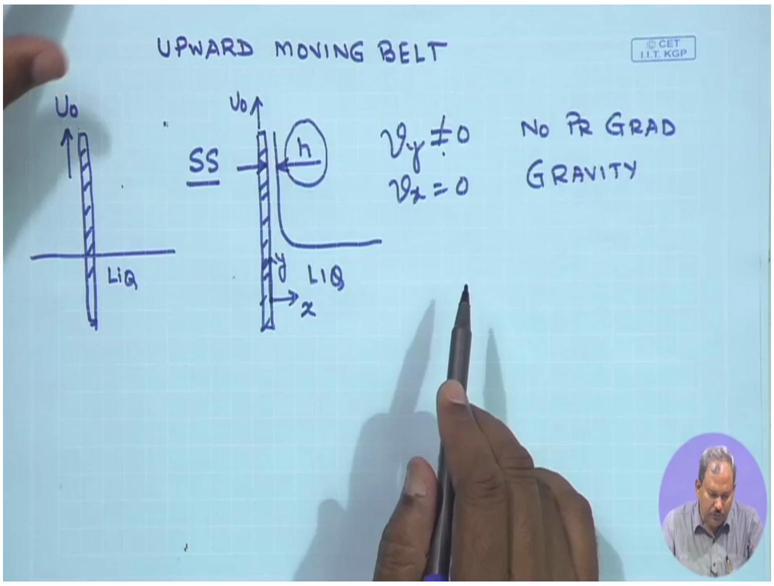

So if that's the case, then we would like to find out what's going to be the velocity distribution, vY in this case, in the thin portion of this liquid.

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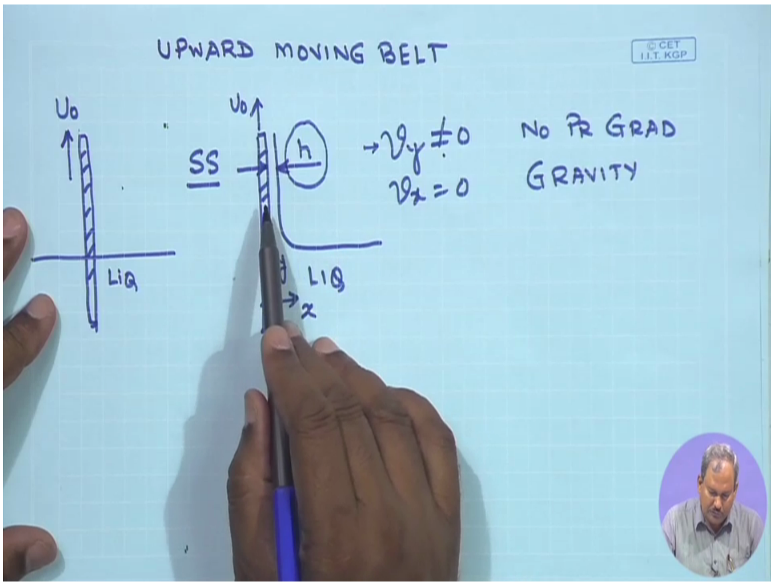

So we have to think of a shell and in order to think of a shell we need to first identify what is the direction in which the velocity is varying. Because whatever be that direction, that's going to be the smaller dimension of the control volume. You can clearly see that the velocity is varying with X. this is moving upwards, this is a free surface, the velocity does not vary with Y and there is no vX in here.

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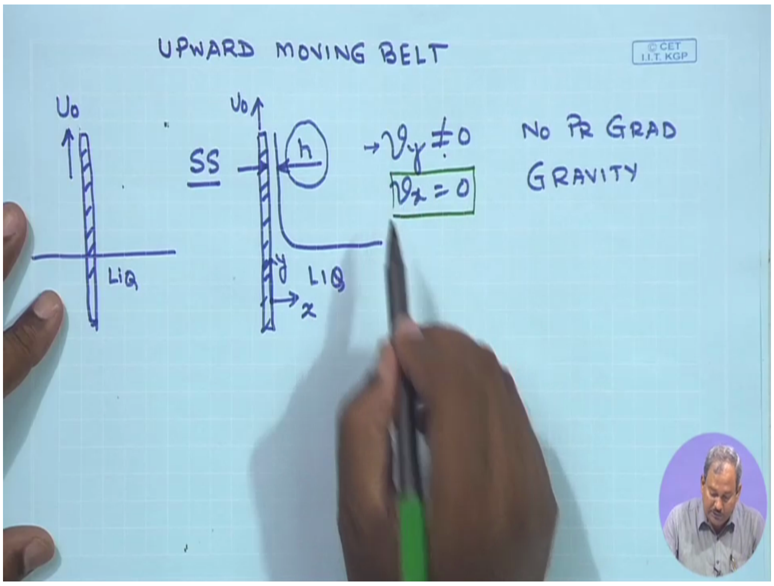

So in order to obtain the shell, if this is my wall and this is the liquid film and this is X and this is Y, then my shell must be something which is of thickness delta x. It could be any length L any width Y, does not matter. Because vY is not a function of Y, it's not a function of Z, it's only a function of X. So that's why Iam taking my shell as this.

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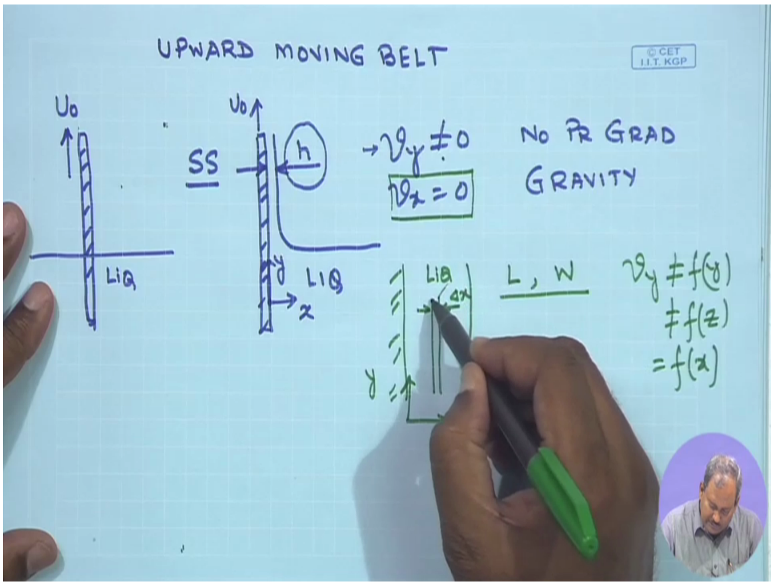

Now when you consider this shell, then we understand there's going to be, if I take this shell of delta x in thickness and on this side I have the wall and on this side I have a free surface. So some amount of momentum, which is convictive momentum will come in. Some amount of convictive momentum will go out. But these two must be equal to each other because of my assumption that vY is not a function of Y. So whatever be the convective momentum here and convective momentum here, they must be equal and opposite.

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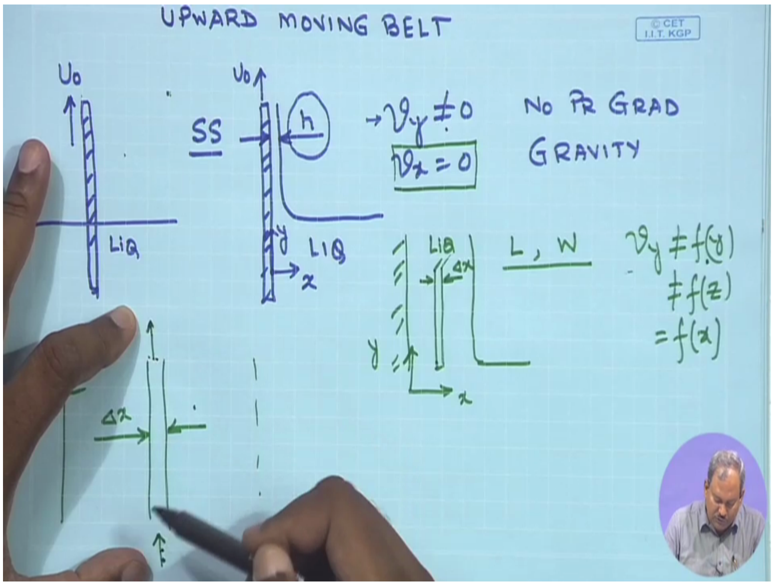

So I do not need to write, when I write the momentum in minus out plus sum of all forces is equal to zero and I identify there's going to be a convective part and there's going to be a conductive part. The convection part, I do not need to write since the velocities at these two points are equal. But I need to take into the account the tau. Now what is going to be the subscript of tau? The subscript of tau, the first subscript would be the direction in which the motion is taking place.

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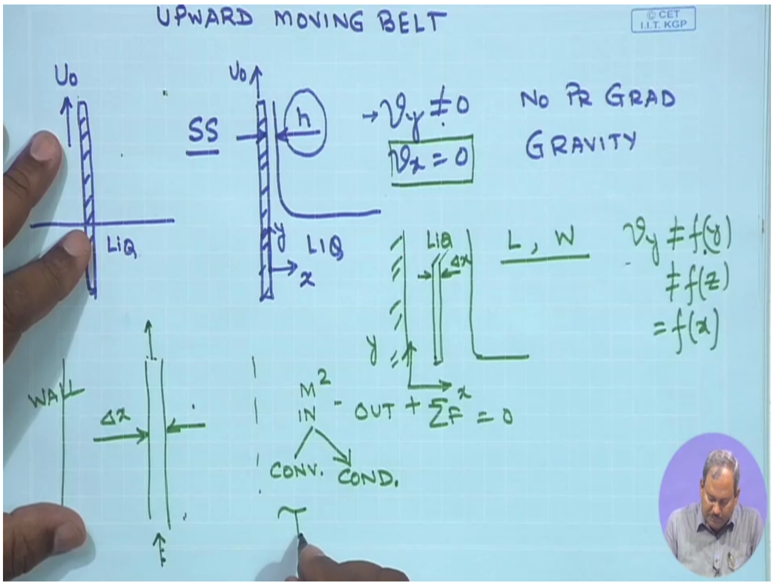

So if you seethis basic problem, the motion is upwards in the plus Y direction. So the first subscript of the shear stress in this case is simply going to be tau Y and as a result of motion in the upward direction and variation in velocity, the momentum gets transported in the X direction. So the subscript of tauwould be tau XY in this case. Y momentum gets transported in the X direction. So tau XYthis is going to be the shear stress.

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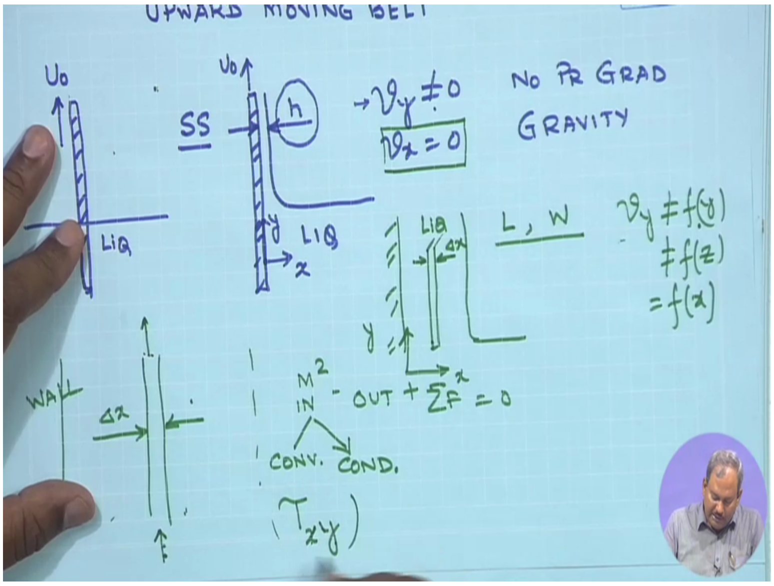

It's acting on an area. This is L and the width of this is W. So it's acting on an area which is L times W and this is evaluated at X. The out one is going to be tau XY multiplied by L and W again. Evaluated at X plus delta x and its no body force is acting on it, its only gravity force. In order to obtain gravity force I first need to find out what is the volume of the liquid which is contained in the control volume, which is L W timesdeltax mass, when I multiply it with rho and I put G in order to obtain the gravitational force.

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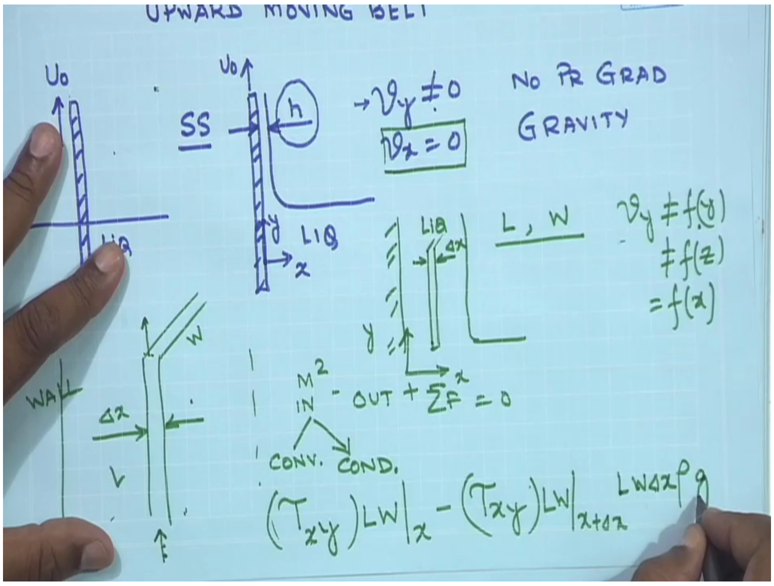

But if you look at the coordinates system in here, I must put minus sign in this case since my G and Y are oppositely directed. So convective momentum in, minus convective momentum out and then the gravitational force acting in the reverse direction, no pressure forces, at steady state the sum of all these must be equal to zero.

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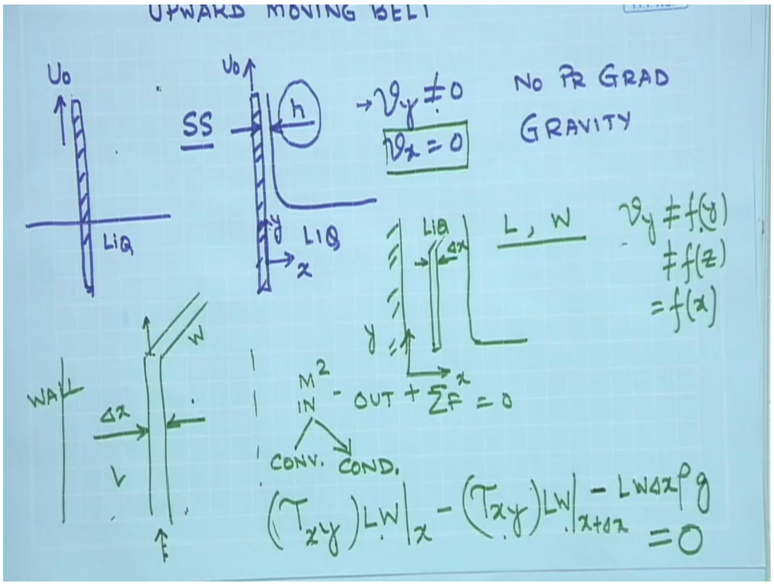

So you see how easy it is now to write the difference equation. It is progressively becoming it easier for you to clearly visualize the flow condition. Write the governing equations. Get rid of all the terms which are not relevant. For example in this case the transport of momentum by convection by flow. Since the velocities are same at the bottom and at the top. So no net contribution of momentum.

The only contribution is the viscous transport in, viscous transport out and the gravity which is acting in the reverse direction. So thealgebraic sum of these three must be equal to zero and that's what I have written in here.

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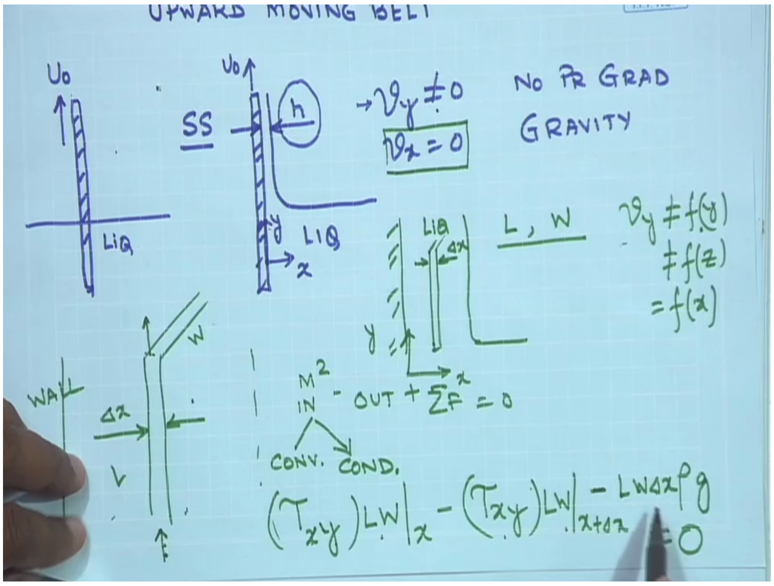

So the next step is simply divide both the sides by del x, taking the limit and obviously cancel LW from all sides. Divide both sides by del x and taking the limit points del x tends to zero.

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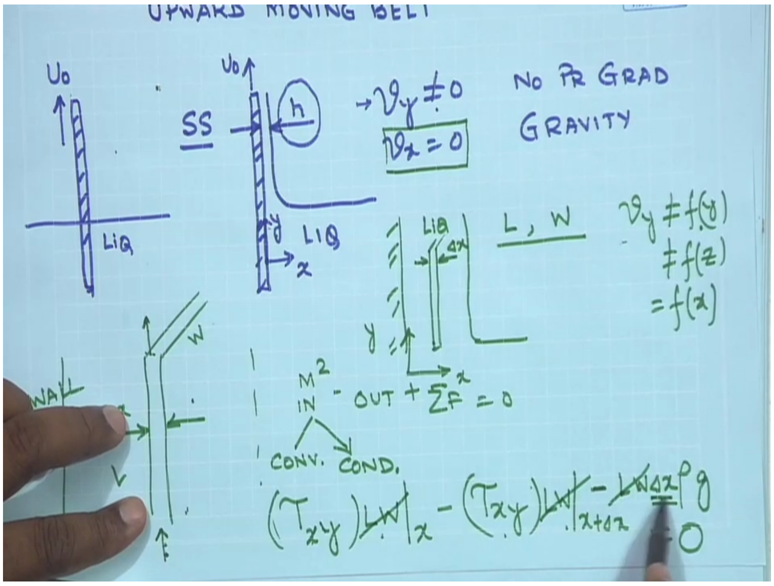

What you would get is D dX of tau XY is equal to minus rho G. So this becomes your governing equation right now.

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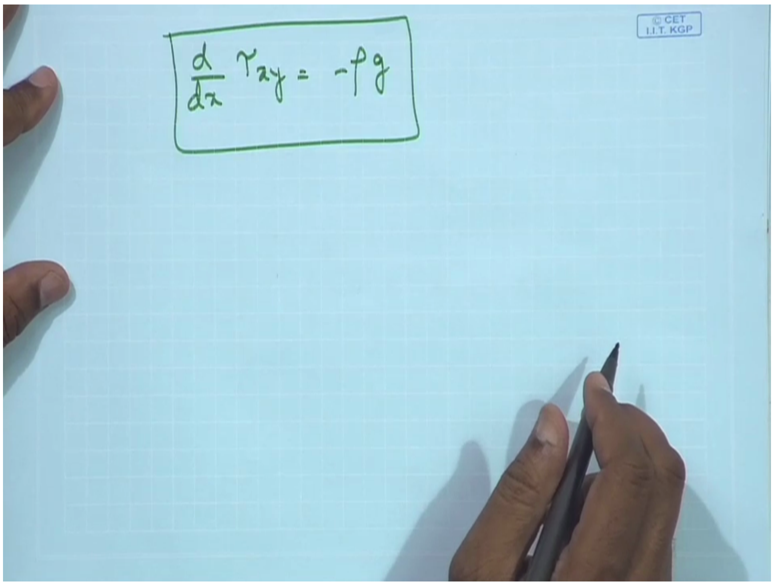

The momentum again, this is the belt and this is the film, this is the liquid, this is thickness of the liquid film which is H. Now this is air. So the first thing that we can do is we can simply integrate it once equals minus rho GX plus C1. Now at X equals H, tau XY must be equal to zero. X equal to H is a liquid-vapor interface and at liquid-vapor interface, the shear stress is zero. If the shearis zero at this point, at X is equal to H tau XYwould be equal to zero.

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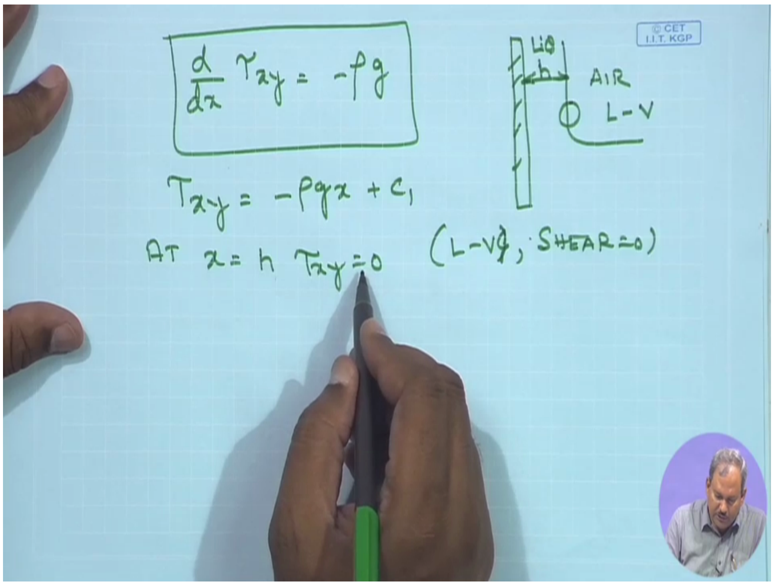

So you C1 would be simply equal to rho GH. And also tau, if I assume it's a Newtonian fluid, it is simply going to be mu timesd vY by dX and this D vY by dX in using plugging in this in here and using the value of C1as rho GH,you can integrate this. I am leaving it for you to do.

VY would be U0 minus rho GH square by 2 mu times 2 X by H minus X by H whole square. This is the distribution you should get.

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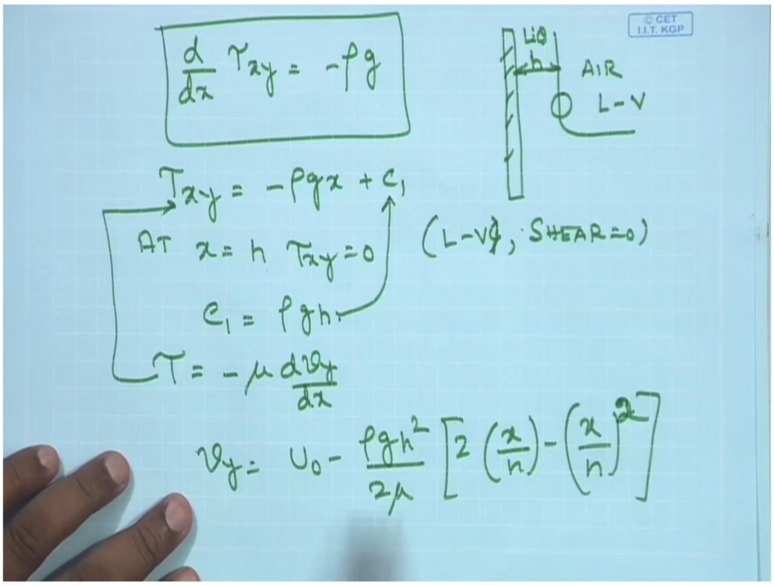

Now you can see that it is slightly more (comp) complicated than the simple parabolic distribution profile. You have a linear term and you have a quadratic term and you have the velocity with whichthe belt has been pulled up. The second boundary condition would be at X is equal to zero. Your vY must be equal to U. This is a very important boundary condition.

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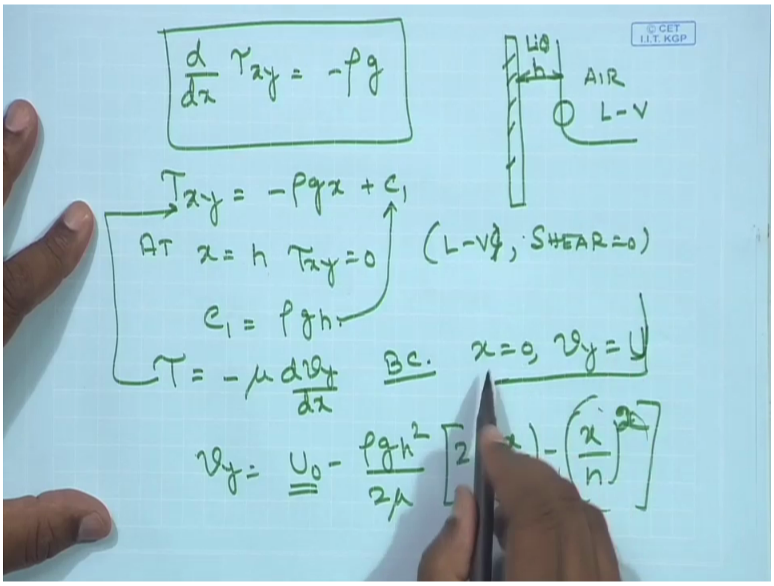

At X is equal to zero that means at the liquid-solid interface there is no relative velocity. So all the molecules of the liquid which I here,they are moving up with the plate with the same velocity as U.

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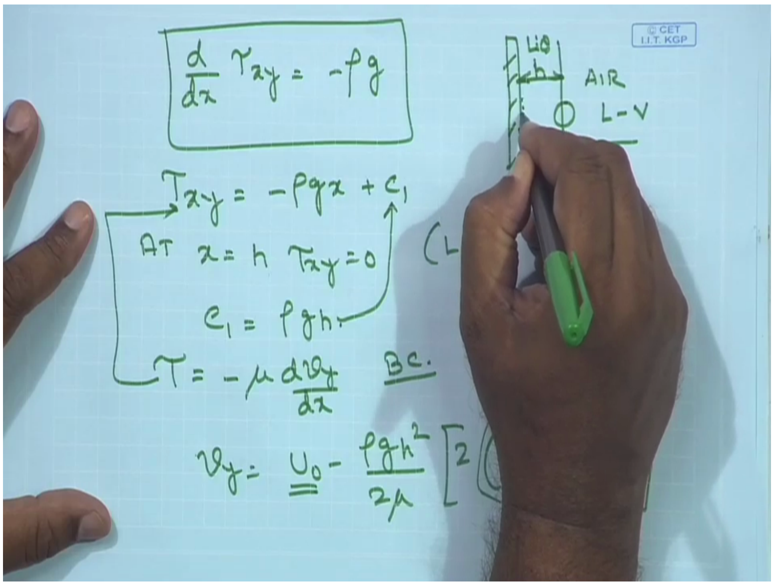

So at the liquid-solid interface, the relative velocity is zero. Which is initially we call it as no slip velocity. Since it is no slip,the velocity must be equal to the velocity X is equal to zero, must be equal to the upward velocity of the belt.

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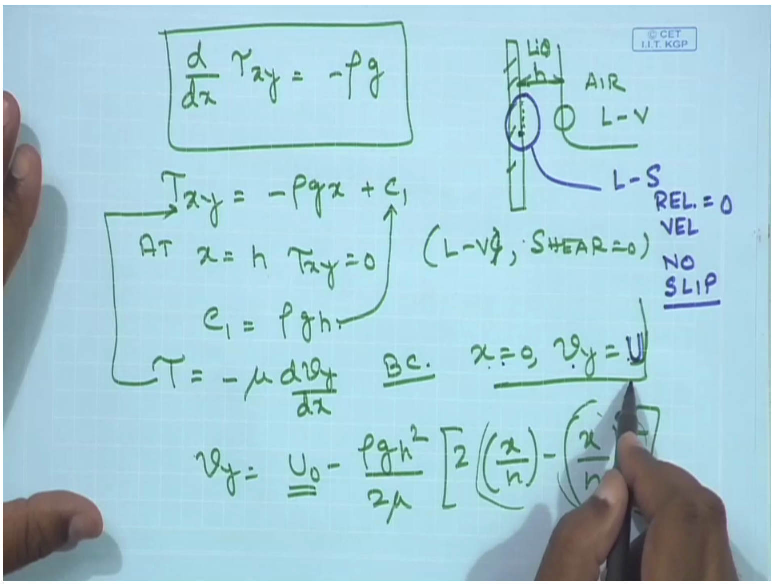

So the belt pulls the liquid, the gravity tries to drain the liquid. A steady state is reached with the condition that the velocity on the belt on the liquid side would be equal to the upward velocity of the belt which is no slip. And if this is the thickness and if this is the liquid-vapor interface, at that point shear stress would be zero. No slip, no relative velocity at the solid-liquid interface and no shear at the liquid-vapor interface. Combination of these two would give you a very clean neat expression for the velocity which is a function of operational parameter, which is U0.

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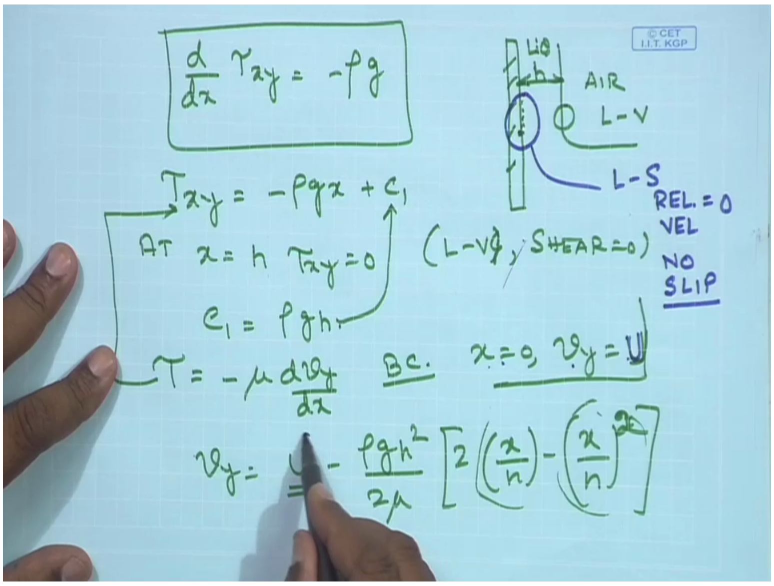

Which is a function of the property, which is mu. The H is again the operational parameters which dictates what's going to be the value of H and as a function of X and so on. So rho G X and so on.

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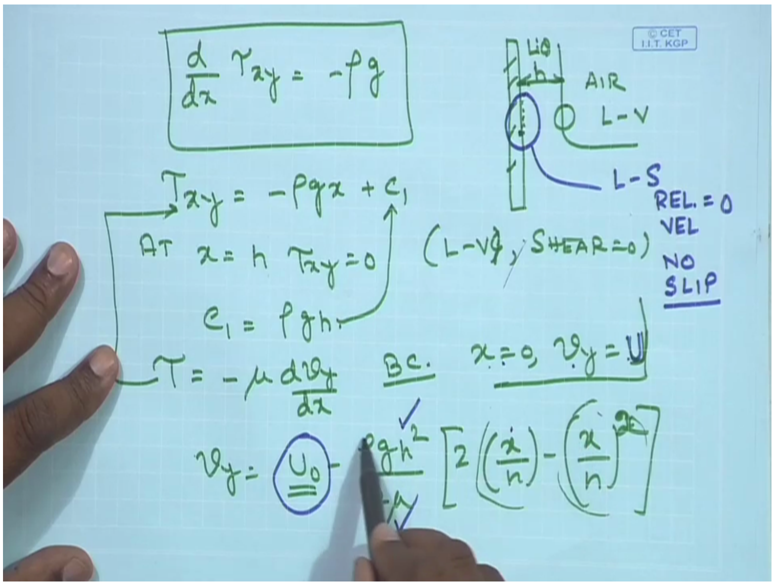

So another example of how to use shell momentum balance to solve problems of momentum transfer and in the next class we will see slightly more different problem. Thank you.